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Word Operation Closure and Primitivity of Languages

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Abstract: Based on the general operation $\circ$ of words, called bw-operation, the notions of $\circ$-primitive words, $\circ$-closed languages, $\circ$-bases of languages and operation-left-quotient-closed languages are defined and investigated. These notions turn out to be generalizations of the classical notions of primitive words, plus-closed (star-closed) languages, minimal generating sets and deletion-closed languages. Properties of the set of all $\circ$-primitive words, the $\circ$-bases of non-empty languages, right $\circ$-residuals and operation-left-quotient closed languages are studied under the general concept of word operation. Properties of bi-catenation and related languages are discussed as examples and also by their own interests.

Key Words: Word operation, primitivity, closure, right residual, base, dense.
Category: F.4.3

1 Introduction

A non-empty closed set for a given binary operation is called a *groupoid*. When the operation is associative, groupoids are called *semigroups*. Within the fundamental researches of the formal language theory, the investigations of the properties concerning operations and the related results when these operations are applied to combine languages or words play a very important role. For example: an *Abelian semigroup* is a groupoid with a commutative and associative binary operation. For any set S, it is important to find a minimal subset M of S such that the operation closure of M contains S and to know whether S has a unique minimal generating set. For example: if a semigroup S has a unique minimal generating set which is a code, then S is a *free semigroup*. Catenation, bi-catenation ([14]), shuffle product ([3], [7]), k-catenation ([11]), insertion, deletion and s-insertion ([10]) are some of the most frequently applied operations. The aim of this paper is to investigate the relations between operations of words, the related closed languages and primitive words.

One of the main results about primitivity is that every non-empty word can be uniquely expressed as a power of a primitive word ([13]). This unique expression property of words makes the properties of primitive words very basic and important in the theory of formal languages. In [12], the ins-primitive words, shuffle-primitive words and the com-shuffle-primitive words for insertion, shuffle product and commutative ordered shuffle pruduct, respectively, are defined similar to the primitive words for catenation. Let $\circ : X^* \times X^* \to 2^{X^*}$ be an operation over a finite non-empty alphabet X. In this paper, the $\circ$-primitive words for the operation $\circ$ of words are defined by an analogous definition. We investigate some general properties concerning $\circ$-primitive words instead of separate properties of primitive words with respect to certain operations.

If we define the $\circ$-primitive words analogous to the primitive words, then every $\circ$-primitive word is primitive whenever $\circ$ is the catenation of words and ins-primitive whenever $\circ$ is the insertion of words. The relationships between words and $\circ$-primitive words are studied in Section 3. Section 4 focuses mainly on the properties of the set of all $\circ$-primitive words over X. In Section 5, we investigate properties of the generating sets of $\circ$-closed languages. The bi-catenation, which is not associative, is considered in Section 6 to elucidate results obtained in this paper. Section 7 is dedicated to study the properties of the right $\circ$-residuals of languages. We show that the right $\circ$-residuals of any given language is $\circ$-closed. A characterization of right $\circ$-residuals for any given language L is given in Section 7. Some general properties of the bw-operation $\triangleleft_\circ$ and $\triangleleft_\circ$-closed languages are concerned in Section 8.

2 Preliminaries

In this paper, let X be a finite non-empty alphabet and 1 denote the empty word. Under the catenation of words, X^* is the free monoid generated by X and $X^+ = X^* \setminus \{1\}$. For a language $L \subseteq X^*$, let $L^n = \{u_1u_2\cdots u_n \mid u_i \in L,\ i = 1, 2, \cdots, n\}$ for $n \geq 1$ and let $L^+ = \bigcup_{i\geq 1} L^i$.

