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Some Notes on Fine Computability

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Abstract: A metric defined by Fine induces a topology on the unit interval which is strictly stronger than the ordinary Euclidean topology and which has some interesting applications in Walsh analysis. We investigate computability properties of a corresponding Fine representation of the real numbers and we construct a structure which characterizes this representation. Moreover, we introduce a general class of Fine computable functions and we compare this class with the class of locally uniformly Fine computable functions defined by Mori. Both classes of functions include all ordinary computable functions and, additionally, some important functions which are discontinuous with respect to the usual Euclidean metric. Finally, we prove that the integration operator on the space of Fine continuous functions is lower semi-computable.

Key Words: Computable analysis, Walsh analysis.
Category: F.1

1 Introduction

The Fine metric is the metric on the unit interval $[0,1]$ which is induced by the Cantor metric on $\{0,1\}^\omega$ via the binary representation of the reals[2]. To be more precise, let us define the binary representation $\rho_2 :\subseteq \{0,1\}^\omega \to [0,1]$ by

$$\rho_2(p) := \sum_{i=0}^{\infty} \frac{p(i)}{2^i}.$$

Here, the inclusion symbol indicates that ρ_2 is partial (only defined for those p such that the infinite sum is actually a value in $[0,1]$). To make ρ_2 injective, we can restrict the representing sequences to those, which do contain infinitely many 0's. In this way, we obtain a restriction $\rho_F :\subseteq \{0,1\}^\omega \to [0,1]$, defined by $\rho_F(p) := \rho_2(p)$ for all p such that $1^\omega = 111...$ is not a suffix of p. In all other cases $\rho_F(p)$ is undefined. We will call this mapping the *Fine representation* of $[0,1]$. Clearly, ρ_F is injective and surjective and hence it admits an inverse $\Psi := \rho_F^{-1} : [0,1] \to \{0,1\}^\omega$. Now, using the *Cantor metric* $d_C : \{0,1\}^\omega \times \{0,1\}^\omega \to \mathbb{R}$, defined by

$$d_C(p,q) := \sum_{i=0}^{\infty} \frac{|p(i)-q(i)|}{2^i},$$

we obtain the *Fine metric* $d_F : [0,1] \times [0,1] \to \mathbb{R}$ by $d_F(x,y) := d_C(\Psi(x), \Psi(y))$. The *Fine topology* τ_F induced by the Fine metric on $[0,1]$ can be generated by

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[2] As general references for Walsh analysis see [Schipp, Wade and Simon 1990, Golubov, Efimov and Skvortsov 1991].

the intervals $[a, b)$ with dyadic rationals $a, b \in \mathbb{Q}_2 := \{n/2^k : n, k \in \mathbb{N}, n \leq 2^k\}$, which form a countable base of this topology. Besides the ordinary open intervals (x, y), $x, y \in \mathbb{R}$ also the intervals $[x, y)$ with dyadic rationals $x \in \mathbb{Q}_2$, and $y \in \mathbb{R}$ are open with respect to the Fine topology. Let us denote by τ_E the *Euclidean topology* on $\mathbb{R}$, induced by the *Euclidean metric* $d_E : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with $d_E(x, y) := |x - y|$ and let us denote by τ'_E the subspace topology of τ_E on $[0, 1]$. Then

$$d_E(\rho_F(p), \rho_F(q)) = \left| \sum_{i=0}^{\infty} \frac{p(i) - q(i)}{2^i} \right| \leq \sum_{i=0}^{\infty} \frac{|p(i) - q(i)|}{2^i} = d_F(\rho_F(p), \rho_F(q)).$$

Altogether we obtain $\tau'_E \subsetneq \tau_F$. In the following we will call (τ_F, τ_E)–continuous functions $f : [0, 1] \to \mathbb{R}$ simply *Fine continuous.*

In a sequence of two papers [Mori, Mori 2000] Mori studied an approach to computability with respect to the Fine metric. His investigation is based on the sequential approach to computable analysis, which has been initiated by Pour-El and Richards [Pour-El and Richards 1989] and which has been further developed by Washihara, Yasugi, Mori and Tsujii, cf. [Washihara 1995, Yasugi, Mori and Tsujii 1999]. One drawback of the sequential approach to computable analysis is that it requires some technical tools to handle computability properties of functions which are not uniformly continuous. This problem especially appears with respect to the Fine metric since it is easy to see that $[0, 1]$ is not complete and hence not compact with respect to the Fine metric d_F. Thus, Fine continuous functions $f : [0, 1] \to \mathbb{R}$ are not uniformly (d_F, d_E)–continuous in general. The purpose of these brief notes is to discuss the notion of Fine computability from the point of view of the representation based approach to computable analysis, which has been developed by Weihrauch and others [Weihrauch 2000] under the name "Type-2 Theory of Effectivity".

We will start to recall some basic facts and notions of the representation based approach to computable analysis; for details see [Weihrauch 2000]. By Σ^ω we denote the set of infinite sequences over some finite alphabet Σ, in the following we typically assume $\Sigma := \{0, 1\}$. A function $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ is called *computable*, if there exists a Turing machine M which computes infinitely long, and which in the long run transforms each input sequence p into the corresponding output sequence $F(p)$ (written on a one-way output tape of the machine). Here the inclusion symbol indicates that F might be partial. This notion of computability can be transferred to other spaces with the help of representations. In general, a *representation* of a set M is a surjective partial mapping $\delta :\subseteq \Sigma^\omega \to M$. If δ and δ' are representations of sets M and M', respectively, then a function $f :\subseteq M \to M'$ is called (δ, δ')*–computable*, if there exists a computable function $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that

$$\delta' F(p) = f\delta(p)$$

for all $p \in \mathrm{dom}(f\delta)$. Correspondingly, f is called (δ, δ')*–continuous*, if there exists a continuous F which fulfills the same equation as stated above. Here, we use the Cantor topology on Σ^ω. If we have two representations δ, δ' of the same set M, then we call δ *reducible* to δ', if the identity $\mathrm{id} : M \to M$ is (δ, δ')–computable (in other words, if there exists a computable function $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that $\delta(p) = \delta' F(p)$ for all $p \in \mathrm{dom}(\delta)$). We also write $\delta \leq \delta'$ to indicate that δ is reducible to δ'. If $\delta \leq \delta'$ and $\delta' \leq \delta$, then we write $\delta \equiv \delta'$. If $\delta \leq \delta'$ and $\delta' \not\leq \delta$, then we write $\delta < \delta'$. Correspondingly, δ is called *topologically reducible* to δ', if the identity is (δ, δ')–continuous. In this case we write $\delta \leq_t \delta'$.

