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On Identification in $\mathbb{Z}^2$ Using Translates of Given Patterns

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Abstract: Given a finite set of patterns, i.e., subsets of $\mathbb{Z}^2$. What is the best way to place translates of them in such a way that every point belongs to at least one translate and no two points belong to the same set of translates? We give some general results, and investigate the particular case when there is only a single pattern and that pattern is a square or has size at most four.

Key Words: Identifying code, square lattice, multiprocessor architecture

Category: E.4

1 Introduction and basics

Assume that $\mathcal{T}$ is a set consisting of a finite number of finite, nonempty subsets of $\mathbb{Z}^2$, which we call *patterns*.

Definition 1. A set

$$\mathcal{A} = \{(c_j, T_j) : j \in J\} \subseteq \mathbb{Z}^2 \times \mathcal{T}$$

(or the covering by $\mathcal{A}$) is called $\mathcal{T}$-*identifying* if for all $x \in \mathbb{Z}^2$, the sets

$$I(x) = \{(c_i, T_i) \in \mathcal{A} : x \in c_i + T_i\}$$

are nonempty, and moreover no two of them coincide.

So we are covering all the vertices of $\mathbb{Z}^2$ using translates of the $|\mathcal{T}|$ given patterns, and the idea is that for every $x \in \mathbb{Z}^2$ it is true that if we know which ones contain x, then we know x, i.e., x is uniquely determined.

We call the set $I(x)$ in Definition 1 the I-set of x (whether or not $\mathcal{A}$ is $\mathcal{T}$-identifying).

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It is natural to assume — and we do so from now on — that

$$c_i + T_i \neq c_j + T_j \text{ whenever } i \neq j.$$

Without loss of generality, we can assume that all the patterns contain the point $(0, 0)$.

We often consider the case in which $\mathcal{T}$ consists of a single set T. In this case the identifying set is completely determined by the pattern T and the set $C = \{c_1, c_2, \ldots, c_{|C|}\}$, and we call C a T-identifying code; its elements are called codewords.

If T is a ball of a given radius, r, in a graph with vertex set $V = \mathbb{Z}^2$, i.e., T is the set of all points within graphic distance r from a given $x \in \mathbb{Z}^2$, then C is called r-identifying. Such r-identifying codes have been studied in particular in the following four graphs:
– the square grid, where the edge set is

$$E_S = \{\{u, v\} : u - v \in \{(0, \pm 1), (\pm 1, 0)\}\}$$

(see, e.g., [3], [5], [8], [9], [12], [14], [16], [17]); for instance, it is known that the density of any 1-identifying code is at least $15/43$, and there is a construction with density $7/20$;
– the triangular grid, where the edge set is

$$E_T = \{\{u, v\} : u - v \in \{(0, \pm 1), (\pm 1, 0), (1, 1), (-1, -1)\}\}$$

(see, e.g., [3], [5], [12], [17]);
– the king grid, where the edge set is

$$E_K = \{\{u, v\} : u - v \in \{(0, \pm 1), (\pm 1, 0), (1, \pm 1), (-1, \pm 1)\}\}$$

(see, e.g., [4], [5], [11], [12], [15]);
– the hexagonal mesh, where the edge set is

$$E_H = \{\{u = (i, j), v\} : u - v \in \{(0, (-1)^{i+j+1}), (\pm 1, 0)\}\}$$

(see, e.g., [3], [5], [10], [12], [17]). Usually — unlike in this paper — a codeword is associated with the centre of the ball, i.e., the pattern T is chosen to be the ball with centre $(0, 0)$.

The problem of r-identifying codes originated (see [17]) in fault diagnosis in multiprocessor systems: such a system can be modeled as a graph $G = (V, E)$ where V is the set of processors and E the set of links between processors. Assume that at most one of the processors is malfunctioning and one wishes to test the system and locate the faulty processor. For this purpose, some processors (which constitute the code) will be selected and assigned the task of testing their

r-neighbourhoods (which are balls of radius r). Whenever a selected processor, i.e., a codeword, detects a fault, it sends an alarm signal. The requirement is that when we know which codewords gave the alarm, this information alone is enough to locate the malfunctioning processor.

The use of paths instead of balls is also suggested in [18].

Our approach also provides useful infrastructure for identifying faulty edges as we shall see in Section 4. For more on the technical background and edge identification, we refer to [2] and [18].

We now give a definition intended to measure the performance of an identifying set. Denote by Q_n the set of vertices $(x, y) \in \mathbb{Z}^2$ with $|x| \leq n$ and $|y| \leq n$.