A *binary word-operation with right identity* (shortly *bw-operation*) is defined as a mapping $\circ : X^* \times X^* \to 2^{X^*}$ with $\circ(u, 1) = \{u\}$. Furthermore, we define $\circ(L_1, L_2) = \bigcup_{u\in L_1, v\in L_2} \circ(u, v)$ and $\circ(L_1, \emptyset) = \emptyset = \circ(\emptyset, L_2)$ for any two languages L_1 and L_2 and often identify singleton sets with their elements. The *iterated bw-operation* $\circ^i$ is defined by $\circ^0(L_1, L_2) = L_1$ and $\circ^i(L_1, L_2) = \circ\big(\circ^{i-1}(L_1, L_2), L_2\big)$ whenever $i \geq 1$ for languages L_1 and L_2. The *i*-th $\circ$-*power* of a non-empty language L is defined as $L^{\circ(0)} = \{1\}$ and $L^{\circ(i)} = \circ^{i-1}(L, L)$ for $i \geq 1$. A non-empty word w is called $\circ$-*primitive* if $w \in u^{\circ(i)}$ for some word u and $i \geq 1$ yields $i = 1$ and $w = u$. Properties concerning insertion-primitive words are studied in [5].

The $+$-*closure* of a non-empty language L with respect to a bw-operation $\circ$, denoted by $L^{\circ(+)}$, is defined as $L^{\circ(+)} = \bigcup_{k\geq 1} L^{\circ(k)}$. A language L is $\circ$-*closed* if $u, v \in L$ imply $\circ(u, v) \subseteq L$.

Given a language L. A word u is a *right* $\circ$-*residual of* L if $\circ(w, u) \subseteq L$ for all $w \in L$, i.e., $\circ(L, u) \subseteq L$ whenever L is not empty. Let $\rho_\circ(L)$ denote the set of all right $\circ$-residuals of L, i.e., $\rho_\circ(L) = \{u \in X^* \mid \forall w \in L, \circ(w, u) \subseteq L\}$. For properties of right shuffle-residuals, one is referred to [7]. Note that $\rho_\circ(\emptyset) = \emptyset$ and $1 \in \rho_\circ(L)$ for any non-empty language L. The $\circ$-*left-quotient*, denoted by $\triangleleft_\circ$, is defined as $\triangleleft_\circ(L_1, L_2) = \{w \in X^* \mid \circ(L_2, w) \cap L_1 \neq \emptyset\}$. The shuffle-closed and shuffle-left-quotient-closed languages are investigated in [8]. The $\circ$-left quotient $\triangleleft_\circ$ is used to characterize the set $\rho_\circ(L)$ of right $\circ$-residuals for any given language

L in Section 7. The following properties of bw-operations are concerned in this paper:

A bw-operation $\circ$ is called *right-commutative* if for any three languages L_1, L_2 and L_3, $\circ\big(\circ(L_1, L_2), L_3\big) = \circ\big(\circ(L_1, L_3), L_2\big)$.

A bw-operation $\circ$ is called *length-incerasing* if for any $u, v \in X^+$ and $w \in \circ(u, v)$, $\lg(w) > \max\{\lg(u), \lg(v)\}$. For a word w and a lettter a, $N_a(w)$ denotes the number of occurrences of a in w. A bw-operation $\circ$ is called *propagating* if for any $u, v \in X^*$ and $w \in \circ(u, v)$, $N_a(w) = N_a(u) + N_a(v)$ for any $a \in X$. Every propagating bw-operation is length-increasing.

A bw-operation $\circ$ is called *left-inclusive* if for any three words u, v, w in X^*, $\circ\big(\circ(u, v), w\big) \supseteq \circ\big(u, \circ(v, w)\big)$. Every associative bw-operation is left-inclusive. A bw-operation $\circ$ is called *star-left-inclusive* if for any $w \in X^*$, $\circ(X^*, w) \supseteq \circ\big(X^*, \circ(X^*, w)\big)$. Every left-inclusive bw-operation is star-left-inclusive. A bw-operation $\circ$ is called $\circ$*-power-left-inclusive* if $\circ(L^{\circ(i+j-1)}, L) \supseteq \circ(L^{\circ(i)}, L^{\circ(j)})$ for any non-empty language L and $i, j \geq 1$. A bw-operation $\circ$ is called *plus-closed* if for any non-empty language L, $L^{\circ(+)}$ is $\circ$-closed. Note that every left-inclusive bw-operation $\circ$ is $\circ$-power-left-inclusive and every $\circ$-power-left-inclusive bw-operation is plus-closed.

