Journal of Universal Computer Science, vol. 9, no. 9 (2003), 1137-1167
submitted: 10/2/03, accepted: 9/6/03, appeared: 28/9/03 © J.UCS

Propositional Interval Neighborhood Temporal Logics

Valentin Goranko
Rand Afrikaans University, South Africa
vfg@rau.ac.za

Angelo Montanari
University of Udine, Italy
montana@dimi.uniud.it

Guido Sciavicco
University of Udine, Italy
sciavicc@dimi.uniud.it

Abstract: Logics for time intervals provide a natural framework for dealing with time in various areas of computer science and artificial intelligence, such as planning, natural language processing, temporal databases, and formal specification. In this paper we focus our attention on propositional interval temporal logics with temporal modalities for neighboring intervals over linear orders. We study the class of propositional neighborhood logics ($\mathcal{PNL}$) over two natural semantics, respectively admitting and excluding point-intervals. First, we introduce interval neighborhood frames and we provide representation theorems for them; then, we develop complete axiomatic systems and semantic tableaux for logics in $\mathcal{PNL}$.

Categories: F.4.1 [Mathematical Logic and Formal Languages]: Temporal Logic; I.2.4 [Knowledge Representation Formalisms and Methods]: Temporal Logic

Keywords: Interval Temporal Logic, Axiomatic Systems, Tableau Systems

1 Introduction

Logics for time intervals provide a natural framework for dealing with time in various areas of computer science and artificial intelligence, such as planning and natural language processing, where reasoning about time intervals rather than time points is far more natural and closer to common sense (differences and similarities between point-based and interval-based temporal logics are systematically analyzed in [Ben91]). Various interval temporal logics have been proposed in the literature. The most important propositional ones are Halpern and Shoham's Modal Logic of Time Intervals (HS) [HS91] and Venema's CDT logic [Ven91], while relevant first-order interval logics are Zhou and Hansen's Neighborhood Logic (NL) [ZH98] and Moszkowski's Interval Temporal Logic (ITL) [Mos83]. (In [GMS03c] we survey the main developments, results, and open problems on interval temporal logics and duration calculi.)

HS features four basic operators: $\langle B\rangle$ (*begin*) and $\langle E\rangle$ (*end*), and their transposes $\langle\overline{B}\rangle$ and $\langle\overline{E}\rangle$. Given a formula φ and an interval $[d_0, d_1]$, $\langle B\rangle\varphi$ holds at $[d_0, d_1]$ if φ holds at $[d_0, d_2]$, for some $d_2 < d_1$, and $\langle E\rangle\varphi$ holds at $[d_0, d_1]$ if φ holds at $[d_2, d_1]$, for some $d_2 > d_0$. All other temporal operators corresponding to Allen's relations can be defined by means of the basic ones. In particular, it is possible to define the (strict) *after* operator $\langle A\rangle$ (resp., its transpose $\langle\overline{A}\rangle$) such that $\langle A\rangle\varphi$ (resp., $\langle\overline{A}\rangle\varphi$) holds at $[d_0, d_1]$ if φ holds at $[d_1, d_2]$ (resp., $[d_2, d_0]$) for some $d_2 > d_1$ (resp., $d_2 < d_0$), and the sub-interval operator $\langle D\rangle$ such that $\langle D\rangle\phi$ holds at a given interval $[d_0, d_1]$ if ϕ holds at a proper sub-interval $[d_2, d_3]$ of $[d_0, d_1]$. Complete axiomatic systems for HS with respect to several classes of structures are given in [Ven90], while the undecidability of HS over various linear orders has been proved in [HS91] by encoding the halting problem in it. Venema's CDT has three binary operators, namely, C (*chop*), D, and T, which correspond to ternary interval relations occurring when an extra point is added in one of the three possible distinct positions with respect to the two endpoints of the current interval (*between, before*, and *after*), and a modal constant π which holds over an interval $[d_0, d_1]$ if $d_0 = d_1$. Axiomatic systems for CDT can be found in [Ven91]. Since HS can be embedded into CDT, the undecidability of the latter follows from that of the former. Furthermore, in [Lod00] Lodaya shows that the fragment of HS that only contains $\langle B\rangle$ and $\langle E\rangle$, interpreted over dense linear orders, is already undecidable. Since both $\langle B\rangle$ and $\langle E\rangle$ can be easily defined in terms of C and π, it immediately follows that a logic only provided with C and π is undecidable as well.

Zhou and Hansen's NL features the two 'expanding' modalities $\Diamond_r$ and $\Diamond_l$ and a special symbol l denoting the length of the current interval. Given a formula φ and an interval $[d_0, d_1]$, $\Diamond_r\varphi$ holds at $[d_0, d_1]$ if φ holds at $[d_1, d_2]$, for some $d_2 \geq d_1$; $\Diamond_l\varphi$ holds at $[d_0, d_1]$ if φ holds at $[d_2, d_0]$, for some $d_2 \leq d_0$; and the valuation of l over $[d_0, d_1]$ is $d_1 - d_0$. Some properties, applications, and extensions of NL are given in [BZ97, Roy97], while a complete axiomatic system can be found in [BRZ00]. NL undecidability can be easily proved by embedding HS in it. Moszkowski's ITL is a first-order interval logic providing two modalities, namely $\bigcirc$ (*next*) and C. In ITL an interval is defined as a finite or infinite sequence of states. Given two formulas φ, ψ and an interval $s_0, \ldots, s_n$, $\bigcirc\varphi$ holds over $s_0, \ldots, s_n$ if φ holds over $s_1, \ldots, s_n$, while $\varphi C\psi$ holds over $s_0, \ldots, s_n$ if there exists i, with $0 \leq i \leq n$, such that φ holds over $s_0, \ldots, s_i$ and ψ holds over $s_i, \ldots, s_n$. Studies of axiomatic systems and completeness for fragments and extensions of ITL include [Dut95, Gue00]. ITL has been proved to be undecidable even at the propositional level by a reduction from the problem of testing the emptiness of the intersection of two grammars in Greibach form [Mos83]. As a matter of fact, the results given in [Lod00] prove the undecidability of (a variant of) ITL with the C and the modal constant ϕ for point-intervals, interpreted over dense linear ordering. A decidable fragment of propositional ITL with quantification over propositional variables has been obtained by imposing a suitable *locality* constraint [Mos83]. Such a constraint states that each propositional variable is true over an interval if (and only if) it is true at its first state. This allows one to collapse all the intervals starting at the same state into the single interval consisting of the first state only. By exploiting such a constraint, decidability of Local ITL can be easily proved by embedding it into Quantified Propositional Linear Temporal Logic.

In order to model duration properties of real-time systems, both NL and ITL have been extended with a notion of 'state variable' that represents an instantaneous observation of the system behavior. In particular, the Duration Calculus (DC) extends ITL by adding *temporal variables* (also called state expressions) as integrals of state variables [ZH98, ZHR91, HZ97]. Temporal variables make it possible to represent the duration of intervals as well as numerical constants. As an example [SRR90], the specification of the behavior of a gas burner can include conditions as the following one: "for any period of 30 seconds the gas may leak, that is, flow and not burn, only once and for 4 seconds at most". Such a condition is expressed by the DC formula: $l > 30 \vee ((\int(\neg Gas \vee Flame) = l); (\int(Gas \wedge \neg Flame) = l \wedge l \leq 4); (\int(\neg\, Gas \vee Flame) = l))$, where Gas (the gas is flowing) and $Flame$ (the gas is burning) are two state variables. In [ZHS93] Zhou et al. show that DC is undecidable, the main source of undecidability being the fact that state changes in real-time systems can occur at any time point.

In this paper we study propositional interval neighborhood temporal logics, the family of which we denote by $\mathcal{PNL}$. Logics in $\mathcal{PNL}$ can express meaningful timing properties, without being excessively expressive to an extent easily leading to high undecidability, a typical phenomenon for interval logics. They feature two modalities which correspond to Allen's *meet* and *met by* relations [All83], intuitively capturing a *right* neighboring interval and a *left* neighboring interval. There are two natural semantics for interval logics interpreted over linearly ordered domains, namely a *non-strict* one, which includes intervals with coincident endpoints (*point-intervals*), and a *strict* one, which excludes them and was already studied in [AH85, Lad87]. To make it easier to distinguish between the two semantics from the syntax, the modal operators $\Diamond_r$ and $\Diamond_l$ are used in the case of non-strict propositional neighborhood logics, generically denoted by $\mathcal{PNL}^+$, while for the strict ones, denoted by $\mathcal{PNL}^-$, $\Diamond_r$ and $\Diamond_l$ are replaced by $\langle A\rangle$ and its transpose $\langle\overline{A}\rangle$, respectively. While the logics in $\mathcal{PNL}^+$are built on the propositional fragment of NL, those in $\mathcal{PNL}^-$can be viewed as based on the $A\overline{A}$-fragment of HS. In fact, the semantics of HS admits point-intervals and hence, according to our classification, it is non-strict. However, the modalities $\langle A\rangle$ and $\langle\overline{A}\rangle$ only refer to strict intervals, and thus the semantics of the fragment

$A\overline{A}$ can be considered essentially strict.

The main contributions of the paper are: (i) representation theorems for strict and non-strict interval neighborhood structures; (ii) complete axiomatic systems for logics in $\mathcal{PNL}$; (iii) complete semantic tableaux for $\mathcal{PNL}$ logics. Unlike classical logic and most modal and temporal logics, where the first-order axiomatic systems are obtained by extending their propositional fragments with relevant axioms for the quantifiers, the first-order NL was axiomatized first, without its propositional fragment having been identified. It now turns out that the latter was hidden into the originally introduced first-order axiomatic system, the propositional axioms of which, taken alone, are substantially incomplete. In particular, a curious feature of NL is that while it can be finitely axiomatized, its propositional fragment involves an infinite axiom scheme. The strict analogue, however, is a finitely axiomatized subsystem of the latter. As for the tableaux, there is no a straightforward way of adapting existing tableaux for point-based propositional and first-order temporal logics to interval temporal logics [Wol85, Eme90]. We develop an original tableau method for $\mathcal{PNL}$ logics which combines features of classical first-order tableau and point-based temporal tableaux. (For a detailed account of the existing tableau methods see [DGHP99].)

There are very few tableau methods for time interval logics and duration calculi in the literature. In [BT03], Bowman and Thompson consider an extension of Local ITL, which, besides the chop operator C, contains a projection operator *proj* and the modal constant π. They introduce a normal form for the formulas of the resulting logic that allows them to exploit a classical tableau method, devoid of any mechanism for constraint label management. In [CSdC00], Chetcuti-Sperandio and Fariñas del Cerro identify a decidable fragment of DC, which is expressive enough to model the above-given condition on the behavior of a gas burner, that imposes no restriction on state expressions, but encompasses a proper subset of DC operators, namely, $\wedge$, $\vee$, and C. The tableau construction for the resulting logic combines application of the rules of classical tableaux with that of a suitable constraint resolution algorithm and it essentially depends on the assumption of *bounded* variability of the state variables. Finally, tableau systems for the propositional and first-order Linear Temporal Logic (LTL), which employ a mechanism for labeling formulas with temporal constraints somewhat similar to ours, are given in [SGL97] and [CMP99], respectively. The main differences between these tableau methods and ours are: (i) they are specifically designed to deal with integer time structures (i.e., linear and discrete) while ours is quite generic; (ii) LTL is essentially point-based, and intervals only play a secondary role in it (viz., a formula is true on an interval if and only if it is true at every point in it), while in our systems intervals are primary semantic objects on which the truth definitions are entirely based; (iii) the closedness of a tableau is defined in terms of unsatisfiability of the associated set of temporal constraints, while in our system it is entirely syntactic.

The rest of the paper is organized as follows. In Section 2 we introduce interval neighborhood frames and structures; then, in Section 3 we provide representation theorems for both strict and non-strict semantics. In Section 4 we give syntax and semantics of $\mathcal{PNL}$, and in Section 5 we define various logics in this class. In Section 6 we briefly discuss the expressive power of $\mathcal{PNL}$ and we give some examples of its use. Sections 7 and 8 are devoted to the axiomatic systems and respective completeness theorems for both semantics. In Section 9 we develop semantic tableaux for $\mathcal{PNL}$ logics, and prove their soundness and completeness. In the last section we provide an assessment of the work done and we briefly discuss our ongoing research on $\mathcal{PNL}$.

2 Interval Neighborhood Frames and Structures

In this section, we introduce the basic notions of interval neighborhood frame and structure.

Definition 1. A **neighborhood frame** is a triple $\mathbf{F} = \langle \mathbb{I}, R, L \rangle$ where $\mathbb{I}$ is a non-empty set and R, L are binary relations on $\mathbb{I}$.