Definition 2. Assume that $\mathcal{A}$ is as in Definition 1. The density of (the covering by) $\mathcal{A}$ is

$$D(\mathcal{A}) = \limsup_{n\to\infty} \frac{\sum_{j\in J} |(c_j + T_j) \cap Q_n|}{|Q_n|}.$$

The density of the underlying multiset $C = \{c_j : j \in J\}$ is

$$D(C) = \limsup_{n\to\infty} \frac{\sum_{j\in J} |\{c_j\} \cap Q_n|}{|Q_n|}.$$

The density of $\mathcal{A}$ measures in how many sets $c_j + T_j$ a point $x \in \mathbb{Z}^2$ lies in average, and therefore $D(\mathcal{A}) \geq 1$ whenever $\mathcal{A}$ is $\mathcal{T}$-identifying. It is a natural objective, given $\mathcal{T}$, to try to find a $\mathcal{T}$-identifying set with as small density as possible. In view of the practical applications, it is also natural to try to minimize the density of the underlying multiset. We will be considering almost exclusively the case in which $\mathcal{T}$ consists of a single pattern T, and as shown in the next theorem, in this case these two objectives coincide. The definition of the density of a T-identifying code C (which is simply a set) reduces to

$$D(C) = \limsup_{n\to\infty} \frac{|C \cap Q_n|}{|Q_n|}.$$

Clearly, if $\mathcal{T}$ contains a singleton set, there is a $\mathcal{T}$-identifying set $\mathcal{A}$ with density one. From now on, except in Theorem 12, we therefore assume that $\mathcal{T}$ does not contain a singleton set. Then — as shown in Theorem 4 — the density of any $\mathcal{T}$-identifying set $\mathcal{A}$ is strictly greater than one.

The density of a T-identifying code C is closely related to the density of the corresponding set $\mathcal{A}$ as proved in the following simple theorem.

Theorem 3. *Let $\mathcal{A}$ be a $\mathcal{T}$-identifying set, with $|T| = k$ for all $T \in \mathcal{T}$, and C the underlying multiset. Then $D(\mathcal{A}) = k\, D(C)$.*

Proof. Take $a > 0$ such that $T \subseteq Q_a$ for all $T \in \mathcal{T}$, and assume that $n > a$. Then

$$\sum_{j\in J} |(c_j + T_j) \cap Q_{n+a}| \geq k \sum_{j\in J} |\{c_j\} \cap Q_n| \geq \sum_{j\in J} |(c_j + T_j) \cap Q_{n-a}|.$$

Dividing by $|Q_n| = (2n+1)^2$ we get

$$\frac{(2n+2a+1)^2}{(2n+1)^2} \frac{\sum_{j\in J} |(c_j + T_j) \cap Q_{n+a}|}{(2n+2a+1)^2} \geq k \frac{\sum_{j\in J} |\{c_j\} \cap Q_n|}{|Q_n|}$$
$$\geq \frac{(2n-2a+1)^2}{(2n+1)^2} \frac{\sum_{j\in J} |(c_j + T_j) \cap Q_{n-a}|}{(2n-2a+1)^2}.$$

The claim follows by letting n tend to infinity. Notice that the difference of the left-hand side and the quantity $\sum_{j\in J} |(c_j + T_j) \cap Q_{n+a}|/(2n+2a+1)^2$ tends to zero; a similar remark applies to the right-hand side. □

Theorem 4. *If $|T| \geq k$ for all $T \in \mathcal{T}$, and $\mathcal{A}$ is $\mathcal{T}$-identifying, then*

$$D(\mathcal{A}) \geq \frac{2k}{k+1}.$$

Proof. Let $\mathcal{A} = \{(c_j, T_j) : j \in J\}$. Choose an integer $a > 0$ such that $\cup_{T\in\mathcal{T}} T \subseteq Q_a$. Assume that $n > a$, and denote by F_i the set of points $x \in Q_n$ which are contained in exactly i of the sets $c_j + T_j$. By counting in two ways the pairs $(x, c_j + T_j)$ such that $x \in Q_n$, $(c_j, T_j) \in \mathcal{A}$ and $x \in c_j + T_j$ we get

$$\sum_{i\geq 1} i|F_i| = \sum_{j\in J} |(c_j + T_j) \cap Q_n|.$$

Moreover, because a set $c_j + T_j$ can contain at most one point of F_1, and $|T| \geq k$ for all $T \in \mathcal{T}$, corresponding to every such pair $(x, c_j + T_j)$ with $x \in F_1 \cap Q_{n-a}$ there are at least $k-1$ such pairs $(y, c_j + T_j)$ with $y \notin F_1$. Hence

$$\sum_{i>1} i|F_i| \geq (k-1)|F_1 \cap Q_{n-a}|.$$

Consequently,

$$\frac{\sum_{j\in J} |(c_j + T_j) \cap Q_n|}{(2n+1)^2}$$
$$= \frac{(2(n-a)+1)^2}{(2n+1)^2} \frac{|F_1| + \sum_{i>1} i|F_i|}{|F_1 \cap Q_{n-a}| + \sum_{i>1} |F_i \cap Q_{n-a}|}$$
$$\geq \frac{(2(n-a)+1)^2}{(2n+1)^2} \frac{|F_1 \cap Q_{n-a}| + \sum_{i>1} i|F_i|}{|F_1 \cap Q_{n-a}| + \sum_{i>1} |F_i \cap Q_{n-a}|}$$