In general, definitions and notations will be given when they are needed. Items not defined in this paper can be founded in [1], [3], [8] and [10]. The definitions, notations and properties of some bw-operations are listed as follows:

– Catenation :
  $w \cdot u = wu$, associative, propagating.
– Shuffle :
  $w \diamond u = \{x_1 y_1 \cdots x_n y_n \mid w = x_1 \cdots x_n, u = y_1 \cdots y_n, x_i, y_i \in X^*, i = 1, 2, \ldots, n\}$, associative, propagating, right-commutative.
– Insertion :
  $w \leftarrow u = \{xuy \mid w = xy, x, y \in X^*\}$, left-inclusive, propagating.
– Bi-catenation :
  $w \bullet u = \{wu, uw\}$, $\bullet$-power-left-inclusive, star-left-inclusive, propagating.
– Left quotient :
  $w \lhd\!. \, u = u^{-1} w = \{v \in X^* \mid w = uv\}$, star-left-inclusive.
– Scattered deletion :
  $w \lhd_{\diamond} u = \{x_1 x_2 \cdots x_n \mid x_i, y_i \in X^*, w = x_1 y_1 \cdots x_n y_n, u = y_1 \cdots y_n\}$, star-left-inclusive, right-commutative.
– Deletion :
  $w \circ_{\mathrm{d}} u = \{xy \mid x, y \in X^*, w = xuy\}$, star-left-inclusive.

3 $\circ$-Primitivity

In this section, we study the primitive-expressions of words and basic properties of $\circ$-primitive words. We establish a construction of $\circ$-primitive words according to any given non-empty word.

Lemma 1. *If $\circ$ is left-inclusive then for any non-empty language L, $L^{\circ(+)}$ is $\circ$-closed.*

Proof. For $u, v \in L^{\circ(+)}$, $u \in L^{\circ(m)}$ and $v \in L^{\circ(n)}$ for some $m, n \geq 1$. We shall prove this assertion by induction on n. If $n = 1$, then $\circ(u, v) \subseteq \circ(L^{\circ(m)}, v) \subseteq$

$\circ(L^{\circ(m)}, L) = L^{\circ(m+1)} \subseteq L^{\circ(+)}$. Assume that $\circ(u, v) \subseteq L^{\circ(+)}$ for $1 \leq n \leq k$. Let $n = k + 1$. Then $v \in L^{\circ(k+1)} = \circ(L^{\circ(k)}, L)$. There exist $x \in L^{\circ(k)}$ and $y \in L$ such that $v \in \circ(x, y)$. Then by the induction hypothesis, $\circ(u, x) \subseteq L^{\circ(+)}$, i.e., $\circ(u, x) \subseteq L^{\circ(j)}$ for some j. From the definition, $\circ\big(\circ(u, x), y\big) \subseteq \circ(L^{\circ(j)}, L) = L^{\circ(j+1)} \subseteq L^{\circ(+)}$. Since $\circ$ is left-inclusive, $\circ(u, v) \subseteq \circ\big(u, \circ(x, y)\big) \subseteq \circ\big(\circ(u, x), y\big) \subseteq L^{\circ(+)}$. The proof of the induction step and therefore of the lemma is complete. □

Lemma 2. *If $\circ$ is plus-closed then for any word u in X^*, $\circ^m\big(\circ^n(u, u), \circ^p(u, u)\big) \subseteq u^{\circ(+)}$, for all $m, n, p \geq 0$.*

Proof. This will be proved by induction on m. By definition, $\circ^0\big(\circ^n(u, u), \circ^p(u, u)\big) = \circ^n(u, u) \subseteq u^{\circ(+)}$. Since $\circ$ is plus-closed, $u^{\circ(+)}$ is $\circ$-closed. For every $v \in \circ^n(u, u) \subseteq u^{\circ(+)}$ and $w \in \circ^p(u, u) \subseteq u^{\circ(+)}$, $\circ(v, w) \subseteq u^{\circ(+)}$. Thus $\circ\big(\circ^n(u, u), \circ^p(u, u)\big) \subseteq u^{\circ(+)}$. Assume that $\circ^k\big(\circ^n(u, u), \circ^p(u, u)\big) \subseteq u^{\circ(+)}$ for some $k \geq 1$. Take a word $y \in \circ^{k+1}\big(\circ^n(u, u), \circ^p(u, u)\big) = \circ\Big(\circ^k\big(\circ^n(u, u), \circ^p(u, u)\big), \circ^p(u, u)\Big)$. There exist $v \in \circ^k\big(\circ^n(u, u), \circ^p(u, u)\big)$ and $w \in \circ^p(u, u)$ such that $y \in \circ(v, w)$. By the induction hypothesis, $v \in \circ^k\big(\circ^n(u, u), \circ^p(u, u)\big) \subseteq u^{\circ(+)}$. This in conjunction with that $\circ$ is plus-closed and $w \in \circ^p(u, u) \subseteq u^{\circ(+)}$ yields $y \in \circ(v, w) \subseteq u^{\circ(+)}$. Hence $\circ^{k+1}\big(\circ^n(u, u), \circ^p(u, u)\big) \subseteq u^{\circ(+)}$. Therefore, the lemma is complete. □