For every sequence $S_1, ..., S_k \in \{R, L\}$, we denote the composition of the relations $S_1, ..., S_k$ by $S_1 ... S_k$. Also, we put:

$\mathbf{B_F} = \{w \in \mathbb{I} \mid \text{there is no } v \in \mathbb{I} \text{ such that } wLv\}$,

$\mathbf{B_F^2} = \{w \in \mathbb{I} \mid \text{there are no } u, v \in \mathbb{I}, \text{ with } u \neq v, \text{ such that } wLv \text{ and } wLu\}$,

$\mathbf{E_F} = \{w \in \mathbb{I} \mid \text{there is no } v \in \mathbb{I} \text{ such that } wRv\}$, and

$\mathbf{E_F^2} = \{w \in \mathbb{I} \mid \text{there are no } u, v \in \mathbb{I}, \text{ with } u \neq v, \text{ such that } wRv \text{ and } wRu\}$.

Consider the following conditions:

(NF1) R and L are mutually inverse;

(NF2) $\forall x \forall y (\exists z (xLz \wedge zRy) \rightarrow \forall z (xLz \rightarrow zRy))$ and
$\forall x \forall y (\exists z (xRz \wedge zLy) \rightarrow \forall z (xRz \rightarrow zLy))$;

(NF3) $RL \subseteq LRR \cup LLR \cup E$ on $\mathbb{I} - \mathbf{B_F^2}$ and $LR \subseteq RLL \cup RRL \cup E$ on $\mathbb{I} - \mathbf{E_F^2}$, where E is the equality, that is,

$\forall x \forall y (\exists z \exists u (xLz \wedge zLu) \wedge \exists z (xRz \wedge zLy) \rightarrow x = y \vee$
$\exists w \exists z ((xLw \wedge wRz \wedge zRy) \vee (xLw \wedge wLz \wedge zRy)))$

and

$\forall x \forall y (\exists z \exists u (xRz \wedge zRu) \wedge \exists z (xLz \wedge zRy) \rightarrow x = y \vee$
$\exists w \, \exists z ((xRw \wedge wLz \wedge zLy) \vee (xRw \wedge wRz \wedge zLy)))$;

(NF4) $RRR \subseteq RR$, i.e. $\forall w \forall x \forall y \forall z(wRx \wedge xRy \wedge yRz \rightarrow \exists u(wRu \wedge uRz))$.

Definition 2. An **interval neighborhood frame** is a neighborhood frame $\mathbf{F} = \langle \mathbb{I}, R, L \rangle$ satisfying the conditions NF1,...,NF4.

Note that, assuming **NF1, NF4** is equivalent to

$$\forall w \forall x \forall y \forall z(wLx \wedge xLy \wedge yLz \rightarrow \exists u(wLu \wedge uLz)).$$

Definition 3. An interval neighborhood frame $\mathbf{F} = \langle \mathbb{I}, R, L \rangle$ is said to be:

- **strict,** if the relation LRR is irreflexive, and **non-strict** if the relation LRR is reflexive (note that 'not strict' does not imply 'non-strict');
- **open,** if it satisfies the condition $\forall x(\exists y(xLy) \wedge \exists y(xRy))$;
- **rich,** if it satisfies the condition $\forall x(\exists y(xRy \wedge yRy) \wedge \exists y(xLy \wedge yLy))$;
- **normal,** if it satisfies the condition $\forall x \forall y(\forall z(zRx \leftrightarrow zRy) \wedge \forall z(zLx \leftrightarrow zLy) \rightarrow x = y)$;
- **tight,** if it satisfies the condition $\forall x \forall y((xRRy \wedge yRRx) \rightarrow x = y)$;
- **weakly left-connected** (resp., **weakly right-connected**) if the relation $LR \cup LRR \cup LLR$ (resp., $RL \cup RRL \cup RLL$) is an equivalence relation on $\mathbb{I} - \mathbf{B_F}$ (resp., $\mathbb{I} - \mathbf{E_F}$); **left-connected** (resp., **right-connected**) if that relation is the universal relation on $\mathbb{I} - \mathbf{B_F}$ (resp., $\mathbb{I} - \mathbf{E_F}$);
- **weakly connected** if each of the relations $LR \cup LRR \cup LLR$ and $RL \cup RRL \cup RLL$ is an equivalence relation on $\mathbb{I}$; **connected**, if each of these relations is the universal relation on $\mathbb{I}$.

Now, consider the following definitions:

(NF5) NF2 implies $LRL \subseteq L$ and $RLR \subseteq R$, that is,
$\forall x \forall y((xLRLy \rightarrow xLy) \wedge (xRLRy \rightarrow xRy))$;

(NF6) assuming NF2, normality implies
$\forall x \forall y(\exists z(zRx \wedge zRy) \wedge \exists z(zLx \wedge zLy) \rightarrow x = y)$, that is,
$\forall x \forall y(xLRy \wedge xRLy \rightarrow x = y)$.
Assuming also openness, normality becomes equivalent to that condition;

(NF7) in every non-strict interval neighborhood frame, $RR = RRR$ and $LL = LLL$;

(NF8) every rich interval neighborhood frame is non-strict and open;

(NF9) every non-strict interval neighborhood frame is weakly connected. Every strict interval neighborhood frame is weakly left- and right-connected;

(SNF) in every strict interval neighborhood frame each of L, R, LLR, RRL, and RLL is irreflexive, too;

(NNF) an interval neighborhood frame is non-strict iff either of $LRR \cup LLR$ and $RLL \cup RRL$ is an equivalence relation on $\mathbb{I}$.

Proposition 4. *NF5, NF6, NF7, NF8, SNF, and NNF are consequences of the definitions.*

Eventually, we are interested in *concrete* interval neighborhood structures.

Definition 5. If $\langle \mathbb{D}, < \rangle$ is a linearly ordered domain, an **interval** in $\mathbb{D}$ is a pair $[d_0, d_1]$ such that $d_0, d_1 \in \mathbb{D}$ and $d_0 \leq d_1$. $[d_0, d_1]$ is a **strict interval** if $d_0 < d_1$, while it is a **point interval** if $d_0 = d_1$.

Definition 6. A **non-strict interval neighborhood structure** is a neighborhood frame $\langle \mathbb{I}(\mathbb{D})^+, R, L \rangle$, where $\mathbb{I}(\mathbb{D})^+$ is the set of all intervals over some linear ordering $\langle \mathbb{D}, < \rangle$ and R, L are mutually inverse binary relations over $\mathbb{I}(\mathbb{D})^+$ such that vRw holds if and only if w is a **right neighbor** of v, i.e. $v = [d_0, d_1]$ and $w = [d_1, d_2]$ for some $d_0, d_1, d_2 \in \mathbb{D}$. Then v is said to be a **left neighbor** of w. The substructure of the interval neighborhood structure $\langle \mathbb{I}(\mathbb{D})^+, R, L \rangle$ containing only the strict intervals will be called **strict interval neighborhood structure**, denoted by $\langle \mathbb{I}(\mathbb{D})^-, R, L \rangle$.

Proposition 7. *Every strict (resp., non-strict) interval neighborhood structure is a strict (resp., non-strict) interval neighborhood frame.*

Proof. Straightforward. $\square$

If a linear order $\langle \mathbb{D}, < \rangle$ has a particular property (i.e. it is dense, discrete, unbounded, etc.), we say that the interval neighborhood structure based on it has that property.

3 Representation Theorems for Interval Neighborhood Frames

In this section, we provide representation theorems for both strict and non-strict semantics (as for the strict case, it must be noted that similar representation results can be found in [Lad87]).

Theorem 8 (Non-strict Representation Theorem). *If* $\mathbf{F}$ *is a tight, rich, connected, and normal interval neighborhood frame, then* $\mathbf{F}$ *is isomorphic to a non-strict interval neighborhood structure.*

Proof. Let $\mathbf{F} = \langle \mathbb{I}, R, L \rangle$ be a tight, rich, connected, and normal interval neighborhood frame. We construct an underlying linear ordering for $\mathbf{F}$ and then we show that $\mathbf{F}$ is isomorphic to the non-strict interval neighborhood structure over that ordering.

Let $\mathbf{P}(\mathbb{I}) = \{u \in \mathbb{I} | uRu\}$. Note that $\mathbf{P}(\mathbb{I})$ is non-empty and uLu for every $u \in \mathbf{P}(\mathbb{I})$. We will show that for every $u, v \in \mathbf{P}(\mathbb{I})$,

$$uLRv \text{ iff } u = v.$$

Indeed, $uLuRu$, i.e. $uLRu$. Conversely, let $uLRv$. Note that, by NF5, LR is an equivalence relation on $\mathbf{P}(\mathbb{I})$. Furthermore, if $uLRv$ then $uRuLRv$, i.e. $uRLRv$, so uRv, hence $uRLv$ and so, likewise, uLv. Now, for every w, vRw implies $uRLRw$, hence uRw. Likewise, uRw implies vRw. Analogously, uLw implies vLw and vice versa. Then, by normality, $u = v$. From this, it follows that for every $w \in \mathbb{I}$ there is a unique $v \in \mathbf{P}(\mathbb{I})$, hereafter denoted by $\mathbf{b}(w)$, such that wLv. Likewise, there is a unique $v \in \mathbf{P}(\mathbb{I})$, hereafter denoted by $\mathbf{e}(w)$, such that wRv. We now define a relation $\preceq$ on $\mathbf{P}(\mathbb{I})$ as follows:

$$u \preceq v \text{ iff } uRRv.$$

The relation $\preceq$ is a linear ordering on $\mathbf{P}(\mathbb{I})$: reflexivity is obvious, transitivity follows from NF7 and NF8, and anti-symmetry follows from tightness. As for the linearity: for any $u, v \in \mathbf{P}(\mathbb{I})$, $uLRRv$ or $uLLRv$ since $LRR \cup LLR$ is the universal relation on $\mathbb{I}$. Suppose $uLRRv$. Then $uRuLRRv$, i.e., $uRLRRv$, hence $uRRv$, i.e., $u \preceq v$. Likewise, if $uLLRv$ then $uLLv$, hence $vRRu$, i.e., $v \preceq u$. Note that for every $w \in \mathbb{I}$, $\mathbf{b}(w)RwR\mathbf{e}(w)$, hence $\mathbf{b}(w) \preceq \mathbf{e}(w)$. Now, we define a mapping μ from $\mathbb{I}$ to the non-strict interval neighborhood structure $\langle \mathbb{I}^+(\mathbf{P}(\mathbb{I})), \mathbf{L}, \mathbf{R} \rangle$ over $\langle \mathbf{P}(\mathbb{I}), \preceq \rangle$ as follows:

$$\mu(w) = (\mathbf{b}(w), \mathbf{e}(w)).$$

1. μ is an injection. If $\mu(w_1) = \mu(w_2)$, then let $\mathbf{b}(w_1) = \mathbf{b}(w_2) = b$ and $\mathbf{e}(w_1) = \mathbf{e}(w_2) = e$. Then, for every $x \in \mathbb{I}$, w_1Rx implies $w_2R\mathbf{e}(w_2)(= \mathbf{e}(w_1))Lw_1Rx$, i.e., w_2RLRx, hence w_2Rx. Likewise, w_2Rx implies w_1Rx. Analogously, w_1Lx iff w_2Lx. Then, by normality, $w_1 = w_2$.

2. μ is onto. If $u, v \in \mathbf{P}(\mathbb{I})$ and $u \preceq v$, then $uRRv$, i.e., $uRwRv$ for some $w \in \mathbb{I}$ and hence $\mu(w) = (u, v)$.

3. μ is an isomorphism. If w_1Rw_2, then $\mathbf{e}(w_1)R\mathbf{e}(w_1)Lw_1Rw_2$, that is, $\mathbf{e}(w_1)$ $RLRw_2$. Hence $\mathbf{e}(w_1)Rw_2$, and thus $\mathbf{e}(w_1) = \mathbf{b}(w_2)$ by uniqueness of $\mathbf{b}(w_2)$. It follows that $\mu(w_1)\mathbf{R}\mu(w_2)$. Conversely, if $\mu(w_1)\mathbf{R}\mu(w_2)$, then $w_1R\mathbf{e}(w_1)L$ $\mathbf{e}(w_1)$ $(= \mathbf{b}(w_2))Rw_2$, i.e., w_1 $RLRw_2$, and hence w_1Rw_2. Likewise, w_1Lw_2 iff $\mu(w_1)$ $\mathbf{L}$ $\mu(w_2)$.

This completes the proof. $\square$

Theorem 9 (Strict Representation Theorem). *We have that:*

1. *Every weakly connected, strict and normal interval neighborhood frame is isomorphic to a strict interval neighborhood structure;*

2. *Every connected, open, strict and normal interval neighborhood frame is isomorphic to a strict unbounded interval neighborhood structure.*

Proof. We prove 2 (the proof can be easily modified for 1). Let $\mathbf{F}^- = \langle \mathbb{I}, R, L \rangle$ be a connected, open, strict, and normal interval neighborhood frame. We construct an underlying point-based linear ordering and we show that $\mathbf{F}^-$ is isomorphic to the strict unbounded interval neighborhood structure over that ordering.