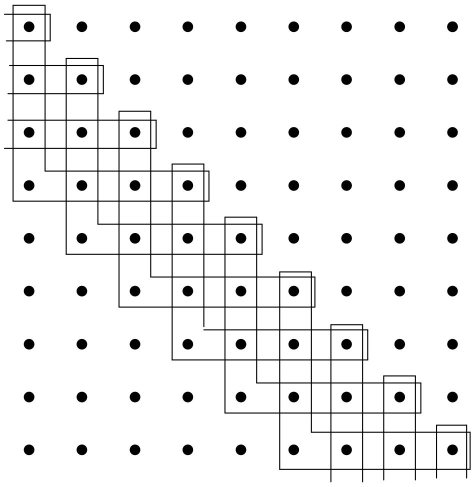

Figure 1: An optimal solution using a pattern of size seven.

$$\geq \frac{(2(n-a)+1)^2}{(2n+1)^2}\left(\frac{|F_1 \cap Q_{n-a}| + \sum_{i>1} i|F_i|}{|F_1 \cap Q_{n-a}| + \frac{1}{2}\sum_{i>1} i|F_i|}\right)$$
$$= \frac{(2(n-a)+1)^2}{(2n+1)^2}\left(2 - \frac{|F_1 \cap Q_{n-a}|}{|F_1 \cap Q_{n-a}| + \frac{1}{2}\sum_{i>1} i|F_i|}\right)$$
$$\geq \frac{(2(n-a)+1)^2}{(2n+1)^2}\left(2 - \frac{|F_1 \cap Q_{n-a}|}{|F_1 \cap Q_{n-a}| + \frac{1}{2}(k-1)|F_1 \cap Q_{n-a}|}\right)$$
$$= \frac{(2(n-a)+1)^2}{(2n+1)^2}\left(2 - \frac{2}{k+1}\right),$$

where in the last inequality we have assumed that $|F_1 \cap Q_{n-a}| > 0$ (if not, then the conclusion trivially holds). The claim follows when we let n tend to infinity. $\square$

The next theorem and the next example show that the lower bound of the previous theorem can be attained.

Theorem 5. *For all values of k there is a pattern T of cardinality k and a set $\mathcal{A}$ which is T-identifying and has density $2k/(k+1)$. If $k = 2s+1$, we can take $T = \{(0,0), (0,1), \ldots, (0,s), (1,0), (2,0), \ldots, (s,0)\}$; if $k = 2s$, we can take $T = \{(0,0), (0,1), \ldots, (0,s), (1,0), (2,0), \ldots, (s-1,0)\}$.*

Proof. The construction for $k = 2s+1 = 7$ is given in Figure 1. We have only drawn $s+1 = 4$ diagonals; we repeat these $s+1$ diagonals in both directions. Clearly the same construction works for all s.

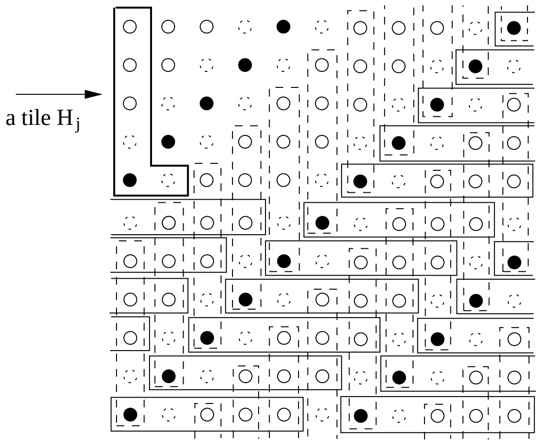

Figure 2: A $\mathcal{T}$-identifying set: $\mathcal{T} = \{\{(0,0), (0,1), (0,2), (0,3), (0,4)\}, \{(0,0), (1,0), (2,0), (3,0), (4,0)\}\}$. The points c_j, $j \in J$, are in black. The points with dotted lines are the points which belong to only one translate.

If k is even ($k = 2s$), we draw $2s+1$ diagonals at a time; first the $s+1$ diagonals obtained from Figure 1 by dropping out the lower right-hand corners of each translate of T, and then taking all the translates obtained by shifting everything upwards by s (the new translates partly overlap the old ones). $\square$

Example 1. If we take $\mathcal{T} = \{T_1, T_2\}$, where $T_1 = \{(0,0), (0,1), \ldots, (0, k-1)\}$ and $T_2 = \{(0,0), (1,0), \ldots, (k-1, 0)\}$, then there is an optimal $\mathcal{T}$-identifying set $\mathcal{A}$ with density $2k/(k+1)$. See Figure 2 for $k = 5$, from which it is easy to see the general construction, in which, in any set $H_j = (c_j + T_1) \cup \{c_j + (1,0)\}$, $j \in J$, two points, $c_j + (0,1)$ and $c_j + (1,0)$, belong to one translate, and the remaining points belong to two translates. Since the sets H_j, $j \in J$, tile the whole space $\mathbb{Z}^2$, the density is $\frac{2+2(k-1)}{k+1}$ as claimed, and the density of the underlying multiset is $2/(k+1)$.