A certain class of representations has specifically nice topological properties: a representation δ of a topological space M is called *admissible*, if it is continuous and maximal among all continuous representations of M, i.e. if $\delta' \leq_t \delta$ holds for all continuous representations δ' of M. If M and M' are T_0–spaces with countable bases and topologies τ and τ' and admissible representations δ and δ', respectively, then a function $f :\subseteq M \to M'$ is (δ, δ')–continuous, if and only if it is (τ, τ')–continuous in the ordinary sense. Thus, for admissible representations the notion of relative continuity and the notion of ordinary continuity coincide. This is called the "Main Theorem" of the representation based approach to computable analysis (cf. [Weihrauch 2000]). If δ is admissible with respect to a T_0–topology τ with countable base, then τ is the final topology of δ.

Finally, we mention that a sequence $(x_n)_{n\in\mathbb{N}}$ in M is called *δ–computable*, if it is $(\delta_{\mathbb{N}}, \delta)$–computable as a function $\mathbb{N} \to M$, where $\delta_{\mathbb{N}}$ denotes some standard representation of the natural numbers $\mathbb{N} := \{0, 1, 2, ...\}$ (e.g. $\delta_{\mathbb{N}}(0^n 1^\omega) := n$). A point $x \in M$ is called *δ–computable*, if the constant sequence with value x is computable.

We close this section with a short survey on the organization of this paper. In the next section we exhibit some basic properties of the Fine representation and in Section 3 we will characterize this representation in a structural way. In Section 4 we characterize the notion of Fine continuity and in Section 5 we study different notions of Fine computability for functions. We close the paper in Section 6 with a discussion of computability properties of the integration operator on the space of Fine continuous functions.

2 The Fine representation

The purpose of this section is to state some elementary facts about the Fine representation. Especially, we will locate it in the lattice of real number representations. We recall that the Fine representation $\rho_F :\subseteq \Sigma^\omega \to [0, 1]$ of the unit interval $[0, 1]$, roughly speaking, represents real numbers by their binary expansion with digits $0, 1$ with the restriction that binary rationals $\mathbb{Q}_2$ are only represented by sequences $p \in \Sigma^\omega$ which contain infinitely many 0's. Since the definition guarantees that ρ_F is injective, one can deduce that ρ_F is an admissible representation of $[0, 1]$ with respect to the Fine topology τ_F (cf. Proposition 14 in [Brattka and Hertling]).

Since the Fine representation is a restriction of the binary representation ρ_2, it is clear that $\rho_F \leq \rho_2$. There are some other related representations of the real numbers. First of all, let ρ_E denote some admissible standard representation of the reals with respect to the Euclidean topology τ_E (cf. [Weihrauch 2000]). Moreover, there are the representations via characteristic functions of left and right cuts, ρ_L and ρ_R, respectively, and the continued fraction representation ρ_{cf} which is known to be the infimum of ρ_L and ρ_R with respect to reducibility. Deil [Deil 1984] has studied representations of real numbers extensively[3] and from his results we obtain the picture given in Figure 1 (we tacitly assume that all representations are restricted to $[0, 1]$). Here a line from a representation δ below

[3] Deil's representation $\delta''''_{(2)}$ is equivalent to ρ_F on $[0, 1]$ and he proved $\rho_R \leq \rho_F$.

to some representation δ' above means $\delta < \delta'$. Besides transitivity the diagram is complete (i.e. if there is no line, then the corresponding representations are incomparable).

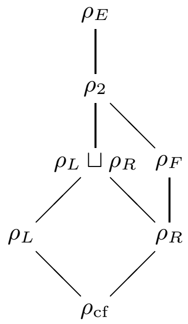

Figure 1: Lattice of real number representations.

The final topologies of ρ_L, ρ_R and ρ_{cf} are generated by the intervals $(a,b]$, $[a,b)$ and $[a,b]$ with $a,b \in \mathbb{Q}$, respectively. The final topology of ρ_E, ρ_2 and $\rho_L \sqcup \rho_R$ is the Euclidean topology τ'_E. Since $\delta \leq \delta'$ implies $\tau' \subseteq \tau$ for the final topologies τ and τ' of δ and δ', respectively, we immediately obtain $\rho_2 \not\leq \rho_F \not\leq \rho_R$ and $\rho_L \sqcup \rho_R \not\leq \rho_F$. Using some non-dyadic rational number one can finally prove $\rho_F \not\leq \rho_L \sqcup \rho_R$. Consequently, $\rho_L \sqcup \rho_R$ and ρ_F are incomparable.

What can be said about the class of ρ_F–computable real numbers? On the one hand, $\delta \leq \delta'$ implies the corresponding inclusion of the associated classes of computable points. On the other hand, $\rho_R < \rho_F < \rho_2$ and it is known that the classes of ρ_R– and ρ_2–computable real numbers coincide with the class of ordinary computable real numbers (i.e. ρ_E–computable real numbers). Thus, we directly obtain the following corollary.

Corollary 1. *A real number $x \in [0,1]$ is ρ_F–computable, if and only if it is computable.*

A direct proof would also be straightforward: obviously, an irrational real number is ρ_F–computable, if and only if it is ρ_2–computable; rational numbers are all computable anyway. Typically, the notion of a computable sequence is much more sensitive to the choice of the representation than the notion of a computable number [Mostowski 1957]. This does also hold in case of the Fine representation, as the following easy example shows (which also proves $\rho_2 \not\leq \rho_F$).