First, for every $w \in \mathbb{I}$, we define $[w]_b = \{v \in \mathbb{I} \mid wLRv\}$ and $[w]_e = \{v \in \mathbb{I} \mid wRLv\}$. By NF5, we have that $LRL \subseteq L$ and $RLR \subseteq R$. Hence, the relations LR and RL are equivalence relations in $\mathbb{I}$, and thus the sets $P_b = \{[w]_b \mid w \in \mathbb{I}\}$ and $P_e = \{[w]_e \mid w \in \mathbb{I}\}$ are partitions of $\mathbb{I}$. Now, we define the mapping $\theta : P_e \mapsto P_b$ as follows:

$$\theta([w]_e) = [v]_b \text{ where } wRv.$$

First, note that the definition is correct: if $[w_1]_e = [w_2]_e$, $[v_1]_b = [v_2]_b$, and w_1Rv_1 then $w_2RLRLRv_2$. By NF5, we obtain w_2RLRv_2 and thus w_2Rv_2 by NF5 again. Then, θ is a function: if wRv_1 and wRv_2 then v_1LRv_2, i.e., $[v_1]_b = [v_2]_b$; also, if wRv and $w_1 \in [w]_e$, then w_1RLw. Hence w_1RLRv, and thus w_1Rv. Furthermore, θ is a bijection between P_e and P_b. Indeed, if $\theta([w_1]_e) = \theta([w_2]_e) = [v]_b$, then w_1Rv and w_2Rv, and hence w_1RLw_2, i.e., $[w_1]_e = [w_2]_e$. The surjectivity immediately follows from the definition of P_b. From now on, we will identify P_e with P_b via θ and we will only deal with P_b. We define a relation $<$ on P_b as follows:

$$[w]_b < [v]_b \text{ iff } wLRRv.$$

Correctness of the definition: if $[w_1]_b = [w_2]_b$, $[v_1]_b = [v_2]_b$, and $w_1LRR\,v_1$, then $w_2LRw_1LRRv_1LRv_2$, i.e., $w_2(LRL)R(RLR)v_2$, and thus w_2LRRv_2 by NF5. Now we show that the relation $<$ is a strict linear ordering on P_b:

1. Irreflexivity holds because $\mathbf{F}^-$ is strict.

2. Transitivity: let $w_1LRRw_2LRRw_3$, i.e., $w_1LR(RLR)Rw_3$. Hence, we have that $w_1\ LRRRw_3$ by NF5, and thus $w_1\ LRRw_3$ by NF4.

3. Linearity: we have to show that for every $[w]_b, [v]_b \in P_b$, $[w]_b < [v]_b$ or $[w]_b = [v]_b$ or $[v]_b < [w]_b$, i.e., $wLRRv$ or $wLRv$ or $vLRRw$, that is, $wLLRv$, which is precisely the connectedness condition on $\mathbf{F}^-$.

Note that $\langle P_b, < \rangle$ is open: for every $[w]_b \in P_b$ there exists $v \in \mathbb{I}$ such that vRw and there exists $u \in \mathbb{I}$ such that vLu. Hence, $vLRRw$, i.e., $[v]_b < [w]_b$. Likewise, there exists $[v]_b$ such that $[w]_b < [v]_b$. It remains to show that the strict interval structure on $\langle P_b, <\rangle$ is isomorphic to $\mathbf{F}^-$. The isomorphism is given by the mapping $\mu : \mathbf{F}^- \mapsto \mathbb{I}(P_b)^-$ determined by

$$\mu(w) = ([w]_b, \theta([w]_e)).$$

Let $\theta([w]_e) = [v]_b$ where wRv. We have that $wLRRv$, and thus $[w]_b < [v]_b$. Hence, μ associates intervals from $\mathbb{I}(P_b)^-$ with every $w \in \mathbf{F}^-$. Now, if $[w_1]_b = [w_2]_b$ and $\theta([w_1]_e) = \theta([w_2]_e)$, then w_1LRw_2, and w_1Rv_1 and w_2Rv_2, for v_1, v_2 such that $[v_1]_b = [v_2]_b$ and thus v_1LRv_2. Hence w_1RLRLw_2, and thus w_1RLw_2 by NF5. From w_1LRw_2 and w_1RLw_2, it follows that $w_1 = w_2$ by NF6, that is, NF2 plus normality. Finally, for every interval $([w]_b, [v]_b)$ in $\mathbb{I}(P_b)^-$, we have $[w]_b < [v]_b$, i.e., $wLRRv$, and thus $wLRu$ and uRv for some $u \in \mathbf{F}^-$. Then $[u]_b = [w]_b$ and $\theta([u]_e) = [v]_b$, i.e., $([w]_b, [v]_b) = \mu(u)$. Thus, μ is an isomorphism and the proof is completed. □

4 Propositional Neighborhood Logics: Syntax and Semantics

The language $\mathbf{L}^+$ for the class $\mathcal{PNL}^+$ of **non-strict propositional neighborhood logics** contains a set of propositional variables $\mathcal{AP}$, the propositional logical connectives $\neg$ and $\rightarrow$, and the modalities $\Box_r$ and $\Box_l$, the dual operators of which will be denoted by $\Diamond_r$ and $\Diamond_l$, respectively. The remaining classical propositional connectives, as well as the logical constants $\top$ (true) and $\perp$ (false), can be considered as abbreviations.

The **formulas** of $\mathcal{PNL}^+$, denoted by $\phi, \psi, \ldots$, are recursively defined as follows:

$$\phi = p \mid \neg\phi \mid \phi \wedge \psi \mid \Box_r\phi \mid \Box_l\phi.$$

The language $\mathbf{L}^-$ for the class $\mathcal{PNL}^-$ of **strict propositional neighborhood logics** differs from $\mathbf{L}^+$ only in the notation for the modalities, now denoted by $[A]$ and $[\overline{A}]$, with dual operators $\langle A\rangle$ and $\langle\overline{A}\rangle$, respectively. The **formulas** of $\mathcal{PNL}^-$ are defined as follows:

$$\phi = p \mid \neg\phi \mid \phi \wedge \psi \mid [A]\phi \mid [\overline{A}]\phi.$$

We use different notations for the modalities in $\mathbf{L}^+$ and $\mathbf{L}^-$ only to reflect their historical links and to make it easier to distinguish between the two semantics from the syntax. Clearly, there is a straightforward translation between the two languages.

The semantics of a propositional neighborhood logic is given in **non-strict** or **strict models**, respectively based on non-strict and strict interval neighborhood frames and equipped with a **valuation function** for the propositional variables. Valuation functions are defined as $V : \mathbb{I} \mapsto 2^{\mathcal{AP}}$ in such a way that, for any $p \in \mathcal{AP}$ and $w \in \mathbb{I}$, if $p \in V(w)$, then p is **true** over w, otherwise it is **false**.

Satisfiability at an interval $w \in \mathbb{I}$ in a non-strict (resp., strict) model $\mathbf{M}$ is defined by induction on the structure of the formulas:

1. $\mathbf{M}, w \Vdash p$ iff $p \in V(w)$, for all $p \in \mathcal{AP}$;
2. $\mathbf{M}, w \Vdash \neg\psi$ iff it is not the case that $\mathbf{M}, w \Vdash \psi$;
3. $\mathbf{M}, w \Vdash \phi \wedge \psi$ iff $\mathbf{M}, w \Vdash \phi$ and $\mathbf{M}, w \Vdash \psi$;
4. $\mathbf{M}, w \Vdash \Box_l \phi$ (resp., $[\overline{A}]\phi$) iff for every interval v such that wLv we have $\mathbf{M}, v \Vdash \phi$;
5. $\mathbf{M}, w \Vdash \Box_r \phi$ (resp., $[A]\phi$) iff for every interval v such that wRv we have $\mathbf{M}, v \Vdash \phi$.

We will also take into consideration the extension of logics in $\mathcal{PNL}^+$ with the modal constant π:

6. $\mathbf{M}^+, w \Vdash \pi$ iff w is a point-interval.

Finally, the relevant notion of **p-morphism** between (non-strict or strict) models for $\mathcal{PNL}$ is defined in a standard way, and it satisfies the usual truth-preservation property well-known from modal/temporal logics (see e.g. [Ben91]).

5 Some Propositional Neighborhood Logics

The logics of the (valid formulas in the) classes of all non-strict, respectively strict, interval neighborhood structures will be denoted by PNL^+, respectively PNL^-. Besides the valid formulas in the class of all interval neighborhood structures, we will be interested in some natural subclasses of non-strict or strict interval structures:

– the **unbounded** linear orderings, denoted by **u**;

– the **dense** linear orderings (between every two different points there is a point), denoted by **de**;

– the **discrete** linear orderings (every point having a successor, respectively, a predecessor, has an immediate one), denoted by **di**;

– the **Dedekind complete** linear orderings (where every non-empty and bounded above set of points has a least upper bound), denoted by **c**;

– the **unbounded and dense** linear orderings, denoted by **ude**;

– the **unbounded and discrete** linear orderings, denoted by **udi**;

– the **unbounded and Dedekind complete** linear orderings, denoted by **uc**.

The logics of these classes will be denoted accordingly: in the class $\mathcal{PNL}^+$ by $\mathrm{PNL}^{\lambda+}$ and in the class $\mathcal{PNL}^-$ by $\mathrm{PNL}^{\lambda-}$, where $\lambda \in \{\mathbf{u}, \mathbf{de}, \mathbf{di}, \mathbf{c}, \mathbf{ude}, \mathbf{udi}, \mathbf{uc}\}$. For example, the logic PNL^{udi+} is the logic of the valid formulas in all non-strict unbounded and discrete neighborhood structures. Furthermore, the logic PNL^+ (resp., $\mathrm{PNL}^{\lambda+}$) endowed with π will be denoted by $\mathrm{PNL}^{\pi+}$ (resp., $\mathrm{PNL}^{\lambda\pi+}$).

Consider the following formulas:

(A-SNF^{ur}) $[A]p\rightarrow\langle A\rangle p$ (or, equivalently, $\langle A\rangle\top$);

(A-SNF^{der}) $(\langle A\rangle\langle A\rangle p\rightarrow\langle A\rangle\langle A\rangle\langle A\rangle p) \wedge (\langle A\rangle[A]p\rightarrow\langle A\rangle\langle A\rangle[A]p)$;

(A-SNF^{aux}) $\langle\overline{A}\rangle\top\rightarrow\langle\overline{A}\rangle\langle\overline{A}\rangle\top$;

(A-SNF^{dir}) $([A]\bot \rightarrow [\overline{A}]([A][A]\bot \vee \langle A\rangle(\langle A\rangle\top \wedge [A][A]\bot)))\wedge$
$((\langle A\rangle\top \wedge [A](p \wedge [\overline{A}]\neg p \wedge [A]p)) \rightarrow [\overline{A}][\overline{A}]\langle A\rangle(\langle A\rangle\neg p \wedge [A][A]p))$;

(A-SNF$^{\mathbf{c}}$) $\langle A\rangle\langle A\rangle[\overline{A}]p\wedge\langle A\rangle[A]\ \neg[\overline{A}]p \rightarrow\langle A\rangle(\langle A\rangle[\overline{A}]\ [\overline{A}]p\wedge[A]\langle A\rangle\neg[\overline{A}]p)$.

Proposition 10. *In the strict semantics:*

1. *The class of all unbounded structures is defined by the formula A-SNF^{ur} and its inverse* ***A-SNF^{ul}*** *(let* ***A-SNF^{u}*** *be A-SNF^{ur} $\wedge$ A-SNF^{ul}).*
2. *The class of all dense structures, extended with the 2-element linear ordering*[1]*, is defined by the formula A-SNF^{der} and its inverse* ***A-SNF^{del}*** *or, alternatively, the formula A-SNF^{aux} (let* ***A-SNF^{de}*** *be A-SNF^{der} $\wedge$ A-SNF^{del}).*
3. *The class of all discrete structures is defined by the formula A-SNF^{dir} and its inverse* ***A-SNF^{dil}*** *(let* ***A-SNF^{di}*** *be A-SNF^{dir} $\wedge$ A-SNF^{dil}).*
4. *The class of all Dedekind complete structures is defined by the formula A-SNF^{c}.*
5. *The class of all unbounded and dense structures is defined by the formulas A-SNF^{ur}, A-SNF^{der}, and their inverses A-SNF^{ul}, A-SNF^{del}.*
6. *The class of all unbounded and discrete structures is defined by the formulas A-SNF^{ur}, A-SNF^{dir}, and their inverses A-SNF^{ul}, A-SNF^{dil}.*
7. *The class of all unbounded and Dedekind complete structures is defined by the formulas A-SNF^{ur} and its inverse A-SNF^{ul}, and A-SNF^{c}.*

Proof. Sketch:

[1] The 2-element linear ordering cannot be separated in the language of PNL^-.