Example 2. Assume that we want to use cycles of length $2k$. Then we can take $T_1 = \{(0,0), (0,1), \ldots, (0, k-1), (1, k-1), (1, k-2), \ldots, (1, 0)\}$ and $T_2 = \{(0,0), (1,0), \ldots, (k-1, 0), (k-1, 1), (k-2, 1), \ldots, (0, 1)\}$. If we take the three translates $(0,0) + T_2$, $(0,1) + T_2$ and $(0,2) + T_2$, we immediately see that if we know that a point is in their union, we know its y-coordinate. If we take C_2 to consist of all the points $(ik, 4j)$, $(ik, 4j+1)$, $(ik, 4j+2)$ where $i, j \in \mathbb{Z}$, then the translates $c + T_2$, $c \in C_2$, together cover all the points, and uniquely identify the y-coordinate of a point. If we similarly define C_1 as the set of all points (x, y)

for which $(y, x) \in C_2$, then the set $\mathcal{A} = \{(c_1, T_1) : c_1 \in C_1\} \cup \{(c_2, T_2) : c_2 \in C_2\}$ is $\{T_1, T_2\}$-identifying and has density 3.

2 Using a square pattern

In the king grid, two vertices in $\mathbb{Z}^2$ are connected by an edge if the Euclidean distance between them is at most $\sqrt{2}$. With respect to the graphic distance, a ball of radius r in the king grid is a square consisting of $(2r + 1) \times (2r + 1)$ vertices. It is known from [11], [5] and [4], that if we take T to be such a square, then the optimal density of a T-identifying code is $2/9$ if $r = 1$ and $1/(4r)$ for all $r > 1$. We can ask what happens if, more generally, T is a square consisting of $s \times s$ vertices, i.e., s is allowed to be even.

We first prove an easy lower bound, using a technique borrowed from [12].

Theorem 6. *Let $T = \{(a, b) : a, b \in \{0, 1, \ldots, s - 1\}\}$. If C is a T-identifying code, then its density is at least $1/(2s)$ for all $s \geq 1$.*

Proof. Let us assume that C is a T-identifying code, and let

$$L = \{(0, 0), (1, 0), \ldots, (n, 0)\}$$

be a line segment in $\mathbb{Z}^2$. We see that

$$R = \{(i, j) \in \mathbb{Z}^2 : -s + 1 \leq i \leq n, -s + 1 \leq j \leq 0\}$$

is a rectangle containing $s(n+s)$ points and is the set of all points c such that $c+T$ can cover at least one element of L. Now for all the n pairs $((u, 0), (u + 1, 0))$, $u \in \{0, 1, \ldots, n - 1\}$, of consecutive points of L there has to be a separating codeword c such that $c + T$ contains one point of the pair but not the other. Since for all codewords $c \in C$, $c + T$ intersects L in a subsegment, a codeword can be separating for at most two pairs, and therefore

$$\frac{n}{2} \leq |C \cap R|.$$

Finally, because we can tile $\mathbb{Z}^2$ using R, we get

$$D(C) \geq \frac{n}{2|R|} = \frac{n}{2s(n + s)}.$$

By letting n tend to infinity, we obtain $D(C) \geq 1/(2s)$. □

We can do better, by using a more sophisticated method from [4] (which was used there for odd $s \geq 5$).

Theorem 7. *Let $T = \{(a, b) : a, b \in \{0, 1, \ldots, s - 1\}\}$. If C is a T-identifying code, then its density is at least $1/(2s - 2)$ for all $s \geq 4$.*

Proof. **(Sketch)** The proof is essentially the same as the one used in [4] for the odd case, and works for all $s \geq 4$. In the proof of [4], a square is referred to by its centre point (which is in $\mathbb{Z}^2$). In the general proof (with our choice of T), each codeword $c \in C$ corresponds to the square $c + T$, i.e., the codeword is the lower left-hand corner of the square. Nevertheless, by taking any four vertices (x, y), $(x, y + 1)$, $(x + 1, y)$ and $(x + 1, y + 1)$, and using the fact that every two of them, say p and q, must be separated, i.e., there is a codeword c such that $c+T$ contains exactly one of p and q, we obtain Lemmas 2 and 3 of [4] using the same argument. In the proof of Theorem 2 of [4], the numerical calculations go through in a similar way, and by the assumption $s \geq 4$, the sets K, which are translates of the set $\{(0, 0), (0, 1), \dots , (0, s), (1, s), (2, s), \dots , (s, s), (s, s - 1), (s, s - 2), \dots , (s, 0), (s - 1, 0), (s - 2, 0), \dots , (1, 0)\}$, have side length at least five, which guarantees that the key step (the marking process) goes through similarly. $\square$