Proposition 2. *Each ρ_F–computable sequence is ρ_2–computable, but there exists a ρ_2–computable sequence which is not ρ_F–computable.*

Proof. The positive part of the statement is a direct consequence of $\rho_F \leq \rho_2$. Let $a : \mathbb{N} \to \mathbb{N}$ be some computable function with non-recursive range(a). The sequence $(x_n)_{n\in\mathbb{N}}$, defined by

$$x_n := \begin{cases} 1 - \frac{1}{2^k} & \text{if } k = \min\{i : a(i) = n\} \\ 1 & \text{if } n \notin \mathrm{range}(a) \end{cases}$$

is ρ_2–computable but not ρ_F–computable. $\square$

3 The Fine structure of the real numbers

In this section we want to characterize the Fine representation by a corresponding structure. Hertling has characterized the standard representation ρ_E in a structural way [Hertling 1999] and we have generalized these results to a larger class of topological structures [Brattka 1999a, Brattka]. Let us start to consider the space $([0,1], d_F)$ as a computable metric space.

Definition 3 (Computable metric space). We will call a triple (X, d, α) a *computable metric space*, if

(1) $d : X \times X \to \mathbb{R}$ is a metric on X,
(2) $\alpha : \mathbb{N} \to X$ is dense in X,
(3) $\mathbb{N} \to \mathbb{R}, \langle i,j\rangle \mapsto d(\alpha(i), \alpha(j))$ is a computable sequence of real numbers.

Here $\langle i,j\rangle := 1/2(i+j)(i+j+1)+j$ is used as a notation for *Cantor pairs* and a sequence of real numbers is called *computable*, if it is ρ_E–computable. An example of a computable metric space is $(\mathbb{R}, d_E, \nu_{\mathbb{Q}})$, where $\nu_{\mathbb{Q}}$ denotes some standard numbering of the rational numbers such as $\nu_{\mathbb{Q}}\langle i,j,k\rangle := (i-j)/(k+1)$. If we define $e : \mathbb{N} \to [0,1]$ by

$$e\langle i,j\rangle := e_{\langle i,j\rangle} := \frac{2^j \dot{-} i}{2^j},$$

where $\dot{-}$ denotes the *arithmetical difference* (with the convention that $j \dot{-} i$ is 0, if $j \le i$), then it is straightforward to see that $([0,1], d_F, e)$ is a computable metric space.

Proposition 4. *$([0,1], d_F, e)$ is a computable metric space.*

Given a computable metric space (X, d, α), we can define the *Cauchy representation* δ_α of X by

$$\delta_\alpha(01^{n_0}01^{n_1}01^{n_2}...) := \lim_{i\to\infty} \alpha(n_i)$$

for all $n_i \in \mathbb{N}$ such that $d(\alpha(n_i), \alpha(n_j)) \le 2^{-i}$ for all $j > i$. In the following we can assume without loss of generality $\rho_E = \delta_{\nu_{\mathbb{Q}}}$ (cf. [Weihrauch 2000]). It is easy to see that in case of the computable metric space $([0,1], d_F, e)$, we obtain $\delta_e \equiv \rho_F$.

Proposition 5. $\delta_e \equiv \rho_F$.

Proof. "$\delta_e \le \rho_F$" We sketch the construction of a Turing machine M which translates δ_e into ρ_F. Given a sequence $p = 01^{n_0}01^{n_1}01^{n_2}...$ with $x := \delta_e(p) \in [0,1]$ we know that $d_F(x, e_{n_i}) \le 2^{-i}$ for all $i \in \mathbb{N}$. Let v_i be the prefix of $\rho_F^{-1}(e_{n_{i+1}})$ of length $i+2$ and let w_i be the prefix of length $i+1$. Then

$$d_F(x, \rho_F(v_i 0^\omega)) \le d_F(x, e_{n_{i+1}}) + d_F(e_{n_{i+1}}, \rho_F(v_i 0^\omega)) < 2^{-i-1} + 2^{-i-1} = 2^{-i}$$

and thus w_i is a prefix of w_{i+1} for all $i \in \mathbb{N}$. Since $(\rho_F^{-1}(e_n))_{n\in\mathbb{N}}$ is a computable sequence in Σ^ω, the machine M can compute w_i in step i and extend the output of the previous step to w_i. In this way, M produces an output sequence $q \in \Sigma^\omega$ such that $\rho_F(q) = x$.

"$\rho_F \leq \delta_e$" We sketch the construction of a Turing machine M which translates ρ_F into δ_e. Given a sequence $p \in \Sigma^\omega$ with $x := \rho_F(p) \in [0,1]$ the machine M works in steps $i = 0, 1, 2, ...$ as follows: in step i it reads the prefix w_i of p of length $i + 2$ and computes some value $n_i \in \mathbb{N}$ with $e_{n_i} = \rho_F(w_i 0^\omega)$ and then M writes 01^{n_i} on the output tape. Altogether, M produces an output sequence $q = 01^{n_0}01^{n_1}01^{n_2}...$ with $d_F(e_{n_i}, x) < 2^{-i-1}$ and thus $d_F(e_{n_i}, e_{n_j}) \leq 2^{-i}$ for all $j > i$. Hence $\delta_e(q) = x$. $\square$

Now we will formulate a theorem which characterizes the Fine representation with the help of a structure. Therefore, we mention that whenever we have two representations δ and δ' of sets X and Y, respectively, then a canonical representation $[\delta, \delta']$ of the Cartesian product $X \times Y$ and a canonical representation δ^ω of the set of sequences $X^\mathbb{N}$ can be defined [Weihrauch 2000]. For short we write $\delta^2 := [\delta, \delta]$. For the following theorem we will use the *limit operation* $\mathrm{Lim}_F :\subseteq [0,1]^\mathbb{N} \to [0,1], (x_i)_{i\in\mathbb{N}} \mapsto \lim_{i\to\infty} x_i$, defined for all rapidly converging Cauchy sequences $(x_i)_{i\in\mathbb{N}}$ with respect to the Fine metric d_F, i.e. $d_F(x_i, x_j) \leq 2^{-i}$ for all $j > i$. Moreover, we will use the operation $\oplus :\subseteq [0,1] \times [0,1] \to [0,1], (x, y) \mapsto d_F(x, y)$, restricted to the following domain of continuity: $\mathrm{dom}(\oplus) := \{(x, y) : |\Psi(x)(i) - \Psi(y)(i)| = 0 \text{ for infinitely many } i\}$.