1. Straightforward.
2. The formula A-SNF^{der} says that every interval with a left neighbor can be split into two sub-intervals. In addition, A-SNF^{aux} guarantees that if there are at least 2 intervals (i.e., at least 3 points), then the left-most interval, if there is one, can be split into two sub-intervals, too.
3. The formula A-SNF^{dir} (resp. A-SNF^{dil}) says that every point which has a successor (resp. predecessor) has an immediate one.
4. The formula A-SNF^{c} says that every non-empty and bounded above set of points has a least upper bound. $\square$

Proposition 11. *The above-defined logics satisfy the following relations:*

1. *For every* $\lambda_1, \lambda_2 \in \{u, de, di, c, ude, udi, uc\}$*,* $\mathrm{PNL}^{\lambda_1 -} \subsetneq \mathrm{PNL}^{\lambda_2 -}$ *if and only if the class of linear orders characterized by the condition* λ_2 *is strictly contained in the class of linear orders characterized by the condition* λ_1*;*
2. $\mathrm{PNL}^{ude-} \subsetneq \mathrm{PNL}^{+}$*, where the inclusion is in terms of the obvious translation between the two languages.*
3. $\mathrm{PNL}^{+} = \mathrm{PNL}^{u+} = \mathrm{PNL}^{de+} = \mathrm{PNL}^{ude+} = \mathrm{PNL}^{di+} = \mathrm{PNL}^{udi+}$.

Proof. Sketch:

1. First, $\mathrm{PNL}^{-} \subsetneq \mathrm{PNL}^{u-}$ because the formula A-SNF$^{u} \in \mathrm{PNL}^{u-} - \mathrm{PNL}^{-}$. Likewise, $\mathrm{PNL}^{de-} \subsetneq \mathrm{PNL}^{ude-}$. $\mathrm{PNL}^{-} \subsetneq \mathrm{PNL}^{de-}$ because the formula A-SNF$^{de} \in \mathrm{PNL}^{de-} - \mathrm{PNL}^{-}$, since it is valid in every strict and dense neighborhood structure, but e.g. not in the one based on $\mathbb{Z}$. Likewise, $\mathrm{PNL}^{u-} \subsetneq \mathrm{PNL}^{ude-}$. $\mathrm{PNL}^{-} \subsetneq \mathrm{PNL}^{di-}$ because the formula A-SNF$^{di} \in \mathrm{PNL}^{di-} - \mathrm{PNL}^{-}$, since it is valid in every strict and discrete neighborhood structure, but not in the one based on $\mathbb{Q}$. Likewise, $\mathrm{PNL}^{u-} \subsetneq \mathrm{PNL}^{udi-}$. $\mathrm{PNL}^{-} \subsetneq \mathrm{PNL}^{c-}$ because the formula A-SNF$^{c} \in \mathrm{PNL}^{c-} - \mathrm{PNL}^{-}$, since it is valid in every Dedekind-complete strict neighborhood structure, but not in the one based on $\mathbb{Q}$. Finally, we have that $\mathrm{PNL}^{di-} \subsetneq \mathrm{PNL}^{udi-}$, $\mathrm{PNL}^{u-} \subsetneq \mathrm{PNL}^{uc-}$, and $\mathrm{PNL}^{c-} \subsetneq \mathrm{PNL}^{uc-}$.
2. Every PNL^{+}-formula satisfiable in a model $\mathbf{M}^{+}$ over non-strict neighborhood structure is satisfiable in a dense and unbounded strict one. Indeed, replacing every point in $\mathbf{M}^{+}$ by a copy of $\mathbb{Q}$ produces a dense and unbounded strict model $\mathbf{M}^{-*}$ such that $\mathbf{M}^{+}$ is a p-morphic copy of $\mathbf{M}^{-*}$.
3. Essentially the same construction works for the equalities $\mathrm{PNL}^{+} = \mathrm{PNL}^{u+} = \mathrm{PNL}^{de+} = \mathrm{PNL}^{ude+}$, but now we take the non-strict version of $\mathbf{M}^{*}$. For the equality $\mathrm{PNL}^{+} = \mathrm{PNL}^{udi+}$, we can similarly replace every point in $\mathbf{M}$ by a copy of $\mathbb{Z}$, and thus produce an unbounded and discrete non-strict model which maps p-morphically onto $\mathbf{M}$. $\square$

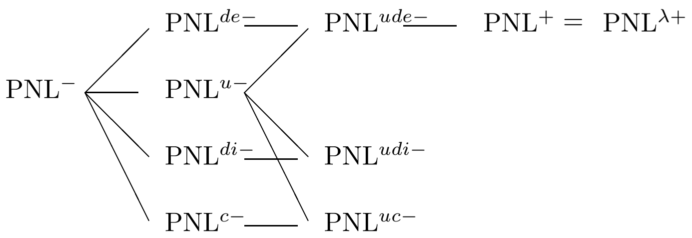

Figure 1: Relative expressive power of $\mathcal{PNL}$ logics.

It is worth noting that the logic PNL^{udi-} does not yet characterize the interval structure of the integers, because the formula

$$\langle A\rangle p\wedge[A](p\rightarrow\langle A\rangle p)\wedge[A][A](p\rightarrow\langle A\rangle p)\rightarrow[A]\langle A\rangle\langle A\rangle p$$

is valid in the integers, but not in PNL^{udi-} since it fails in a PNL^{udi-}-model based on $\mathbb{Z}+\mathbb{Z}$.

The above proposition shows that there is a collapse of the expressiveness in the non-strict semantics, while the strict one is at least as expressive as the point-based temporal logic over linear orders. The situation is graphically depicted in Figure 1, where $\lambda\in\{u, de, di, ude, udi\}$.

6 Expressing Timing Properties in $\mathcal{PNL}$

Here we give some simple examples of properties that can be expressed in $\mathcal{PNL}$. First of all, note that PNL^{-}, besides distinguishing among different properties of the underlying linear order, is powerful enough to express the **difference** operator:

$$\overline{D}(q)\equiv[\overline{A}][\overline{A}][A]q\wedge[\overline{A}][A][A]q\wedge[A][A][\overline{A}]q\wedge[A][\overline{A}][\overline{A}]q,$$

and consequently to simulate **nominals**: $n(q)\equiv q\wedge[\neq](\neg q)$, that is, to express the fact that q holds in the current interval and nowhere else. Therefore, every universal property of strict interval structures can be expressed in PNL^{-}.

The following more practical examples are borrowed from typical application domains in Artificial Intelligence. As a first example, consider the case of a robot that, in order to accomplish a given goal, must pick a finite set of objects $a_1, a_2, \ldots, a_n$ in whatever order. Moreover, assume that the robot cannot pick up more than one object at a time. Such a scenario can be modeled as follows. Let the propositional variable p_{a_i}, with $1\leq i\leq n$, denote the action "the robot is picking up the objects a_i" and the propositional variable $h_{a_{i_1},\ldots,a_{i_k}}$, with $a_{i_j} \in \{a_1,\ldots,a_n\}$ and $1 \leq k \leq n$, denote the state "the robot holds the objects $a_{i_1},\ldots,a_{i_k}$". The constraint that picking up and holding each object is a necessary pre-condition of any situation in which the robot simultaneously holds all objects can be expressed in PNL$^+$ as follows:

$$h_{a_1,\ldots,a_n} \rightarrow \Diamond_l\Diamond_l(p_{a_1} \wedge \Diamond_r h_{a_1}) \wedge \ldots \wedge \Diamond_l\Diamond_l(p_{a_n} \wedge \Diamond_r h_{a_n})$$

Note that such a formulation does not constrain "picking up" actions to be instantaneous. However, such a condition can be easily expressed in PNL$^{\pi+}$:

$$h_{a_1,\ldots,a_n} \rightarrow \Diamond_l\Diamond_l(p_{a_1} \wedge \pi \wedge \Diamond_r h_{a_1}) \wedge \ldots \wedge \Diamond_l\Diamond_l(p_{a_n} \wedge \pi \wedge \Diamond_r h_{a_n}).$$

As another example, we note that both PNL$^+$ and PNL$^-$ allows one to define an interval version of the **until** operator by means of the formulas

$$\phi\Diamond_u\psi \equiv \Diamond_r(\phi\wedge\Diamond_r\psi) \quad \text{and} \quad \phi\langle U\rangle\psi \equiv \langle A\rangle(\phi\wedge\langle A\rangle\psi),$$

respectively. Such an operator can be used to express conditions of the form "The flight from Milano to Johannesburg initiates a period of time during which the traveler is in Johannesburg" as follows:

$$\textit{Milano-to-Johannesburg}\langle U\rangle\textit{Stay-in-Johannesburg}.$$

Moreover, in PNL$^{\pi+}$ one can express the constraint that "the non-instantaneous period of time during which the light is on is initiated (resp. terminated) by an instantaneous action of switch on (resp. switch off)" as follows:

$$\textit{Switch-On} \wedge \pi \wedge ((\textit{Light-On} \wedge \neg\pi)\Diamond_u(\textit{Switch-Off} \wedge \pi)).$$

As a matter of fact, the proposed interval version of the until operator suffers from some limitations. In particular, to obtain a decomposable version of it we should force homogeneity either implicitly (via the assumption of the homogeneity principle [AF94]) or explicitly (by means of sub-interval operators). Besides the well-known fields of planning and natural language processing, successful applications of interval temporal logics can be found in the areas of digital system design and verification [Mos83] and of model validation phase support [PPH98]. As for Moszkowski's ITL, the logic PNL$^{di\pi+}$ can be exploited to express various interesting statements about digital systems. As an example, one can constrain "the output q of a device to strictly follow the input p" (being p and q two non-instantaneous states of the device) as follows:

$$(\neg\pi \wedge p) \;\rightarrow\; (\neg\pi\Diamond_u(\neg\pi \wedge q)).$$

Other useful statements about digital systems can be captured by exploiting the difference operator. As for the model validation task, interval temporal logics have been used to keep significantly low the number of states to be checked in HSTS Planner, a model-based planning system of the Remote Agent autonomous system architecture [PPH98].

7 Axiomatic Systems for $\mathcal{PNL}^+$

7.1 An Axiomatic System for PNL$^+$

We propose the following axioms for PNL$^+$, where the inverse of a formula is obtained by interchanging $\Box_r$ and $\Box_l$:

(A-NT) enough propositional tautologies;

(A-NK) the K axioms for $\Box_r$ and $\Box_l$;

(A-NNF0) $\Box_r p \rightarrow \Diamond_r p$, and its inverse;

(A-NNF1) $p \rightarrow \Box_r \Diamond_l p$, and its inverse;

(A-NNF2) $\Diamond_r \Diamond_l p \rightarrow \Box_r \Diamond_l p$, and its inverse;

(A-NNF3) $\Box_r \Diamond_l p \rightarrow \Diamond_l \Diamond_r \Diamond_r p \vee \Diamond_l \Diamond_l \Diamond_r p$, and its inverse;

(A-NNF4) $\Diamond_r \Diamond_r \Diamond_r p \rightarrow \Diamond_r \Diamond_r p$, and its inverse;

(A-NNF$_\infty$) $\Box_r q \wedge \Diamond_r p_1 \wedge \ldots \wedge \Diamond_r p_n \rightarrow \Diamond_r(\Box_r q \wedge \Diamond_r p_1 \wedge \ldots \wedge \Diamond_r p_n)$, and its inverse, for each $n \geq 1$.

The rules of inference are, as usual, Modus Ponens, Uniform Substitution, and $\Box_r$ and $\Box_l$ Generalization.

Proposition 12. *A neighborhood frame $\mathbf{F}^+ = \langle \mathbb{I}, R, L \rangle$ is an interval neighborhood frame if and only if the axioms A-NNF1,...,A-NNF4 are valid in $\mathbf{F}^+$.*

Proof. It is simple to check that the axioms A-NNF1, ..., A-NNF4 modally define the semantic conditions NF1-NF4 in the non-strict semantics. $\Box$

We show that the given axiomatic system for PNL$^+$ is sound and complete.

Lemma 13. *The following formulas and their inverses are derivable in* PNL$^+$*:*

1. $\Diamond_r p \rightarrow \Box_r \Box_l \Diamond_r p$;
2. $\Diamond_r \Diamond_l \Diamond_r p \rightarrow \Diamond_r p$;
3. $\Diamond_l \Diamond_r p \rightarrow \Diamond_r \Diamond_l \Diamond_l p \vee \Diamond_r \Diamond_r \Diamond_l p$.