Theorem 8. *With T as in the previous theorem, there is a T-identifying code with density $\frac{s}{2(s-1)^2}$ for all even $s \geq 4$.*

Proof. Let $S = \{0, 1, 3, \dots , 2i + 1, \dots , s - 3\}$,

$$C_0 = \{(j(s - 1), 0) : j \in \mathbb{Z}\}$$

and

$$C = \bigcup_{j\in\mathbb{Z}, i\in S} \big(C_0 + (i + (s - 1)j, i + (s - 1)j)\big)$$

(see Figure 3 for the case $s = 6$). Clearly, C is doubly periodic with periods $(0, s-1)$ and $(s-1, 0)$; and the set $\{(i, j) : 0 \leq i \leq s-2, 0 \leq j \leq s-2\}$ contains exactly $s/2$ codewords. Therefore, C has density $\frac{s}{2(s-1)^2}$. We claim that C is T-identifying.

First, we consider all the points (p, i), $i \in \mathbb{Z}$, lying on the same vertical line p, and show that none of them can be covered in the same way (i.e., have the same I-set) as a point on some other vertical line. Without loss of generality, we can assume that $s - 1 \leq p \leq 2s - 3$. We distinguish between two cases.

(i) $p = s - 1$ **or** $p > s - 1$ **is even**: Then any point (p, i) is contained in two sets, $c_1 + T$ and $c_2 + T$, where c_1 and c_2 are codewords belonging to the vertical lines $p - (s - 1)$ and p, respectively. On the other hand, no point (q, j), $q < p$, belongs to $c_2 + T$, and no point (q, j), $q > p$, belongs to $c_1 + T$.

(ii) $p > s - 1$ **is odd**: Given (p, i), it suffices to show by (i) that there is no (q, j), $q \neq p$, q odd, which has the same I-set as (p, i). But this is clear, since (p, i) belongs to a set $c + T$, where c is a codeword lying on the vertical line $p-1$, whereas there is no such codeword on the vertical line $p+1$. This uniquely identifies p.

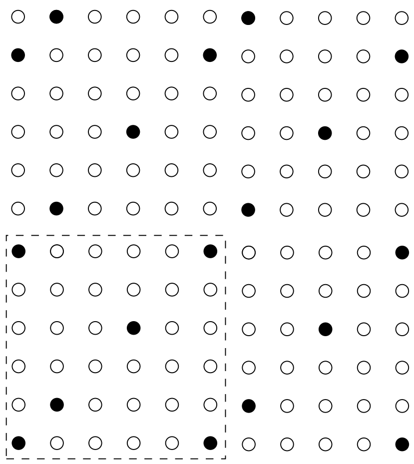

Figure 3: A T-identifying code: the case $s = 6$. Codewords are in black.

Therefore we have just proved that all points in $\mathbb{Z}^2$ are covered, and that two points lying on different vertical lines cannot be covered in the same way.

For reasons of symmetry, the same is true with horizontal lines, which shows that C is T-identifying. □

Remark. In the odd case (see [4]), the construction is analogous to the above construction, the difference being that the set S is now

$$\{0, 2, 4, \ldots, 2i, \ldots, s-3\},$$

and the density of the code is $1/2(s-1)$, which meets the lower bound for $s \geq 5$. In the even case, $s \geq 4$, the best density of a T-identifying code lies between $\frac{s-1}{2(s-1)^2}$ and $\frac{s}{2(s-1)^2}$. In the case $s = 2$, the following theorem gives the exact value.

Theorem 9. *Let* $T = \{(0,0), (0,1), (1,0), (1,1)\}$. *The optimal density of a* T*-identifying code is* $2/5$.

Proof. The lower bound comes from Theorem 4 for $k = 4$ and Theorem 3. The upper bound comes from the construction given in Figure 4, which is easy to check. □

We call a point $(x, y) \in \mathbb{Z}^2$ *even*, if $x + y$ is even, and *odd*, otherwise. A code is called even, if all its codewords are even. The codes in the next theorem have slightly higher density than the ones in Theorem 8, but they are even. Such codes are needed in the final section.

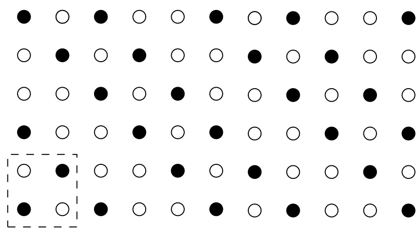

Figure 4: An optimal T-identifying code: the case $s = 2$.

Theorem 10. *Let T be as in Theorem 7.*

(i) If $s = 4t + 2$, $t \geq 1$, then there is an even code which is T-identifying and has density $\frac{1}{2s-4}$.