Theorem 6 (Fine structure). *Up to equivalence, ρ_F is the only representation such that the following structure becomes effective:*

$$\left([0,1], 0, 1, x \oplus y, \frac{1}{2}x, d_F, \mathrm{Lim}_F\right)$$

More precisely, if a representation ρ of $[0,1]$ has the property that

(1) $0, 1$ *are ρ–computable numbers,*
(2) $\oplus :\subseteq [0,1] \times [0,1] \to [0,1]$ *is (ρ^2, ρ)–computable,*
(3) $[0,1] \to [0,1], x \mapsto \frac{1}{2}x$ *is (ρ, ρ)–computable,*
(4) $d_F : [0,1] \times [0,1] \to \mathbb{R}$ *is (ρ^2, ρ_E)–computable,*
(5) $\mathrm{Lim}_F :\subseteq [0,1]^\mathbb{N} \to [0,1]$ *is (ρ^ω, ρ)–computable,*

then ρ is equivalent to the Fine representation, i.e. $\rho \equiv \rho_F$.

Proof. We sketch the proof using techniques from [Brattka 1999a, Brattka]. There we have proved that whenever (X, d, α) is a computable metric space, then δ_α is, up to computable equivalence, the only representation such that $(X, \alpha, \mathrm{id}, d, \mathrm{Lim})$ becomes effective. Using Propositions 4 and 5, it follows that ρ_F is, up to computable equivalence, the only representation such that the structure $S = ([0,1], e, \mathrm{id}, d_F, \mathrm{Lim}_F)$ becomes effective. It remains to prove that structure S is equivalent to the structure $S' = ([0,1], 0, 1, x \oplus y, \frac{1}{2}x, d_F, \mathrm{Lim}_F)$. Since $\mathrm{id}(x) = x \oplus 0$, it follows that id is recursive over S'. Moreover, $e_{\langle 0,0\rangle} = 1$, $e_{\langle i+1,0\rangle} = 0$, $e_{\langle 2i,j+1\rangle} = e_{\langle i,j\rangle}$ for all $i, j \in \mathbb{N}$ and

$$e_{\langle 2i+1,j+1\rangle} = \frac{2^{j+1} - (2i+2)}{2^{j+1}} + \frac{1}{2^{j+1}} = e_{\langle i+1,j\rangle} \oplus \left(\frac{1}{2}\right)^{j+1}$$

for all $i, j \in \mathbb{N}$ such that $2i + 1 < 2^{j+1}$, $e_{\langle 2i+1, j+1\rangle} = 0$ otherwise. Thus, e is recursive over S' too. Additionally, this shows that $0, 1$ are computable over S. Finally, $\rho_F(p) \oplus \rho_F(q) = \rho_F(r)$ with $r(i) := |p(i) - q(i)|$ for all p, q such that $(\rho_F(p), \rho_F(q)) \in \mathrm{dom}(\oplus)$, i.e. $\oplus$ is (ρ_F^2, ρ_F)–computable and $\frac{1}{2}\rho_F(p) = \rho_F(0p)$ for all $p \in \mathrm{dom}(\rho_F)$, i.e. $x \mapsto \frac{1}{2}$ is (ρ_F, ρ_F)–computable and thus both functions are computable over S. Altogether, it follows by the Equivalence Theorem 3.2.13 in [Brattka 1999a] that S is equivalent to S'. $\square$

4 Fine continuous functions

In [Mori] Mori has proved that the class $\mathcal{D} = \{f : [0, 1] \to \mathbb{R}$: there exists a total continuous $g : \Sigma^\omega \to \mathbb{R}$ such that $f(x) = g\Psi(x)$ for all $x \in [0, 1)\}$ coincides with the class of uniformly Fine continuous functions (i.e. with the class of uniformly (τ_F, τ_E)–continuous functions). We derive a similar characterization of the class of Fine continuous functions as an easy corollary of the Main Theorem [Weihrauch 2000]. The only difference will be the fact that g is allowed to be partial.

Theorem 7 (Fine continuity). *A function $f : [0, 1] \to \mathbb{R}$ is Fine continuous, if and only if there exists some continuous $g :\subseteq \Sigma^\omega \to \mathbb{R}$ such that $f(x) = g\Psi(x)$ for all $x \in [0, 1)$.*

Proof. Let $f : [0, 1] \to \mathbb{R}$ be Fine continuous, i.e. (τ_F, τ_E)–continuous. By the Main Theorem and by admissibility of ρ_F and ρ_E this implies that f is (ρ_F, ρ_E)–continuous and thus there exists a continuous function $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that $\rho_E F(p) = f\rho_F(p)$ for all $p \in \mathrm{dom}(\rho_F)$. This implies the first direction of the claim since $g := \rho_E F :\subseteq \Sigma^\omega \to \mathbb{R}$ is continuous (because ρ_E is continuous) and $\Psi = \rho_F^{-1}$. The situation of the proof is illustrated in Figure 2.

$$\begin{tikzcd}
\Sigma^\omega \arrow[r, "F"] \arrow[d, "\rho_F"'] \arrow[dr, "g"] & \Sigma^\omega \arrow[d, "\rho_E"] \\
{[0,1]} \arrow[r, "f"'] & \mathbb{R}
\end{tikzcd}$$

Figure 2: Relative continuity of a function $f : [0, 1] \to \mathbb{R}$.