Proof. For 1, use A-NNF1 and A-NNF2. For 2, observe that PNL$^+ \vdash \Diamond_r \Diamond_l \Diamond_r p \rightarrow \Diamond_r \Box_l \Diamond_r p$ by Axiom A-NNF2 (and Axiom A-NNF0), hence PNL$^+ \vdash \Diamond_r \Diamond_l \Diamond_r p \rightarrow \Diamond_r p$ by Axiom A-NNF1. Finally, 3 follows from A-NNF2 and A-NNF3. $\Box$

Lemma 14. *Let $\mathbf{M}^{+*} = \langle \mathbb{I}^*, R^*, L^*, V^* \rangle$ be any generated sub-model of the canonical model for* PNL$^+$ *and let $w \in \mathbb{I}^*$. Then there is $w_b \in \mathbb{I}^*$ such that $\{\phi \mid \Box_l \phi \in w\} \cup \{\Diamond_l \psi \mid \Diamond_l \psi \in w\} \cup \{\Box_l \xi \mid \Box_l \xi \in w\} \subseteq w_b$, and $w_e \in \mathbb{I}^*$ such that $\{\phi \mid \Box_r \phi \in w\} \cup \{\Diamond_r \psi \mid \Diamond_r \psi \in w\} \cup \{\Box_r \xi \mid \Box_r \xi \in w\} \subseteq w_e$.*

Proof. It suffices to show that the set $\Gamma = \{\phi \mid \Box_l\phi \in w\} \cup \{\Diamond_l\psi \mid \Diamond_l\psi \in w\} \cup \{\Box_l\xi \mid \Box_l\xi \in w\}$ is PNL$^+$-consistent. Suppose otherwise. Then for some ϕ such that $\Box_l\phi \in w$, $\Box_l\xi \in w$, and $\{\Diamond_l\psi_1, \ldots \Diamond_l\psi_n\} \subseteq w$, the set $\{\phi, \Diamond_l\psi_1, \ldots \Diamond_l\psi_n, \Box_l\xi\}$ is PNL$^+$-inconsistent, i.e., PNL$^+\vdash$ $\phi \rightarrow \neg(\Box_l\xi \wedge \Diamond_l\psi_1 \wedge \ldots \wedge \Diamond_l\psi_n)$. Hence PNL$^+\vdash$ $\Box_l\phi \rightarrow \Box_l\neg(\Box_l\xi \wedge \Diamond_l\psi_1 \wedge \ldots \wedge \Diamond_l\psi_n)$. Thus $\Box_l\neg(\Box_l\xi \wedge \Diamond_l\psi_1 \wedge \ldots \wedge \Diamond_l\psi_n) \in w$, i.e., $\neg\Diamond_r(\Box_l\xi \wedge \Diamond_l\psi_1 \wedge \ldots \Diamond_l\psi_n) \in w$. On the other hand, $\Diamond_l(\Box_l\xi \wedge \Diamond_l\psi_1 \wedge \ldots \wedge \Diamond_l\psi_n) \in w$ by A-NF$_\infty$, which is a contradiction. Thus, Γ is contained in a maximal PNL$^+$-consistent set w_b in $\mathbb{I}^*$. The existence of w_e is proved likewise. $\Box$

Theorem 15 (Soundness and Completeness). PNL$^+$ *is (sound and) complete for the class of all non-strict interval neighborhood structures.*

Proof. Soundness is straightforward. Note that the truth of most axioms, including the axiom scheme A-NNF$_\infty$, hinges on the inclusion of point intervals.

For the completeness, we take any PNL$^+$-consistent formula ϕ. It is satisfied at the root w of some generated sub-model $\mathbf{M}^+$ of the canonical model for PNL$^+$. Regarding that generated sub-model as a first-order structure of the language with $=$, R, L, and unary predicates corresponding to the atomic propositions occurring in ϕ, we take (using the Downwards Löwenheim-Skolem theorem) a countable elementary substructure $\mathbf{M}^{+^*}$ of $\mathbf{M}^+$ containing w. Let $\mathbf{M}^{+^*} = \langle \mathbf{F}^{+^*}, V^* \rangle$, where $\mathbf{F}^{+^*} = \langle \mathbb{I}^*, R^*, L^* \rangle$. The elements of $\mathbb{I}^*$ will henceforth be called 'intervals'. Note that $\mathbf{M}^{+^*}, w \Vdash \phi$ since truth of an interval formula at a given interval of a given PNL$^+$-model is a first-order property. Furthermore, Lemma 14 implies the truth of the first-order formulas $\forall x(\exists y(xLy \wedge \forall t(xLt \leftrightarrow yLt)))$ and $\forall x(\exists z(xRz \wedge \forall t(xRt \leftrightarrow zRt)))$ in $\mathbf{M}^+$ and hence in $\mathbf{M}^{+^*}$. Thus, with every interval v, $\mathbf{M}^{+^*}$ contains intervals v_b and v_e satisfying the conditions of Lemma 14. Note also that the 'point intervals' in $\mathbf{M}^{+^*}$ are distinguished by being both R^* -reflexive and L^*-reflexive. (In fact, one reflexivity implies the other since R^* and L^* are mutually inverse.)

Now, let w_b and w_e be as in Lemma 14. We are going to build step-by-step an interval neighborhood structure, mapping p-morphically over $\mathbf{F}^{+^*}$. We will inductively define a chain of interval neighborhood structures $\mathbf{F}_0^+ \subseteq \ldots \mathbf{F}_n^+ \subseteq \ldots$, where $\mathbf{F}_n^+ = \langle \mathbb{I}(\mathbb{D})_n, R_n, L_n \rangle$, and a sequence of mappings $f_n : \mathbf{F}_n^+ \mapsto \mathbf{F}^{+^*}$, for $n = 0, 1, 2, \ldots$, satisfying the conditions: (i) $yR_nz \rightarrow f_n(y)R^*f_n(z)$, and (ii) $yL_nz \rightarrow f_n(y)L^*f_n(z)$, as follows. Let $\mathbb{D}_0 = \{d_0, d_1\}$, with $d_0 < d_1$. R_0 and L_0 are standard right neighbor and left neighbor relations on $\mathbb{I}(\mathbb{D})_0^+$. $f_0([d_0, d_1]) = w$, $f_0([d_0, d_0]) = w_b$, and $f_0([d_1, d_1]) = w_e$. Clearly, the function f_0 preserves the right and left neighbor relations.

Suppose now that $\mathbf{F}_n^+$ and f_n are defined and satisfy the conditions (i) and (ii). Let $\mathbb{D}_n = \{d_0, \ldots, d_n\}$, where $d_0 < \ldots < d_n$. In general, f_n is not a p-morphism from $\mathbf{F}_n^+$ to $\mathbf{F}^{+^*}$ because there are *p-morphism defects* in $\mathbf{F}_n^+$ which we will have to repair during the construction, viz.: the image under f_n of an interval $[d_k, d_m]$ in $\mathbf{F}_n^+$ has a right neighbor (resp., a left neighbor) v in $\mathbf{F}^{+^*}$, which is 'missing' in $\mathbf{F}_n^+$, i.e., v is not an f_n-image of any interval from $\mathbb{I}(\mathbb{D})_n^+$, related likewise to $[d_k, d_m]$. Let all possible defects, i.e., pairs of neighboring intervals from $\mathbf{F}^{+^*}$ (which are countably many since $\mathbf{F}^{+^*}$ is countable), each repeated countably many times, be listed in a sequence $\mathcal{D} = \{\delta_n\}_{n<\omega}$, and let δ be the first one in the sequence, which has not been dealt with yet, and which occurs in $\mathbf{F}_n^+$. We are going to expand $\mathbf{F}_n^+$ to $\mathbf{F}_{n+1}^+$ in such a way that the defect δ will be fixed.

Suppose that δ relates the (image of the) interval $[d_k, d_m]$ from $\mathbf{F}_n^+$ and, say, a right neighbor v of $f_n([d_k, d_m])$ in $\mathbf{F}^{+^*}$, which is not an image of any interval from $\mathbf{F}_n^+$. (In particular, that means that $f_n([d_k, d_m]) \neq v$.) We then extend $\mathbf{F}_n^+$ to $\mathbf{F}_{n+1}^+$ with a new point d_h and f_n to f_{n+1} so that $f_{n+1}([d_m, d_h]) = v$. We must still find an appropriate place of d_h in the linear ordering $\mathbb{D}_n$ and define f_{n+1} over all other intervals with an endpoint d_h in a way which preserves the neighborhood relations. Note that $f_n([d_k, d_m])R^*R^*L^*v$, hence $f_n([d_0, d_m])R^*R^*R^*L^*v$, and so $f_n([d_0, d_m])R^*R^*L^*v$ by axiom A-NNF4. Let d_{m+i} be the greatest element of $\mathbf{F}_n^+$ such that $f_n([d_0, d_{m+i}])\ R^*R^*L^*v$. Then, for each $j = 0, \ldots, m+i$, $f_n([d_0, d_j])$ $R^*R^*L^*\ f_n([d_0, x_{m+i}])$, so $f_n([d_0, d_j])\ R^*R^*L^*R^*R^*L^*v$, and hence $f_n([d_0, d_j])$ $R^*R^*L^*v$ by Lemma 13 (part 2) and axiom A-NF4. Therefore, for each $j = 0, \ldots, m+i$, there is $w_j \in \mathbf{F}^{+^*}$ such that $f_n([d_0, d_j])R^*w_j$ and $w_jR^*L^*v$.

We now place d_h between d_{m+i} and d_{m+i+1} (if $m+1 \leq n$, otherwise we place d_h to the right of d_n) and extend f_n over all new intervals as follows. First, we put $f_{n+1}([d_m, d_h]) = v$. Then, for each $j = 1, \ldots, m+i$, $j \neq m$, we define $f_{n+1}([d_j, d_h]) = w_j$. For $j > m+i$, it is not the case that $f_n([d_m, d_j])R^*R^*L^*v$ (otherwise, $f_n([d_0, d_j])R^*R^*L^*v$). On the other hand, $f_n([d_m, d_j])L^*R^*v$ because $f_n([d_m, d_j])L^*f_n([d_k, d_m])$ and $f_n([d_k, d_m])R^*v$ by assumption. Then, by Lemma 13 (part 3), $f_n([d_m, d_j])R^*L^*L^*v$. Therefore, there exists $w_j \in \mathbf{F}^{+^*}$ such that $f_n([d_m, d_j])R^*L^*w_j$ and w_jL^*v. We define $f_{n+1}([d_h, d_j]) = w_j$. Finally, choose $v_e \in \mathbf{F}^{+^*}$ satisfying the condition of Lemma 14 and put $f_{n+1}([d_h, d_h]) = v_e$. It is straightforward to check that conditions (i) and (ii) still hold for $\mathbf{F}_{n+1}^+$. For example, if $d_j < d_h < d_l$, then $[d_j, d_h]R_{n+1}[d_h, d_h]$, and thus $f_{n+1}([d_j, d_h])R^*$ L^*v and $f_{n+1}([d_h, d_l])L^*v$. Hence $f_{n+1}([d_j, d_h])\ R^*L^*R^*f_{n+1}\ ([d_h, d_l])$, and therefore $f_{n+1}([d_j, d_h])\ R^*f_{n+1}\ ([d_h, d_l])$. This completes the inductive procedure.

Now, we define $\mathbb{D}_\omega = \bigcup_{n<\omega} \mathbb{D}_n$, $L_\omega = \bigcup_{n<\omega} L_n$, $R_\omega = \bigcup_{n<\omega} R_n$, $f_\omega = \bigcup_{n<\omega} f_n$ and $\mathbf{F}_\omega^+ = \langle \mathbb{I}(\mathbb{D})_\omega^+, R_\omega, L_\omega \rangle$. Finally, we define a valuation V_ω in $\mathbf{F}_\omega^+$ according to V^* in $\mathbf{F}^{+^*}$, viz. for all $p \in \mathcal{AP}$, $V_\omega(p) = \{i \in \mathbb{I}(\mathbb{D})_\omega^+ \mid f_\omega(i) \in V^*(p)\}$. Let $\mathbf{M}_\omega^+ = \langle \mathbf{F}_\omega^+, V_\omega \rangle$. Then $f_\omega : \mathbf{M}_\omega^+ \mapsto \mathbf{M}^{+^*}$ is a surjective p-morphism, hence $\mathbf{M}_\omega^+, [d_0, d_1] \Vdash \phi$. $\square$

7.2 An Axiomatic System for PNL$^{\pi+}$

We extend the axiomatic system for PNL$^+$ to PNL$^{\pi+}$ by adding the following axioms:

(A-π1) $\Diamond_l \pi \wedge \Diamond_r \pi$;

(A-π2) $\Diamond_r(\pi \wedge p) \rightarrow \Box_r(\pi \rightarrow p)$ and its inverse $\Diamond_l(\pi \wedge p) \rightarrow \Box_l(\pi \rightarrow p)$;

(A-π3) $\Diamond_r p \wedge \Box_r q \rightarrow \Diamond_r(\pi \wedge \Diamond_r p \wedge \Box_r q)$ and its inverse
$\Diamond_l p \wedge \Box_l q \rightarrow \Diamond_l(\pi \wedge \Diamond_l p \wedge \Box_l q)$.

By induction on n, one can show that all formulas $\Diamond_r(\pi \wedge p_1) \wedge ... \wedge \Diamond_r(\pi \wedge p_n) \rightarrow \Diamond_r(\pi \wedge p_1 \wedge ... \wedge p_n)$ and their inverses are derivable in PNL$^{\pi+}$, and thus that $\Box_r q \wedge \Diamond_r p_1 \wedge ... \wedge \Diamond_r p_n \rightarrow \Diamond_r(\pi \wedge \Box_r q \wedge \Diamond_r p_1 \wedge ... \wedge \Diamond_r p_n)$ is derivable as well. Therefore, the infinite scheme A-NNF$_\infty$ becomes derivable, hence redundant, in PNL$^{\pi+}$.

The completeness proof for PNL$^+$ is readily adaptable to PNL$^{\pi+}$.