(ii) If $s = 4t$, $t \geq 1$, then there is an even code which is T-identifying and has density $\frac{s}{2(s-2)^2}$.

Proof. We only give the constructions. The proofs are very similar to the proof of Theorem 8.

In (i) we take $S = \{0, 1, 4, 5, 8, 9, \ldots, s-6, s-5\}$, and in (ii) we take $S = \{0, 1, 4, 5, 8, 9, \ldots, s-4, s-3\}$. In both cases, we take $C_0 = \{(j(s-2), 0) : j \in \mathbb{Z}\}$ and

$$C = \bigcup_{j \in \mathbb{Z}, i \in S} \left(C_0 + (i + (s-2)j, i + (s-2)j)\right).$$

The code C is even and T-identifying. □

For $s = 4$, the following theorem improves on the upper bound $2/9$ in Theorem 8, and the upper bound $1/2$ in Theorem 10. From Theorem 7 we have the lower bound $1/6$.

Theorem 11. *Let T be the 4×4 square $T = \{(a, b) : a, b \in \{0, 1, 2, 3\}\}$.*

(i) There is a T-identifying code with density $2/11$.

(ii) There is an even T-identifying code with density $3/16$.

Proof. Denote

$$D_k = \{(i, i + k) : i \in \mathbb{Z}\}.$$

(i) We take $C = \cup D_k$ where the union is taken over all indices $k \equiv 0, 4 \pmod{11}$. The density of the code is $2/11$. The union of the squares $c + T$, $c \in C$, is clearly the whole set $\mathbb{Z}^2$.

Assume that $v \in \mathbb{Z}^2$ is an unknown point. Choose an index k such that $v \in (i, i+k)+T$ for some i, and then take the largest i such that $v \in (i, i+k)+T$. Because $v \notin (i+1, i+1+k)+T$, we know that $v \in \{(i, i+k), (i, i+k+1), (i, i+$

$k+2), (i, i+k+3), (i+1, i+k), (i+2, i+k), (i+3, i+k)\}$. By checking, whether or not v belongs to $(i-3, i-3+k)+T$, $(i-2, i-2+k)+T$, $(i-1, i-1+k)+T$, we can determine, which one of the sets $\{(i, i+k)\}$, $\{(i, i+k+1), (i+1, i+k)\}$, $\{(i, i+k+2), (i+2, i+k)\}$, $\{(i, i+k+3), (i+3, i+k)\}$ contains v. Separation between the remaining at most two vertices is easy, because by definition, either all the vertices in D_{k-4} are in the code and the squares $c+T$, $c \in D_{k-4}$, contain all the vertices in $D_{k-1} \cup D_{k-2} \cup D_{k-3}$ but none in $D_{k+1} \cup D_{k+2} \cup D_{k+3}$; or the other way round if all the vertices in D_{k+4} are in the code.

(ii) Take

$$E_k = \{(i, i+k) \in D_k : i \equiv 0, 2, 5 \text{ or } 7 \pmod 8\},$$

and define the code C as the union of all D_{8m}, $m \in \mathbb{Z}$, and E_{8m+4}, $m \in \mathbb{Z}$. The density of the code is $3/16$. Again the squares $c+T$, $c \in C$, together contain all the elements in $\mathbb{Z}^2$.

Assume that $v \in \mathbb{Z}^2$ is unknown.

If $v \in c+T$, $c \in D_{8m}$ for some m, then we proceed in exactly the same way as in case (i), and find which one of the sets $\{(i, i+8m)\}$, $\{(i, i+8m+1), (i+1, i+8m)\}$, $\{(i, i+8m+2), (i+2, i+8m)\}$, $\{(i, i+8m+3), (i+3, i+8m)\}$ contains v. Let us assume that we know that $v \in \{(i, i+8m+1), (i+1, i+8m)\}$. The other two cases can be treated similarly. To separate between the two remaining vertices, it is enough that at least one of the vertices $(i-3, i+8m+1) \in D_{8m+4}$ and $(i+1, i+8m-3) \in D_{8m-4}$ is in the code, and this is true by the definition of the sets E_k: either $i-3 \equiv 0, 2, 5$ or $7 \pmod 8$ or $i+1 \equiv 0, 2, 5$ or $7 \pmod 8$.

Assume then that $v \notin c+T$ for all $m \in \mathbb{Z}$ and $c \in D_{8m}$. Then we know that v lies in one of the diagonals D_{8m+4} — and if $v \in c+T$ for some $c \in D_{8m+4}$, then $v \in D_{8m+4}$. The codewords in D_{8m+4} (and the fact that we already know that $v \in D_{8m+4}$) are now enough to determine v — in fact, in exactly the same way as in the proof of Theorem 12 (cf. Figure 6). □

3 Connected patterns of size at most four

Consider the case $\mathcal{T} = \{T\}$ when $|T| \leq 4$. Up to obvious symmetries, there are nine connected patterns of size at most four, see Figure 5. For the patterns 1, 2, 3 and 5, which consist of rectangles of height one, we give a more general result from [1].