On the other hand, the existence of a continuous $g :\subseteq \Sigma^\omega \to \mathbb{R}$ such that $f(x) = g\Psi(x)$ for all $x \in [0, 1)$ implies that the function $g' :\subseteq \Sigma^\omega \to \mathbb{R}$, defined by

$$g'(p) := \begin{cases} g(p) \text{ if } p \in \mathrm{dom}(\rho_F) \text{ and } \rho_F(p) \neq 1 \\ f(1) \text{ if } \rho_F(p) = 1 \\ \uparrow \quad \text{else} \end{cases},$$

is continuous, since $\rho_F^{-1}(1)$ is an isolated point in the subspace topology of the Cantor topology on $\mathrm{dom}(\rho_F)$. Thus, $f(x) = g'\rho_F^{-1}(x)$ for all $x \in [0,1]$ and f is Fine continuous since ρ_F^{-1} is continuous. $\square$

5 Fine computable functions

In this section we will compare different notions of Fine computable functions. On the one hand, we will call a function $f : [0,1] \to \mathbb{R}$ *Fine computable*, if it is (ρ_F, ρ_E)–computable. On the other hand, we will use the notion of a *locally uniformly Fine computable* function, as it has been defined by Mori [Mori 2000]. We will start to recall his definition. In the following we denote the open balls with respect to the Fine metric d_F by $B_F(x,r) := \{y \in [0,1] : d_F(x,y) < r\}$.

Definition 8 (Local uniform Fine computability). A real-valued function $f : [0,1] \to \mathbb{R}$ is called *locally uniformly Fine computable*, if

(1) f is sequentially computable, i.e. if it maps ρ_F–computable sequences to ρ_E–computable sequences,
(2) there are recursive functions $\gamma, \alpha : \mathbb{N} \to \mathbb{N}$ with $\gamma\langle i,j\rangle \geq j$ such that
   (a) $\bigcup_{i=0}^{\infty} B_F(e_i, 2^{-\gamma(i)}) = [0,1]$,
   (b) $|f(x) - f(y)| \leq 2^{-k}$ for all $x, y \in B_F(e_i, 2^{-\gamma(i)})$ with $d_F(x,y) \leq 2^{-\alpha\langle i,k\rangle}$.

For the following, intervals are always considered as subsets of $[0,1]$, for instance $[a,b) := \{x \in [0,1] : a \leq x < b\}$ for all $a, b \in \mathbb{R}$. It should be mentioned that $\gamma\langle i,j\rangle \geq j$ implies

$$B_F(e_{\langle i,j\rangle}, 2^{-\gamma\langle i,j\rangle}) = [e_{\langle i,j\rangle}, e_{\langle i,j\rangle} + 2^{-\gamma\langle i,j\rangle}).$$

Since $\rho_F \leq \rho_E$ we can directly conclude that all ordinary computable functions $f : [0,1] \to \mathbb{R}$ are Fine computable too. Moreover, this also implies that all ρ_F–computable sequences are ρ_E–computable. Since $d_E(x,y) \leq d_F(x,y)$ and a function $f : [0,1] \to \mathbb{R}$ is computable in the ordinary sense, if and only if it maps ρ_E–computable sequences to ρ_E–computable sequences and if it admits a uniform computable modulus of continuity with respect to d_E, we can conclude that all ordinary computable functions are locally uniformly Fine computable.

Theorem 9. *All computable functions $f : [0,1] \to \mathbb{R}$ are Fine computable, as well as locally uniformly Fine computable.*

It is also easy to see that definition by cases with dyadic border leads to Fine computable functions (because intervals $[0,a)$ with dyadic a are decidable with respect to the Fine representation ρ_F).

Proposition 10. *If $f, g : [0,1] \to \mathbb{R}$ are Fine computable functions and $a \in \mathbb{Q}_2$ is a dyadic rational, then $h : [0,1] \to \mathbb{R}$, defined by*

$$h(x) := \begin{cases} f(x) \text{ if } x < a \\ g(x) \text{ if } x \geq a \end{cases}$$

is Fine computable too.

Now the question appears, how Fine computable and locally uniformly Fine computable functions are related. The following theorem shows that all locally uniformly Fine computable functions are Fine computable too.

Theorem 11. *All locally uniformly Fine computable functions $f : [0,1] \to \mathbb{R}$ are Fine computable.*

Proof. Let us assume that $f : [0,1] \to \mathbb{R}$ is locally uniformly Fine computable with respect to computable functions $\gamma, \alpha : \mathbb{N} \to \mathbb{N}$. Thus, (1) and (2) of Definition 8 hold. We sketch the construction of a Turing machine M which proves that f is (ρ_F, ρ_E)–computable. Given a ρ_F–name p of some point $x := \rho_F(p) \in [0,1]$ the machine M first determines a number $i \in \mathbb{N}$ such that p and $\Psi(e_i)$ agree on a prefix of length $\gamma(i)+1$. Such an i must exist, since $\bigcup_{i=0}^{\infty} B_F(e_i, 2^{-\gamma(i)}) = [0,1]$ by (2)(a). Now the machine M continues to work in steps $k = 0, 1, 2, ...$ to produce rational approximations q_k of $f(x)$ with precision 2^{-k-1}. In step k the machine M determines some $j \in \mathbb{N}$ such that $e_j \in B_F(e_i, 2^{-\gamma(i)})$ and $d_F(x, e_j) < 2^{-\alpha\langle i,k+2\rangle}$. Since f is sequentially computable by (1), we know that $(f(e_n))_{n\in\mathbb{N}}$ is a ρ_E–computable sequence of real numbers and we can effectively determine some rational number $q_k \in \mathbb{Q}$ with $|q_k - f(e_j)| < 2^{-k-2}$. It follows by (2)(b)

$$|f(x) - q_k| \leq |f(x) - f(e_j)| + |f(e_j) - q_k| < 2^{-k-2} + 2^{-k-2} = 2^{-k-1}.$$

Thus, it suffices if M in step k writes q_k (properly encoded) on the output tape. $\square$

The next results shows that the converse statement is not true by constructing an explicit counterexample.