8 Axiomatic Systems for $\mathcal{PNL}^-$

8.1 An Axiomatic System for PNL$^-$

Except for the scheme A-NF$_\infty$, which is no longer valid, the axioms for PNL$^-$ are very similar to those for PNL$^+$ (accordingly modified to reflect the fact that point-intervals are now excluded), where $\Diamond_r, \Diamond_l$ are replaced by $\langle A\rangle, \langle\overline{A}\rangle$, and $\Box_r, \Box_l$ accordingly by $[A], [\overline{A}]$. We propose the following system for PNL$^-$:

(A-ST) enough propositional tautologies;

(A-SK) the K axioms for $[A]$ and $[\overline{A}]$;

(A-SNF1) $p \rightarrow [A]\langle\overline{A}\rangle p$ and its inverse;

(A-SNF2) $\langle A\rangle\langle\overline{A}\rangle p \rightarrow [A]\langle\overline{A}\rangle p$ and its inverse;

(A-SNF3) $(\langle\overline{A}\rangle\langle\overline{A}\rangle\top \wedge \langle A\rangle\langle\overline{A}\rangle p) \rightarrow p \vee \langle\overline{A}\rangle\langle A\rangle\langle A\rangle p \vee \langle\overline{A}\rangle\langle\overline{A}\rangle\langle A\rangle p$ and its inverse;

(A-SNF4) $\langle A\rangle\langle A\rangle\langle A\rangle p \rightarrow \langle A\rangle\langle A\rangle p$ and its inverse.

Proposition 16. *A neighborhood frame* $\mathbf{F} = \langle \mathbb{I}, R, L\rangle$ *is an interval neighborhood frame if and only if the axioms A-SNF1, ..., A-SNF4 are valid in* $\mathbf{F}$.

Proof. As in Proposition 12. □

Note that the axioms cannot guarantee strictness of the neighborhood frame as irreflexivity is not definable in the language of PNL$^-$.

Theorem 17 (Soundness and Completeness). PNL$^-$ *is (sound and) complete for the class of all strict interval neighborhood structures.*

Proof. We closely follow the technique applied in the proof of Theorem 15. Again, the soundness is straightforward. For the completeness, we take any PNL$^-$-consistent formula ϕ. It is satisfied at the root w of some generated sub-model of the canonical model for PNL$^-$. We then pick a countable elementary sub-model $\mathbf{M}^{-*} = \langle \mathbf{F}^{-*}, V^* \rangle$ which contains w and satisfies ϕ there. Let $\mathbf{F}^{-*} = \langle \mathbb{I}^*, R^*, L^* \rangle$. Note that $\mathbf{F}^*$ is a weakly connected interval neighborhood frame in which the axioms A-SNF1, ..., A-SNF4 are valid since they are canonical (being of Sahlqvist type, up to tautological equivalence) and first-order definable. We then build step-by-step a model over a strict interval neighborhood structure, which maps p-morphically over $\mathbf{M}^{-*}$ very much like in the proof of Theorem 15, but easier, because we need not worry about point-intervals. □

8.2 Axiomatic Systems for Extensions of PNL$^-$

Theorem 18 (Soundness and Completeness). *We have the following completeness results:*

1. *the axiomatic system for* PNL$^-$ *extended with A-SNF^u is sound and complete for* PNL^{u-}*;*
2. *the axiomatic system for* PNL$^-$ *extended with A-SNF^{de}, is sound and complete for* PNL^{de-}*;*
3. *the axiomatic system for* PNL$^-$ *extended with A-SNF^{di} is sound and complete for* PNL^{di-}*;*
4. *the axiomatic system for* PNL$^-$ *combining* PNL^{u-} *and* PNL^{de-} *is sound and complete for* PNL^{ude-}*;*
5. *the axiomatic system for* PNL$^-$ *combining* PNL^{u-} *and* PNL^{di-} *is sound and complete for* PNL^{udi-}.

Proof. All proofs are adaptations of the one for PNL$^-$, because the respective axioms are canonical and define semantic conditions which either are reflected by p-morphisms (unboundedness) or can be forced during the step-by step construction to hold in the limit structure (density or discreteness).

9 Semantic Tableau for $\mathcal{PNL}$

In this section we devise a classical tableau method for $\mathcal{PNL}$. We will do the work in detail for the logic PNL$^+$, and then we will present the necessary modifications for PNL^-. The method can also be adapted to include π. (In [GMS03a] we generalize the method to a large class of propositional interval temporal logics.)

First, some basic terminology. A **finite tree** is a finite directed connected graph in which every node, apart from one (the **root**), has exactly one incoming arc. A **successor** of a node $\mathbf{n}$ is a node $\mathbf{n}'$ such that there is an edge from $\mathbf{n}$ to $\mathbf{n}'$. A **leaf** is a node with no successors; a **path** is a sequence of nodes $\mathbf{n_1}, \ldots, \mathbf{n_k}$ such that, for all $j = 0, \ldots, k-1$, $\mathbf{n_{j+1}}$ is a successor of $\mathbf{n_j}$; a **branch** is a path from the root to a leaf. The **height** of a node $\mathbf{n}$ is the maximum length (number of edge) of a path from $\mathbf{n}$ to a leaf. If $\mathbf{n}, \mathbf{n}'$ belong to the same branch and the height of $\mathbf{n}$ is less than or equal to the height of $\mathbf{n}'$, we write $\mathbf{n} \prec \mathbf{n}'$. Let $\langle \mathbb{C}, < \rangle$ be a finite linear order. A **labeled formula**, with label in $\mathbb{C}$, is a pair $(\phi, [c_i, c_j])$, where ϕ is a formula in the language of PNL^+ ($\phi \in \mathrm{PNL}^+$ for short) and $[c_i, c_j] \in \mathbb{I}(\mathbb{C})^+$. For a node $\mathbf{n}$ in a tree, the **decoration** $\nu(\mathbf{n})$ is a triple $((\phi, [c_i, c_j]), \mathbb{C}, u_{\mathbf{n}})$, where $\langle \mathbb{C}, < \rangle$ is a finite linear order, $(\phi, [c_i, c_j])$ is a labeled formula, with label in $\mathbb{C}$, and $u_{\mathbf{n}}$ is a **local flag function** which associates the values 0 or 1 with every branch B containing $\mathbf{n}$. Intuitively, the value 0 for a node n with respect to a branch B means that n can be expanded on B. For the sake of simplicity, we will often assume the interval $[c_i, c_j]$ to consist of the elements $c_i < c_{i+1} < \cdots < c_j$, and sometimes, with a little abuse of notation, we will write $\mathbb{C} = \{c_1 < c_2 < \ldots\}$. A **decorated tree** is a tree in which every node has a decoration $\nu(\mathbf{n})$. For every decorated tree, we define a **global flag function** u acting on pairs *(node, branch through that node)* as $u(\mathbf{n}, B) = u_{\mathbf{n}}(B)$. Sometimes, for convenience, we will include in the decoration of the nodes the global flag function instead of the local ones. For any branch B in a decorated tree, we denote by $\mathbb{C}_B$ the ordered set in the decoration of the leaf B, and for any node $\mathbf{n}$ in a decorated tree, we denote by $\Phi(\mathbf{n})$ the formula in its decoration. If B is a branch, then $B \cdot \mathbf{n}$ denotes the result of the expansion of B with the node $\mathbf{n}$ (addition of an edge connecting the leaf of B to $\mathbf{n}$). Similarly, $B \cdot \mathbf{n_1} \mid \ldots \mid \mathbf{n_k}$ denotes the result of the expansion of B with k immediate successor nodes $\mathbf{n_1}, \ldots, \mathbf{n_k}$ (which produces k branches extending B). A tableau for PNL^+ will be defined as a special decorated tree. We note again that $\mathbb{C}$ remains finite throughout the construction of the tableau.

Definition 19. Given a decorated tree $\mathcal{T}$, a branch B in $\mathcal{T}$, and a node $\mathbf{n} \in B$ such that $\nu(\mathbf{n}) = ((\phi, [c_i, c_j]), \mathbb{C}, u)$, with $u(\mathbf{n}\ , B) = 0$, the **branch-expansion rule** for B and $\mathbf{n}$ is defined as follows (in all the considered cases, $u(\mathbf{n}'\ , B') = 0$ for all new pairs $(\mathbf{n}'\ , B')$ of nodes and branches).

- If $\phi = \neg\neg\psi$, then expand the branch to $B \cdot \mathbf{n_1}$, with $\nu(\mathbf{n_1}) = ((\psi, [c_i, c_j]), \mathbb{C}_B, u)$.
- If $\phi = \psi_0 \wedge \psi_1$, then expand the branch to $B \cdot \mathbf{n_1} \cdot \mathbf{n_2}$, with $\nu(\mathbf{n_1}) = ((\psi_0, [c_i, c_j]), \mathbb{C}_B, u)$ and $\nu(\mathbf{n_2}) = ((\psi_1, [c_i, c_j]), \mathbb{C}_B, u)$.

– If $\phi = \neg(\psi_0 \wedge \psi_1)$, then expand the branch to $B \cdot \mathbf{n_1}|\mathbf{n_2}$, with $\nu(\mathbf{n_1}) = ((\neg\psi_0, [c_i, c_j]), \mathbb{C}_B, u)$ and $\nu(\mathbf{n_2}) = ((\neg\psi_1, [c_i, c_j]), \mathbb{C}_B, u_1)$.

– If $\phi = \Box_r\psi$ and c is the least element of $\mathbb{C}_B$, with $c_j \leq c$, which has not been used yet to expand $\mathbf{n}$ on B, then expand the branch to $B \cdot \mathbf{n_1}$ with $\nu(\mathbf{n_1}) = ((\psi, [c_j, c]), \mathbb{C}_B, u)$.

– If $\phi = \Box_l\psi$ and c is the greatest element of $\mathbb{C}_B$, with $c \leq c_i$, which has not been used yet to expand $\mathbf{n}$ on B, then expand the branch to $B \cdot \mathbf{n_1}$ with $\nu(\mathbf{n_1}) = ((\psi, [c, c_i]), \mathbb{C}_B, u)$.

– If $\phi = \neg\Box_r\psi$, then expand the branch to $B \cdot \mathbf{n_j} \,|\ldots|\mathbf{n_n}|\mathbf{n'_j}|\ldots|\mathbf{n'_n}$, where

1. for all $j \leq k \leq n$, $\nu(\mathbf{n_k}) = ((\neg\psi, [c_j, c_k]), \mathbb{C}_B, u)$, and
2. for all $j \leq k \leq n$, $\nu(\mathbf{n'_k}) = ((\neg\psi, [c_j, c]), \mathbb{C}_k, u)$, where, for $j \leq k \leq n-1$, $\mathbb{C}_k$ is the linear ordering obtained by inserting a new element c between c_k and c_{k+1} in $\mathbb{C}_B$, and, for $k = n$, $\mathbb{C}_k$ is the linear ordering obtained by inserting a new element c after c_n in $\mathbb{C}_B$.

– if $\phi = \neg\Box_l\psi$, then expand the branch to $B \cdot \mathbf{n_1}|\ldots|\mathbf{n_i}|\mathbf{n'_1}|\ldots|\mathbf{n'_i}$, where:

1. for all $1 \leq k \leq i$, $\nu(\mathbf{n_k}) = ((\neg\psi, [c_k, c_i]), \mathbb{C}_B, u)$, and
2. for all $1 \leq k \leq i$, $\nu(\mathbf{n'_k}) = ((\neg\psi, [c, c_i]), \mathbb{C}_k, u)$, where, for $2 \leq k \leq i$, $\mathbb{C}_k$ is the linear ordering obtained by inserting a new element c between c_{k-1} and c_k in $\mathbb{C}_B$, and, for $k = 1$, $\mathbb{C}_1$ is the linear ordering obtained by inserting a new element c before c_1 in $\mathbb{C}_B$.

Finally, for any node $\mathbf{m}$ ($\neq \mathbf{n}$) in B and any branch B' extending B, let $u(\mathbf{m}, B')$ be equal to $u(\mathbf{m}, B)$, and for any branch B' extending B, $u(\mathbf{n}, B') = 1$, unless $\phi = \Box_l\psi$ or $\phi = \Box_r\psi$ (in such cases $u(\mathbf{n}, B') = 0$).

The universal formula $\Box_r\psi$ (the same holds for $\Box_l\psi$) states that, for all $c_j \leq c$, ψ holds over $[c_j, c]$. As a matter of fact, the expansion rule imposes such a condition for a single element c in $\mathbb{C}_B$ (the least element which has not been used yet), and it does not change the flag (which remains equal to 0). In this way, all elements will be eventually taken into consideration, including those elements greater than c_j that will be added to $\mathbb{C}_B$ in some subsequent steps of the tableau construction.

Let us define now the notions of open and closed branch. We say that a node $\mathbf{n}$ in a decorated tree $\mathcal{T}$ is **available on a branch** B to which it belongs if and only if $u(\mathbf{n}, B) = 0$. The branch-expansion rule is **applicable** to a node $\mathbf{n}$ on a branch B if the node is available on B and the application of the rule generates at least one successor node with a new labeled formula. This second condition is needed to avoid looping of the application of the rule on universal formulas.