Theorem 12. *[1] Let* $T = \{(0,0), (1,0), \ldots, (s-1, 0)\}$. *The best density of a* T*-identifying code is* 1 *if* $s = 1$, $2/3$ *if* $s = 2$, *and* $1/2$ *if* $s > 2$.

Proof. The case $s = 1$ is trivial. The lower bound for $s = 2$ is from Theorems 4 and 3. To prove that the density is at least half for all $s > 2$, we show that for every $(x, y) \in \mathbb{Z}^2$ at least half of the points in the set $\{(x, y), (x-1, y), \ldots, (x-$

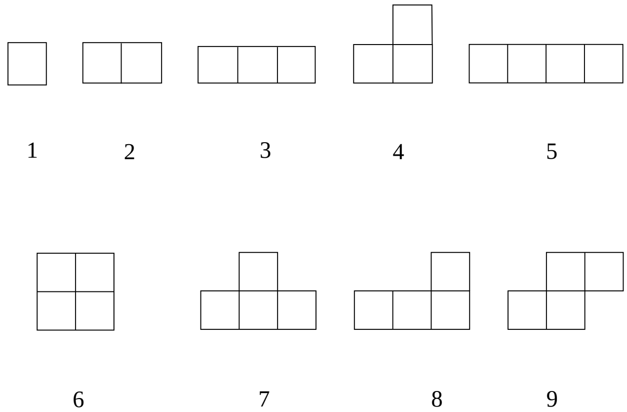

Figure 5: All connected patterns of size at most four up to obvious symmetries.

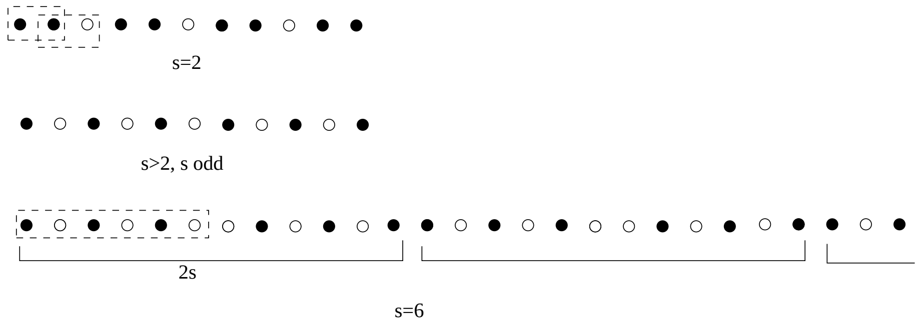

Figure 6: Optimal T-identifying codes for rectangles of height one.

$2s+1, y)\}$ are codewords. It suffices to prove that every pair $\{(i, y), (i-s, y)\}$ contains a codeword, and this follows, because if (i, y) is not a codeword, then the only codeword that can separate between (i, y) and $(i-1, y)$ is $(i-s, y)$.

The constructions from [1] are shown in Figure 6 in the cases $s = 2$, $s > 2$ odd, and $s > 2$ even (illustrated for $s = 6$): in each case every horizontal line is treated as in the figure. □

For the pattern 4, the best density of a T-identifying code is 1/2 by Theorem 5. The pattern 6 was dealt with in Theorem 9 (density 2/5). For the remaining patterns, 7, 8 and 9, a lower bound for the density is 2/5, given by Theorems 4 and 3. The next theorem shows that the best density is exactly 2/5.

Theorem 13. *Let* $T = \{(0,0), (1,0), (2,0), (1,1)\}$, $T = \{(0,0), (1,0), (2,0),$

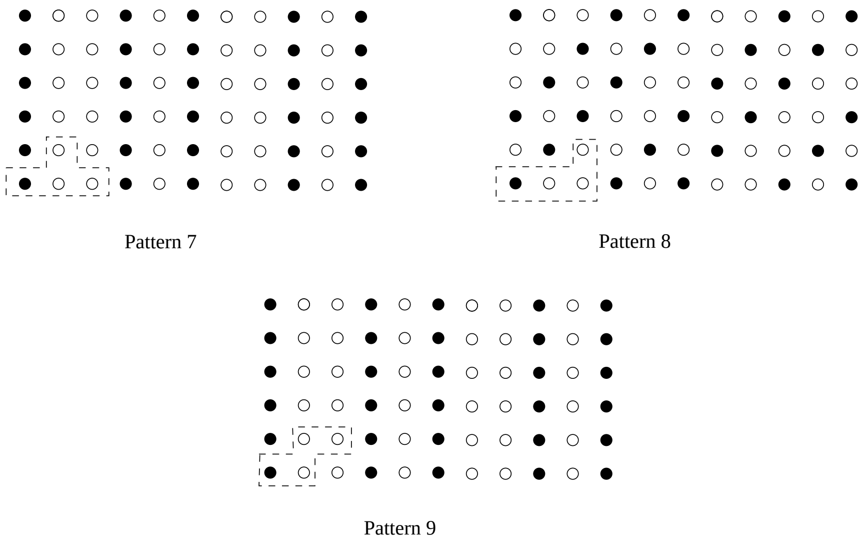

Figure 7: Optimal T-identifying codes for patterns 7, 8 and 9.