Theorem 12. *There exists a Fine computable function $f : [0,1] \to \mathbb{R}$ which is not locally uniformly Fine computable.*

Proof. A point $p \in \mathrm{dom}(\rho_F)$ either contains infinitely many 0's and infinitely many 1's, i.e. p is of type

$$p = 0^{n_0}1^{k_0}0^{n_1}1^{k_1}0^{n_2}1^{k_2}...$$

or p contains infinitely many $0's$ but only finitely many $1's$, i.e. p is of type

$$p = 0^{n_0}1^{k_0}0^{n_1}1^{k_1}...0^{n_m}1^{k_m}0^{\omega},$$

where $n_0, k_0, m \in \mathbb{N}$ and $n_{i+1}, k_{i+1} \in \mathbb{N} \setminus \{0\}$ for all $i \in \mathbb{N}$, and $k_0 = 0$ implies $n_0 = 0$. Using these normal forms we define a function $F :\subseteq \Sigma^{\omega} \to \mathbb{R}$ by

$$F(p) := \begin{cases} \sum\limits_{i=0}^{\infty} (k_i \bmod 2) \cdot 2^{-n_i - \sum\limits_{j=0}^{i-1}(n_j+k_j)} & \text{if } p = 0^{n_0}1^{k_0}0^{n_1}1^{k_1}... \\ \sum\limits_{i=0}^{m} (k_i \bmod 2) \cdot 2^{-n_i - \sum\limits_{j=0}^{i-1}(n_j+k_j)} & \text{if } p = 0^{n_0}1^{k_0}0^{n_1}1^{k_1}...0^{n_m}1^{k_m}0^{\omega} \end{cases}$$

Here $k \bmod 2$ is defined to be 0, if and only if k is even, and 1 otherwise. It is straightforward to construct a Turing machine which proves that the function $f := F\rho_F^{-1} : [0,1] \to \mathbb{R}$ is (ρ_F, ρ_E)–computable. The machine works in steps

$j = 0, 1, 2, ...$ and in the jth step it determines the output with precision 2^{-j}. In order to do this the machine simply has to verify whether a consecutive word of 1's with odd length starts in position j of the input sequence p. Since the machine only has to work correctly for inputs $p \in \mathrm{dom}(\rho_F)$, the machine does not have to produce a reasonable output in the case that the input p ends with an infinite suffix of 1's.

Now we will show that f is not locally uniformly Fine computable. Let us assume that f is locally uniformly Fine computable with corresponding functions $\alpha, \gamma : \mathbb{N} \to \mathbb{N}$. Consider some $i \in \mathbb{N}$. Then there is some word $w \in \Sigma^*$ such that $\rho_F(w\Sigma^\omega) = B_F(e_i, 2^{-\gamma(i)})$ (since $\gamma\langle i_1, i_2\rangle \geq i_2$ for all $i_1, i_2 \in \mathbb{N}$). Let $k := |w|+2$, where $|w|$ denotes the length of w. Let $n := \alpha\langle i, k\rangle$ and let $p := w01^{2n+1}0^\omega$, $q := w01^{2n}0^\omega$, $x := \rho_F(p)$ and $y := \rho_F(q)$. Then $x, y \in \rho_F(w\Sigma^\omega) = B_F(e_i, 2^{-\gamma(i)})$ and $d_F(x, y) = d_C(p, q) < 2^{-n} = 2^{-\alpha\langle i,k\rangle}$, but $|f(x) - f(y)| = 2^{-|w|-1} > 2^{-k}$. Contradiction! $\square$

In the proof by contradiction we have neither used the fact that f is supposed to be sequentially computable nor the fact that α, γ are supposed to be computable. Thus, the proof even shows that the constructed function f is not locally uniformly Fine continuous (defined in the obvious sense).

Finally, we want to characterize Fine computable functions in a way similar to the definition of locally uniformly Fine computable functions. The second half of the proof will be analogous to the proof of Theorem 11. The condition given in the following theorem is related to the notion of relative computability, as it has been defined by Tsujii, Yasugi and Mori [Tsujii, Yasugi and Mori 2001].

Theorem 13 (Fine computability). *A function $f : [0, 1] \to \mathbb{R}$ is Fine computable, if and only if*

(1) f is sequentially computable,
(2) there is a recursive function $\gamma : \mathbb{N} \to \mathbb{N}$ with $\gamma\langle\langle i, j\rangle, k\rangle \geq j$ such that
   (a) $\bigcup\limits_{i=0}^{\infty} B_F(e_i, 2^{-\gamma\langle i,k\rangle}) = [0, 1]$ for all $k \in \mathbb{N}$,
   (b) $|f(x) - f(y)| \leq 2^{-k}$ for all $x, y \in B_F(e_i, 2^{-\gamma\langle i,k\rangle})$.

Proof. Let $f : [0, 1] \to \mathbb{R}$ be Fine computable. It directly follows that f maps ρ_F–computable sequences to ρ_E–computable sequences, i.e. (1) holds. Furthermore, there is a computable function $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that $\rho_E F(q) = f\rho_F(q)$ for all $q \in \mathrm{dom}(\rho_F)$. Consequently, there exists a computable monotone word function $\varphi : \Sigma^* \to \Sigma^*$ such that $F(q) = \sup_{w \sqsubseteq q} \varphi(w)$ for all $q \in \mathrm{dom}(F)$ [Weihrauch 2000]. We recall that we assume $\rho_E(01^{n_0}01^{n_1}01^{n_2}...) := \lim_{i\to\infty} \nu_\mathbb{Q}(n_i)$ for all $n_i \in \mathbb{N}$ such that $|\nu_\mathbb{Q}(n_i) - \nu_\mathbb{Q}(n_j)| \leq 2^{-i}$ for all $j > i$. Then there exists a computable function $\gamma : \mathbb{N} \to \mathbb{N}$ such that $\gamma\langle\langle i_1, i_2\rangle, k\rangle$ is the smallest number $n \geq i_2$ such that the prefix w of $\Psi(e_{\langle i_1,i_2\rangle})$ of length $n + 1$ is mapped to a word $\varphi(w)$ with at least $k + 4$ symbols 0. Now let $I_{i,k} := [e_i, e_i + 2^{-\gamma\langle i,k\rangle}) = B_F(e_i, 2^{-\gamma\langle i,k\rangle})$. Then $f(I_{i,k}) \subseteq B(f(e_i), 2^{-k-1})$ and (2)(b) holds. We claim $[0, 1] = \bigcup_{i=0}^{\infty} I_{i,k}$ for all $k \in \mathbb{N}$, i.e. (2)(a) holds too. This follows, since for any $q \in \mathrm{dom}(\rho_F)$ and $k \in \mathbb{N}$ there is some finite prefix w of q of minimal length $j + 1$ such that $\varphi(w)$ contains at least $k + 4$ symbols 0. Thus, if we choose $i \in \mathbb{N}$ such that $e_{\langle i,j\rangle} = \rho_F(w0^\omega)$, then $\gamma\langle\langle i, j\rangle, k\rangle = j$ and

$$\rho_F(q) \in \rho_F(w\Sigma^\omega) = [e_{\langle i,j\rangle}, e_{\langle i,j\rangle} + 2^{-\gamma\langle\langle i,j\rangle,k\rangle}) = I_{\langle i,j\rangle,k}.$$