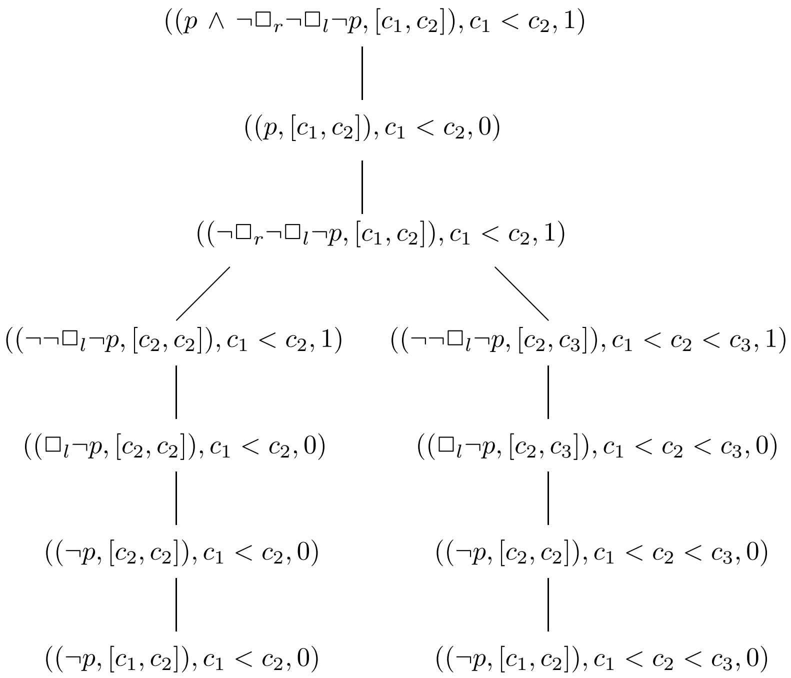

Figure 2: Example of the tableau method for PNL$^+$

Definition 20. A branch B in a tableau for ϕ is **closed** if and only if there are two nodes $\mathbf{n}, \mathbf{n}' \in B$ such that $\nu(\mathbf{n}) = ((\psi, [c_i, c_j]), \mathbb{C}, u)$ and $\nu(\mathbf{n}') = ((\neg\psi, [c_i, c_j]), \mathbb{C}', u)$ for some formula ψ and $c_i, c_j \in \mathbb{C} \cap \mathbb{C}'$, otherwise it is **open**.

Definition 21. The **branch-expansion strategy** for a branch B in a decorated tree $\mathcal{T}$ is as follows: (1) apply the branch-expansion rule to a branch B only if it is open; (2) if B is open, apply the branch-expansion rule to B on the closest to the root available node for which the branch-expansion rule is applicable.

Definition 22. A **tableau** for a given formula $\phi \in$PNL$^+$ is any finite decorated tree $\mathcal{T}$ obtained by expanding the one-node decorated tree $((\phi, [c_1, c_2]), \{c_1, c_2\}, u)$, where the (only) value of u is 0, through successive applications of the branch-expansion strategy to currently existing branches.

Definition 23. A tableau for a given formula $\phi \in$PNL$^+$ is **closed** if and only if every branch in it is closed, otherwise it is **open**.

In Figure 2 we showed the case of the tableau for the formula $p \wedge \neg\Box_r\neg\Box_l\neg p$, that is, $\neg$ A-NNF1. As one can expect, all branches of the tableau are closed, meaning that the formula is not satisfiable.

9.1 Soundness and Completeness

Definition 24. Given a set S of labeled formulas with labels in a linear ordering $\langle \mathbb{C}, < \rangle$, we say that S is **satisfiable over** $\mathbb{C}$ if there exists a non-strict model $\mathbf{M}^+ = \langle \mathbb{I}(\mathbb{D})^+, R, L, V \rangle$ such that $\langle \mathbb{D}, < \rangle$ is an extension of $\langle \mathbb{C}, < \rangle$ and $\mathbf{M}^+, [c_i, c_j] \Vdash \psi$ for all $(\psi, [c_i, c_j]) \in S$.

Clearly, the above notion is equivalent to the notion of satisfiability of a formula in the case that S contains only one labeled formula.

Theorem 25 (Soundness). *If $\phi \in$PNL$^+$and a tableau $\mathbf{T}$ for ϕ is closed, then ϕ is not satisfiable.*

Proof. We prove by induction on the height h of a node $\mathbf{n}$ in the tableau $\mathbf{T}$ a stronger claim: if every branch containing $\mathbf{n}$ is closed, then the set $S(\mathbf{n})$ of all labeled formulas in the decorations of the nodes between $\mathbf{n}$ and the root is not satisfiable over $\mathbb{C}$, where $\mathbb{C}$ is the linear ordering in the decoration of $\mathbf{n}$.

If $h = 0$, then $\mathbf{n}$ is a leaf and the unique branch B containing $\mathbf{n}$ is closed. Then $S(\mathbf{n})$ contains $(\psi, [c_k, c_l])$ and $(\neg\psi, [c_k, c_l])$ for some PNL$^+$-formula ψ. Take any model $\mathbf{M}^+ = \langle (\mathbb{I}(\mathbb{D})^+, R, L, V \rangle$, where $\langle \mathbb{D}, < \rangle$ is an extension of $\langle \mathbb{C}, < \rangle$. $\mathbf{M}^+, [c_k, c_l] \vDash \psi$ iff $\mathbf{M}^+, [c_k, c_l] \nVdash \neg\psi$, and thus $S(\mathbf{n})$ is not satisfiable over $\mathbb{C}$.

Suppose $h > 0$. Then either the branch-expansion rule has been applied to some labeled formula $(\psi, [c_i, c_j]) \in S(\mathbf{n})$ to extend the branch at $\mathbf{n}$, or $\mathbf{n}$ has been generated as the first of the two successors obtained by an application of a $\wedge$-rule. We will only consider in detail the former case, as the latter is subsumed by it.

Let $\mathbb{C} = \{c_1, \ldots, c_n\}$, where $c_1 < \ldots < c_n$, be the linear ordering from the decoration of $\mathbf{n}$. Note that every branch passing through any successor of $\mathbf{n}$ must be closed, so the inductive hypothesis applies to all successors of $\mathbf{n}$.

We consider the possible cases for the branch-expansion rule applied at $\mathbf{n}$.

– Let $\psi = \neg\neg\xi$. Then there exists $\mathbf{n_1}$ such that $\nu(\mathbf{n_1}) = ((\xi, [c_i, c_j]), \mathbb{C}, u)$ and $\mathbf{n_1}$ is a successor of $\mathbf{n}$. Since every branch containing $\mathbf{n}$ is closed, then every branch containing $\mathbf{n_1}$ is closed. By the inductive hypothesis, $S(\mathbf{n_1})$ is not satisfiable over C (since $\mathbf{n_1} \prec \mathbf{n}$). Since ξ_0 and $\neg\neg\xi_0$ are equivalent, $S(\mathbf{n})$ cannot be satisfiable over $\mathbb{C}$.

– Let $\psi = \xi_1 \wedge \xi_2$. Then there are two nodes $\mathbf{n_1} \in B$ and $\mathbf{n_2} \in \mathbf{B}$ such that $\nu(\mathbf{n_1}) = ((\xi_1, [c_i, c_j]), \mathbb{C}, u)$, $\nu(\mathbf{n_2}) = ((\xi_2, [c_i, c_j]), \mathbb{C}, u)$, and, without loss of generality, $\mathbf{n_1}$ is the successor of $\mathbf{n}$ and $\mathbf{n_2}$ is the successor of $\mathbf{n_1}$.

Since every branch containing $\mathbf{n}$ is closed, then every branch containing $\mathbf{n_2}$ is closed. By the inductive hypothesis, $S(\mathbf{n_2})$ is not satisfiable over C since $\mathbf{n_2} \prec \mathbf{n}$. Since every model over $\mathbb{C}$ satisfying $S(\mathbf{n})$ must, in particular, satisfy $(\xi_1 \wedge \xi_2, [c_i, c_j])$, and hence $(\xi_1, [c_i, c_j])$ and $(\xi_2, [c_i, c_j])$, it follows that $S(\mathbf{n})$, $S(\mathbf{n_1})$, and $S(\mathbf{n_2})$ are equi-satisfiable over $\mathbb{C}$. Therefore, $S(\mathbf{n})$ is not satisfiable over $\mathbb{C}$.

– let $\psi = \neg(\xi_1 \wedge \xi_2)$. Then there exist two successor nodes $\mathbf{n_1}$ and $\mathbf{n_2}$ of $\mathbf{n}$, such that $\nu(\mathbf{n_1}) = ((\xi_0, [c_i, c_j]), \mathbb{C}, u_0)$, $\nu(\mathbf{n_2}) = ((\xi_1, [c_i, c_j]), \mathbb{C}, u_1)$, $\mathbf{n_1}, \mathbf{n_2} \prec \mathbf{n}$. Since every branch containing $\mathbf{n}$ is closed, then every branch containing $\mathbf{n_1}$ and every branch containing $\mathbf{n_2}$ is closed. By the inductive hypothesis, $S(\mathbf{n_1})$ and $S(\mathbf{n_2})$ are not satisfiable over $\mathbb{C}$. Since every model over $\mathbb{C}$ satisfying $S(\mathbf{n})$ must also satisfy $(\xi_0, [c_i, c_j])$ or $(\xi_1, [c_i, c_j])$, it follows that $S(\mathbf{n})$ cannot be satisfiable over $\mathbb{C}$.

– Let $\psi = \neg\Box_r\xi$. Assuming that $S(\mathbf{n})$ is satisfiable over $\mathbb{C}$, there is a model $\mathbf{M}^+ = \langle(\mathbb{I}(\mathbb{D})^+, R, L, V\rangle$, where $\langle\mathbb{D}, <\rangle$ is an extension of $\langle\mathbb{C}, <\rangle$, such that $\mathbf{M}^+, [c_i, c_j] \Vdash \theta$ for all $(\theta, [c_i, c_j]) \in S(\mathbf{n})$. In particular, $\mathbf{M}^+, [c_j, d] \models \neg\xi$ for some $d \geq c_j$. Consider 2 cases:

  1. $d \in \mathbb{C}$. Then $d = c_m$ for some $m \geq j$. But one of the successor nodes of $\mathbf{n}$ is $\mathbf{n_m}$, where $\nu(\mathbf{n_m}) = ((\neg\xi, [c_j, c_m]), \mathbb{C}, u)$, and since $\mathbf{n_m} \prec \mathbf{n}$,, by the inductive hypothesis, $S(\mathbf{n_m}) = S(\mathbf{n}) \cup \{(\neg\xi, [c_j, c_m])\}$ is not satisfiable over $\mathbb{C}$, which is a contradiction.
  2. $d \notin \mathbb{C}$. Then there is an m such that $j \leq m \leq n-1$ and $c_m < d < c_{m+1}$, or $m = n$ and $c_n < d$. In either case, there is a successor node $\mathbf{n'_m}$ of $\mathbf{n}$ such that $\nu(\mathbf{n'_m}) = ((\neg\xi, [c_j, d]), \mathbb{C} \cup \{d\}, u)$, and since $\mathbf{n'_m} \prec \mathbf{n}$, by the inductive hypothesis $S(\mathbf{n'_m}) = S(\mathbf{n}) \cup \{(\neg\xi, [c_j, d])\}$ is not satisfiable over $\mathbb{C} \cup \{d\}$, which, again, is a contradiction.

  Thus, in either case $S(\mathbf{n})$ is not satisfiable over $\mathbb{C}$.

– The case of $\psi = \neg\Box_l\xi$ is analogous.

– Let $\psi = \Box_r\xi$. Then $\nu(\mathbf{n_1}) = ((\xi, [c_j, c_m]), \mathbb{C}, u)$, with $j \leq m \leq n$, for the successor $\mathbf{n_1}$ of $\mathbf{n}$. Now, any model over $\mathbb{C}$ satisfying $S(\mathbf{n_1})$ must, in particular, satisfy $(\Box_r\xi, [c_i, c_j])$, and hence $(\xi, [c_j, c_m])$. Thus, the sets $S(\mathbf{n})$ and $S(\mathbf{n_1}) = \mathbf{S}(\mathbf{n}) \cup \{(\xi, [\mathbf{c_j}, \mathbf{c_m}])\}$ are equi-satisfiable over $\mathbb{C}$. Since $\mathbf{n_1} \prec \mathbf{n}$, by the inductive hypothesis $S(\mathbf{n_1})$ is not satisfiable over $\mathbb{C}$, and thus $S(\mathbf{n})$ is not satisfiable over $\mathbb{C}$.