$(2,1)\}$ *or* $T = \{(0,0),(1,0),(1,1),(2,1)\}$. *The best density of a* T*-identifying code is* $2/5$.

Proof. All we need to prove is the upper bound. See Figure 7. □

The following table summarizes the results obtained in this section, on the patterns 1 to 9.

pattern number	1	2	3	4	5	6	7	8	9
pattern size	1	2	3	3	4	4	4	4	4
best density $D(C)$	1	2/3	1/2	1/2	1/2	2/5	2/5	2/5	2/5
best density $D(\mathcal{A})$	1	4/3	3/2	3/2	2	8/5	8/5	8/5	8/5

4 An application: identifying an edge

Consider the following problem in the square grid. Each vertex contains a processor, and each edge represents a link between two processors. We assume that at most one of the links is malfunctioning — and that none of the processors is out of order — and wish to identify the malfunctioning link, if there is one. We use the following simple scheme. Some of the vertices v (marked as black squares in Figure 8) are assigned the task of checking all the four edges starting from v. All those among the chosen vertices that have detected a problem send

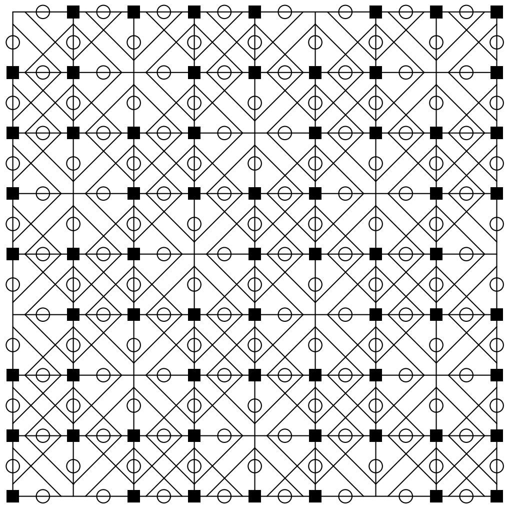

Figure 8: Identifying an edge in the square grid: an optimal solution when $r = 1$. The chosen vertices are denoted by black squares.

us an alarm signal, i.e., just one bit of information telling us that among the four edges, one of them is malfunctioning. Based on the answers we would like to know exactly the location of the malfunctioning link, if there is one. What is the best way of choosing the processors that take care of the checking?

As we have done in Figure 8, we can denote each edge by a circle located in the middle point of the edge, and the area checked by a processor is a 2×2 square in the lattice formed by the circles. In terms of this new lattice, we take T to be a 2×2 square, and from Theorem 4, we know that the best density of an identifying set $\mathcal{A}$ consisting of such squares is at least $8/5$, and the best density of the code formed by the lower left-hand corners of the squares is at least $2/5$. Notice that here our problem is not the same as in Theorem 9, though: we are only allowed to take translates that correspond to sets of four edges all having a common end point. In fact, the requirement is simply that the code in the new lattice should be even. It is easy to check that the construction of Figure 8 gives an optimal solution to the problem. In terms of the original grid, we have chosen $4/5$ of the processors to do the checking.

Given $r > 1$, we can consider the same problem when each of the chosen processors v checks all the edges that belong to at least one path of length r starting from v. Figure 9 illustrates the case $r = 4$. Then each of the chosen processors (vertices) checks $2r \times 2r$ edges, and again we have the extra condition

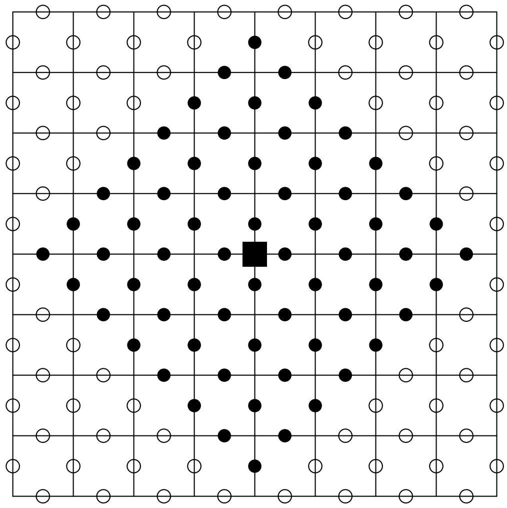

Figure 9: The solid circles mark the edges, which are on a path of length at most $r = 4$ starting from the black square.

that in the lattice formed by the edges (the circles in Figure 9), the code should be even.

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