Now let us assume that $f : [0,1] \to \mathbb{R}$ fulfills (1) and (2) with respect to some computable function $\gamma : \mathbb{N} \to \mathbb{N}$. We sketch the construction of a Turing machine M which proves that f is (ρ_F, ρ_E)–computable. Given a ρ_F–name p of some point $x := \rho_F(p) \in [0,1]$ the machine M works in steps $k = 0, 1, 2, ...$ in order to compute rational approximations q_k of $f(x)$ with precision 2^{-k-1}, i.e. $|f(x) - q_k| < 2^{-k-1}$. In step k the machine determines a number $i \in \mathbb{N}$ such that p and $\Psi(e_i)$ agree on a prefix of length $\gamma\langle i, k+2\rangle + 1$. Such an i must exist, since $\bigcup_{i=0}^{\infty} B_F(e_i, 2^{-\gamma\langle i,k+2\rangle}) = [0,1]$ by (2)(a). Since f is sequentially computable by (1), it follows that $(f(e_n))_{n\in\mathbb{N}}$ is a ρ_E–computable sequence of real numbers and the machine M can effectively determine some rational number $q_k \in \mathbb{Q}$ with $|q_k - f(e_i)| < 2^{-k-2}$. It follows by (2)(b)

$$|f(x) - q_k| \leq |f(x) - f(e_i)| + |f(e_i) - q_k| < 2^{-k-2} + 2^{-k-2} = 2^{-k-1}.$$

Thus, it suffices if M in step k writes q_k (properly encoded) on the output tape. $\square$

6 Integration of Fine continuous functions

In this section we want to study computability properties of operators on Fine continuous functions such as integration. Therefore we need a representation of the space of Fine continuous functions.

We briefly mention that whenever δ and δ' are admissible representations of spaces X and Y with second-countable T_0–topologies τ and τ', respectively, then there is a canonical representation $[\delta \to \delta']$ of the set $\mathcal{C}(X,Y)$ of total (τ, τ')–continuous functions [Weihrauch 2000]. A characteristic feature of this representation is that it allows evaluation and type conversion. *Evaluation* means that the *evaluation function* $\mathcal{C}(X,Y) \times X \to Y, (f,x) \mapsto f(x)$ is $([[\delta \to \delta'], \delta], \delta')$–computable and *type conversion* means that a function $f : Z \times X \to Y$ is $([\delta'', \delta], \delta')$–computable, if and only if the associated function $f' : Z \to \mathcal{C}(X,Y)$ with $f'(z)(x) := f(z,x)$ is $(\delta'', [\delta \to \delta'])$–computable (these operations are also known as currying and uncurrying). It follows from results of Schröder [Schröder] that $[\delta \to \delta']$ is admissible with respect to (the sequentialization of) the compact-open topology on the function space $\mathcal{C}(X,Y)$.

Now let $\mathcal{C}_F[0,1]$ denote the *set of Fine continuous functions* $f : [0,1] \to \mathbb{R}$. Then $[\rho_F \to \rho_E]$ is an admissible representation of $\mathcal{C}_F[0,1]$ (endowed with the sequentialization of the compact-open topology with respect to τ_F and τ_E). Using evaluation and type conversion we can immediately conclude that whenever $f : \mathbb{R}^n \to \mathbb{R}$ is a computable function, then the corresponding pointwise defined operation $F : \mathcal{C}_F[0,1]^n \to \mathcal{C}_F[0,1], F(f_1, ..., f_n)(x) := f(f_1(x), ..., f_n(x))$ on the function space is computable with respect to $[\rho_F \to \rho_E]$. The following proposition summarizes some typical examples.

Proposition 14. *The following operations are computable with respect to the representation* $[\rho_F \to \rho_E]$ *of* $\mathcal{C}_F[0,1]$*:*

(1) $\mathcal{C}_F[0,1] \times \mathcal{C}_F[0,1] \to \mathcal{C}_F[0,1], (f,g) \mapsto f + g,$
(2) $\mathcal{C}_F[0,1] \times \mathcal{C}_F[0,1] \to \mathcal{C}_F[0,1], (f,g) \mapsto f \cdot g,$
(3) $\mathcal{C}_F[0,1] \to \mathcal{C}_F[0,1], f \mapsto |f|.$

Since the $[\rho_F \to \rho_E]$–computable points are exactly the Fine computable functions and computable operations map computable points to computable points, we directly obtain the following closure properties of the class of Fine computable functions.

Corollary 15. *If $f, g : [0,1] \to \mathbb{R}$ are Fine computable functions, then the functions $f+g, f-g, f \cdot g, |f| : [0,1] \to \mathbb{R}$ are Fine computable too.*

As a further operation we will treat integration. We will see that in general integration has weaker computability properties for Fine continuous functions as the forementioned algebraic operations.

Since any open set with respect to the Fine topology can be represented as a union of intervals $[a,b)$, it follows that all Fine open sets are F_σ–sets with respect to the Euclidean topology τ_E. Thus, all Fine continuous functions are Borel measurable with respect to the Euclidean topology. Hence a Fine continuous function $f : [0,1] \to \mathbb{R}$ is Lebesgue integrable if $\int_0^1 |f(x)|\, dx < \infty$. We simply consider the integration operator as operator from $\mathcal{C}_F[0,1]$ to the extended real line $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$. As a representation of $\overline{\mathbb{R}}$ we use the *Dedekind left cut representation* $\rho_<$, defined by

$$\rho_<(p) = x :\iff p = 01^{n_0}01^{n_1}01^{n_2}... \text{ and } \{q \in \mathbb{Q} : q < x\} = \{\nu_{\mathbb{Q}}(n_i) : i \in \mathbb{N}\}.$$

Using this representation we can prove the following uniform result on the Lebesgue integral (defined with respect to the Lebesgue measure μ on the real line) which shows that the integration operator on the space of Fine continuous functions is lower semi-computable. The first part of the proof is a uniform version of the first part of the proof of Theorem 13.