– The case of $\psi = \Box_l\xi$ is analogous. $\Box$

Definition 26. If $\mathbf{T}_0$ is the one-node tableau $((\phi, [c_1, c_2]), \{c_1, c_2\}, 0)$ for a given PNL$^+$-formula ϕ, the **limit tableau** $\overline{\mathbf{T}}$ for ϕ is the (possibly infinite) decorated tree obtained as follows. First, for all i, $\mathbf{T}_{i+1}$ is the tableau obtained by simultaneous application of the branch-expansion strategy to every branch in $\mathbf{T}_i$. Then, we ignore all flags from the decorations of the nodes in every $\mathbf{T}_i$. Thus we obtain a chain by inclusion of decorated trees: $\mathbf{T}_0 \subseteq \mathbf{T}_1 \subseteq \ldots$. Now we define $\overline{\mathbf{T}} := \bigcup_{i=0}^{\infty} \mathbf{T}_i$.

Note that the chain above may stabilize at some $\mathbf{T}_i$ if it closes, or if the branch-expansion rule is not applicable to any of its branches. We associate with each branch B in $\overline{\mathbf{T}}$ the linear ordering $\mathbb{C}_B = \bigcup_{i=0}^{\infty} \mathbb{C}_{B_i}$, where, for all i, $\mathbb{C}_{B_i}$ is the linear ordering from the decoration of the leaf of the (sub-)branch B_i of B in $\mathbf{T}_i$. The definitions of closed and open branches readily apply to $\overline{\mathbf{T}}$.

Definition 27. A branch in a (limit) tableau is **saturated** if there are no nodes on it to which the branch-expansion rule is applicable on the branch. A (limit) tableau is **saturated** if every open branch in it is saturated.

In what follows we will show that the set of all labeled formulas on an open branch in a limit tableau has the saturation properties of a Hintikka set in first-order logic.

Lemma 28. *Every limit tableau is saturated.*

Proof. Given a node $\mathbf{n}$ in a limit tableau $\overline{\mathbf{T}}$, we denote by $d(\mathbf{n})$ the distance (number of edges) between $\mathbf{n}$ and the root of $\overline{\mathbf{T}}$. Now, given a branch B in $\overline{\mathbf{T}}$, we will prove by induction on $d(\mathbf{n})$ that after every step of the expansion of that branch at which the branch-expansion rule becomes applicable to $\mathbf{n}$ (because $\mathbf{n}$ has just been introduced, or because a new point has been introduced in the linear ordering on B) that rule is subsequently applied on B to that node.

Suppose the inductive hypothesis holds for all nodes with distance to the root less than m. Let $d(\mathbf{n}) = m$ and the branch-expansion rule has become applicable to $\mathbf{n}$. If there are no nodes between the root (including the root) and $\mathbf{n}$ (excluding $\mathbf{n}$) to which the branch-expansion rule is applicable at that moment, the next application of the branch-expansion rule on B is to $\mathbf{n}$. Otherwise, consider the closest to $\mathbf{n}$ node $\mathbf{n}^*$ between the root and $\mathbf{n}$ to which the branch-expansion rule is applicable, or becomes applicable on B at least once thereafter. (Such node exists because there are only finitely many nodes between $\mathbf{n}$ and the root.) Since $d(\mathbf{n}^*) < d(\mathbf{n})$, by the inductive hypothesis the branch-expansion rule has been subsequently applied to $\mathbf{n}^*$. Then the next application of the branch-expansion rule on B must have been to $\mathbf{n}$ and that completes the induction. Now, assuming that a branch in a limit tableau is not saturated, consider the closest to the root node $\mathbf{n}$ on that branch B to which the branch-expansion rule is applicable on that branch. If $\Phi(\mathbf{n})$ is neither $\Box_r\psi$ nor $\Box_l\psi$, then the branch-expansion rule has become applicable to $\mathbf{n}$ at the step when $\mathbf{n}$ is introduced, and by the claim above, it has been subsequently applied, at which moment the node has become unavailable thereafter, which contradicts the assumption. If $\Phi(\mathbf{n}) = \Box_r\psi$ or $\Phi(\mathbf{n}) = \Box_l\psi$, then an application of the rule on B must create at least one successor with a new label $(\psi, [c_i, c_j])$ on B. But c_i, c_j have already been introduced at some (finite) step of the construction of B and at the first step when both of them, as well as $\mathbf{n}$, have appeared on the branch, the branch-expansion rule has become applicable to $\mathbf{n}$, hence is has been subsequently applied on B and that application must have introduced the label $(\psi, [c_i, c_j])$ on B, which again contradicts the assumption. □

Corollary 29. *Let ϕ be a* PNL$^+$*-formula, and $\overline{\mathbf{T}}$ the limit tableau for ϕ. Then, for every open branch B in $\overline{\mathbf{T}}$:*

- *if there is a node $\mathbf{n} \in B$ such that $\nu(\mathbf{n}) = ((\neg\neg\psi, [c_i, c_j]), \mathbb{C}, u)$, then there is a node $\mathbf{n_1} \in B$ such that $\nu(\mathbf{n_1}) = ((\psi, [c_i, c_j]), \mathbb{C}, u_1)$;*
- *if there is a node $\mathbf{n} \in B$ such that $\nu(\mathbf{n}) = ((\psi_1 \wedge \psi_2, [c_i, c_j]), \mathbb{C}, u)$, then there is a node $\mathbf{n_1} \in B$ such that $\nu(\mathbf{n_1}) = ((\psi_1, [c_i, c_j]), \mathbb{C}, u_1)$ and a node $\mathbf{n_2} \in B$ such that $\nu(\mathbf{n_2}) = ((\psi_2, [c_i, c_j]), \mathbb{C}, u_2)$;*
- *if there is a node $\mathbf{n} \in B$ such that $\nu(\mathbf{n}) = ((\neg(\psi_1 \wedge \psi_2), [c_i, c_j]), \mathbb{C}, u)$, then there is a node $\mathbf{n_1} \in B$ such that $\nu(\mathbf{n_1}) = ((\neg\psi_1, [c_i, c_j]), \mathbb{C}, u_1)$ or a node $\mathbf{n_2} \in B$ such that $\nu(\mathbf{n_2}) = ((\neg\psi_2, [c_i, c_j]), \mathbb{C}, u_2)$;*
- *if there is a node $\mathbf{n} \in B$ such that $\nu(\mathbf{n}) = ((\neg\Box_r\psi, [c_i, c_j]), \mathbb{C}, u)$, then, for some $c \in \mathbb{C}_B$ such that $c_j \leq c$ there is a node $\mathbf{n}' \in B$ such that $\nu(\mathbf{n}') = ((\neg\psi, [c_j, c]), \mathbb{C}', u')$;*
- *likewise for every node $\mathbf{n}$ with $\Phi(\mathbf{n}) = \neg\Box_l\psi$;*
- *if there is a node $\mathbf{n} \in B$ such that $\nu(\mathbf{n}) = ((\Box_r\psi, [c_i, c_j]), \mathbb{C}, u)$, then for all $c \in \mathbb{C}_B$ such that $c_j \leq c$, there is a node $\mathbf{n}' \in B$ such that $\nu(\mathbf{n}') = ((\psi, [c_j, c]), \mathbb{C}', u')$;*
- *likewise for every node $\mathbf{n}$ with $\Phi(\mathbf{n}) = \Box_l\psi$.*

Lemma 30. *If the limit tableau for some formula $\phi \in$*PNL$^+$ *is closed, then some finite tableau for ϕ is closed.*

Proof. Suppose the limit tableau for ϕ is closed. Then every branch closes at some finite step of the construction and then remains finite. Since the branch-expansion rule always produces finitely many successors, every finite tableau is finitely branching, and hence so is the limit tableau. Then, by König's lemma, the limit tableau, being a finitely branching tree with no infinite branches, must be finite, hence its construction stabilizes at some finite stage. At that stage a closed tableau for ϕ is constructed. □

Theorem 31 (Completeness). *Let* $\phi \in$PNL$^+$*be a valid formula. Then there is a closed tableau for* $\neg\phi$.

Proof. We will show that the limit tableau $\overline{\mathbf{T}}$ for $\neg\phi$ is closed, whence the theorem follows by the previous lemma.

By contraposition, suppose that $\overline{\mathbf{T}}$ has an open branch B. Let $\mathbb{C}_B$ be the linear ordering associated with B and $\Phi(B)$ be the set of all labeled formulas on B. Consider the model $\mathbf{M}^+ = \langle \mathbb{I}(\mathbb{C}_B)^+, R, L, V\rangle$ where, for every $[c_i, c_j] \in \mathbb{I}(\mathbb{C}_B)^+$ and $p \in \mathcal{AP}$,

$$p \in V([c_i, c_j]) \text{ iff } (p, [c_i, c_j]) \in \Phi(B).$$

We are going to show by induction on ψ that, for every $(\psi, [c_i, c_j]) \in \Phi(B)$,

$$\mathbf{M}^+, [c_i, c_j] \Vdash \psi.$$

1. If $\psi = p$ or $\psi = \neg p$ where $p \in \mathcal{AP}$, the claim follows by definition, because if $(\neg p, [c_i, c_j]) \in \Phi(B)$, then $(p, [c_i, c_j]) \notin \Phi(B)$ since B is open (the same for $(p, [c_i, c_j]) \in \Phi(B)$).

2. Let $\psi = \neg\neg\xi$. Then by Lemma 29, $(\xi, [c_i, c_j]) \in \Phi(B)$, and by inductive hypothesis $\mathbf{M}^+, [c_i, c_j] \Vdash \xi$. So $\mathbf{M}^+, [c_i, c_j] \Vdash \psi$.

3. Let $\psi = \xi_0 \wedge \xi_1$. Then by Lemma 29, $(\xi_0, [c_i, c_j]) \in \Phi(B)$ and $(\xi_1, [c_i, c_j]) \in \Phi(B)$. By inductive hypothesis, $\mathbf{M}^+, [c_i, c_j] \Vdash \xi_0$ and $\mathbf{M}^+, [c_i, c_j] \Vdash \xi_1$, so $\mathbf{M}^+, [c_i, c_j] \Vdash \psi$.

4. Let $\psi = \neg(\xi_0 \wedge \xi_1)$. Then by Lemma 29, $(\neg\xi_0, [c_i, c_j]) \in \Phi(B)$ or $(\neg\xi_1, [c_i, c_j]) \in \Phi(B)$. By inductive hypothesis $\mathbf{M}^+, [c_i, c_j] \Vdash \neg\xi_0$ or $\mathbf{M}^+, [c_i, c_j] \Vdash \neg\xi_1$, so $\mathbf{M}^+, [c_i, c_j] \Vdash \psi$.

5. Let $\psi = \neg\Box_r\xi$. Then by Lemma 29, $(\neg\xi, [c_j, c]) \in \Phi(B)$ for some $c \in \mathbb{C}_B$ such that $c_j \leq c$. Thus, by inductive hypothesis, $\mathbf{M}^+, [c_j, c] \Vdash \neg\xi$. So, $\mathbf{M}^+, [c_i, c_j] \Vdash \psi$.

6. The case of $\psi = \neg\Box_l\xi$ is similar.

7. Let $\psi = \Box_r\xi$. Then by Lemma 29, $(\xi, [c_j, c]) \in \Phi(B)$ for all $c \in \mathbb{C}_B$ such that $c_j \leq c$. Hence, by inductive hypothesis, for all $c \in \mathbb{C}_B$ such that $c_j \leq c$ $\mathbf{M}^+, [c_j, c] \Vdash \xi$, and so $\mathbf{M}^+, [c_i, c_j] \Vdash \psi$.

8. The case of $\psi = \Box_l\xi$ is similar.

This completes the induction. In particular, we obtain that $\neg\phi$ is satisfied in $\mathbf{M}^+$, which is in contradiction with the assumption that ϕ is valid. $\Box$

The tableau method for PNL$^+$ developed here can be easily adapted to the case of PNL$^-$. Indeed, the method requires the following straightforward modification of the branch-expansion rule: in the cases of $\neg[A]$, $\neg[\overline{A}]$, $[A]$, and $[\overline{A}]$ the rule does not introduce a successor node with label $[c_j, c_j]$. With that modification, all theorems and their proofs included in this section can be accordingly adapted for PNL$^-$.

10 Conclusion and Possible Developments

In this paper we have studied the class of Propositional Neighborhood Logics ($\mathcal{PNL}$), and we have provided complete axiomatic systems and a classical tableau method for them. Currently the questions about the decidability of $\mathcal{PNL}$ logics are open (as a matter of fact, the technique suggested by Montanari and Sciavicco in [MS02], as it stands, does not work). Generally speaking, the problem of finding decidable fragments of interval temporal logics has been raised by several authors, including Halpern and Shoham (cf. Problem 4 in [HS91]) and Venema (cf. Question 3.20 in [Ven91]). Possible approaches to prove decidability are via finite model property or interpretation into other decidable logics. Note, however, that the finite model property with respect to standard models fails in both semantics [GMS03b], and thus one could only hope for a finite model properrty with respect to non-standard models.

Acknowledgments

The authors would like to thank the Italian Ministero degli Affari Esteri and the National Research Foundation of South Africa for the research grant, under the Joint Italy/South Africa Science and Technology Agreement, that they received for the project: "Theory and applications of temporal logics to computer science and artificial intelligence". We also thank Alberto Casagrande and Raffaella Gentilini who pointed out a minor imprecision in the original formulation of the formula defining the class of all discrete structures [GMS03b].

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