Theorem 16 (Integration). *The operator*

$$T : \mathcal{C}_F[0,1] \to \overline{\mathbb{R}}, f \mapsto \int_0^1 |f(x)|\, dx$$

is $([\rho_F \to \rho_E], \rho_<)$–computable.

Proof. By Proposition 14(3) it suffices to consider non-negative Fine continuous functions $f : [0,1] \to \mathbb{R}$. Let p be a $[\rho_F \to \rho_E]$–name of a non-negative function $f : [0,1] \to \mathbb{R}$. Given p, we can effectively determine a monotone word function $\varphi : \Sigma^* \to \Sigma^*$ such that $F(q) = \sup_{w \sqsubseteq q} \varphi(w)$ for all $q \in \mathrm{dom}(F)$, where the function $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ is some continuous (ρ_F, ρ_E)–realization of f, i.e. $\rho_E F(q) = f \rho_F(q)$ for all $q \in \mathrm{dom}(\rho_F)$. This allows us to determine a function $\gamma : \mathbb{N} \to \mathbb{N}$ such that $\gamma\langle\langle i_1, i_2\rangle, k\rangle$ is the smallest number $n \geq i_2$ such that the prefix w of $\Psi(e_{\langle i_1, i_2\rangle})$ of length $n+1$ is mapped to a word $\varphi(w)$ with at least $k+3$ symbols 0. Let $I_{i,k} := [e_i, e_i + 2^{-\gamma\langle i,k\rangle}) = B_F(e_i, 2^{-\gamma\langle i,k\rangle})$. Then $f(I_{i,k}) \subseteq B(f(e_i), 2^{-k})$. We claim $[0,1] = \bigcup_{i=0}^\infty I_{i,k}$ for all $k \in \mathbb{N}$. This follows, since for any $q \in \mathrm{dom}(\rho_F)$ and $k \in \mathbb{N}$ there is some finite prefix w of q of minimal length $j+1$ such that $\varphi(w)$ contains at least $k+3$ symbols 0. Thus, if we choose $i \in \mathbb{N}$ such that $e_{\langle i,j\rangle} = \rho_F(w0^\omega)$, then $\gamma\langle\langle i,j\rangle, k\rangle = j$ and

$$\rho_F(q) \in \rho_F(w\Sigma^\omega) = [e_{\langle i,j\rangle}, e_{\langle i,j\rangle} + 2^{-\gamma\langle\langle i,j\rangle,k\rangle}) = I_{\langle i,j\rangle,k}.$$

Now we compute a sequence $y_{k,n}$ of real numbers for any $k \in \mathbb{N}$ as follows. Let

$$y_{k,n} := \sum_{i=0}^{n} \mu \left(I_{i,k} \setminus \bigcup_{j=0}^{i-1} I_{j,k} \right) \cdot (f(e_i) - 2^{-k})$$

and let $y_k := \sup_{n\in\mathbb{N}} y_{k,n}$. Then we obtain either $y_k = \int_0^1 |f(x)|\ dx = \infty$ or $\int_0^1 |f(x)|\ dx - y_k < 2^{-k+1}$ and thus $y := \sup_{k\in\mathbb{N}} y_k = \int_0^1 |f(x)|\ dx$. Altogether, given p we can effectively compute a $\rho_<$–name r of y. □

Since the so-called *left-computable real numbers* are just the $\rho_<$–computable real numbers, we directly obtain the following corollary.

Corollary 17. *If $f : [0,1] \to \mathbb{R}$ is a Fine computable and Lebesgue integrable function, then the integral $\int_0^1 |f(x)|\ dx$ is a left-computable real number.*

The following simple example shows that the integral operator on Fine continuous functions cannot be computable from above. Thus the previous results are the best possible in a certain sense.

Proposition 18. *There exists a locally uniformly Fine computable function $f : [0,1] \to \mathbb{R}$ such that $\int_0^1 |f(x)|\ dx$ is a non-computable real number.*

Proof. Let $a \in \mathbb{R}$ be some non-negative left-computable but non-computable real number. Without loss of generality, we can assume that there exists a computable sequence $(a_n)_{n\in\mathbb{N}}$ of non-negative real numbers such that $a = \sum_{n=0}^{\infty} a_n$. Consider the intervals $I_n := [1 - 2^{-n}, 1 - 2^{-n-1})$ and define $f : [0,1] \to \mathbb{R}$ by

$$f(x) := \begin{cases} 2^{n+1}a_n & \text{if } x \in I_n \\ 0 & \text{if } x = 1 \end{cases}$$

It is easy to see that f is locally uniformly Fine computable. We obtain

$$\int_0^1 |f(x)|\ dx = \sum_{n=0}^{\infty} \mu(I_n) f(1 - 2^{-n}) = \sum_{n=0}^{\infty} a_n = a.$$

□

The proof can be extended to a proof which shows that the integration operator T admits a $(\rho_<, [\rho_F \to \rho_E])$–computable right-inverse (on non-negative real numbers). Moreover, this implies that T is not $([\rho_F \to \rho_E], \rho_E)$–continuous since otherwise $\rho_< \leq_t \rho_E$ would follow (on non-negative real numbers).

Since the Fine topology is not locally compact, it follows that the compact-open topology on $\mathcal{C}_F[0,1]$ is not first-countable and thus especially neither second-countable nor metrizable (cf. Exercise 3.4.E in [Engelking 1989]). Results of Matthias Schröder seem to suggest that the sequentialization of the compact-open topology on $\mathcal{C}_F[0,1]$ is also neither metrizable nor second-countable (personal communication). Thus, the result on integration shows that the representation based approach to computable analysis captures also complicated topological spaces in a reasonable way.

However, tasks which require integration, like computing the Walsh-Fourier coefficients, should be restricted to some proper subclass of the class of Fine continuous functions which admit fully computable integration (cf. [Mori, Mori 2000])

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