It is known that every non-empty word is a power of a unique primitive word. The following proposition shows that a similar result holds for the case of plus-closed and length-incerasing bw-operations, with the exception of uniqueness.

Proposition 3. *Let $\circ$ be plus-closed and length-incerasing. Then for every word $w \in X^+$ there exist a $\circ$-primitive word u and an integer $n \geq 1$ such that $w \in u^{\circ(n)}$.*

Proof. Suppose w is not $\circ$-primitive. Then there exist $u_1 \in X^+$ and $n_1 > 1$ such that $w \in u_1^{\circ(n_1)} = \circ^{n_1-1}(u_1, u_1)$. If u_1 is not $\circ$-primitive, then $u_1 \in u_2^{\circ(n_2)} = \circ^{n_2-1}(u_2, u_2)$ for some $u_2 \in X^+$ and $n_2 > 1$. This implies $w \in \circ^{n_1-1}\big(\circ^{n_2-1}(u_2, u_2), \circ^{n_2-1}(u_2, u_2)\big)$ which, according to Lemma 2, implies $w \in u_2^{\circ(+)}$. Since $\circ$ is length-incerasing, $\lg(u_1) > \lg(u_2)$. By repeatedly applying the procedure and Lemma 2, after a finite number of steps, we have a $\circ$-primitive word u such that $w \in u^{\circ(+)}$. This means that there is an integer $n \geq 1$ such that $w \in u^{\circ(n)}$. □

Corollary 4. *Let $\circ$ be plus-closed and propagating. Then for every word $w \in X^+$ there exist a $\circ$-primitive word u and a unique integer $n \geq 1$ such that $w \in u^{\circ(n)}$.*

Proof. By Proposition 3, for every word $w \in X^+$ there exists a $\circ$-primitive word u and an integer $n \geq 1$ such that $w \in u^{\circ(n)}$. Take $a \in X$ such that $N_a(u) \neq 0$. As $\circ$ is propagating, for any $w_1 \in u^{\circ(m)}$ with $m \neq n$, $N_a(w_1) = mN_a(u) \neq nN_a(u) = N_a(w)$. Thus $w \notin u^{\circ(m)}$ for any $m \neq n$. □

A $\circ$-primitive word u such that $w \in u^{\circ(n)}$ for some $n \geq 1$ is called a $\circ$-*root* of w. In general, a word may have several $\circ$-roots.

Lemma 5. *Let $\circ$ be plus-closed and propagating and let $|X| \geq 2$. If a word $w \in X^+$ is not $\circ$-primitive then for any $a, b \in X$, $N_a(w)$ and $N_b(w)$ have a common factor $n > 1$.*

Proof. If w is not $\circ$-primitive, then, according to Proposition 3, $w \in u^{\circ(n)}$ for some $\circ$-primitive word $u \in X^+$ and $n > 1$. Since $\circ$ is propagating, $N_a(w) = nN_a(u)$ for all $a \in X$. Thus for any $a, b \in X$, the numbers of a's and b's in w have the common factor $n > 1$. □

Proposition 6. *Let $\circ$ be plus-closed and propagating and let $|X| \geq 2$. If $w \in X^+$, $a \in X$, $w \notin a^+$ then there is an integer $m \geq 1$ such that all the words $v_1 \in \circ(w, w^{m-1}a)$, $v_2 \in \circ(aw^{m-1}, w)$, $v_3 = w^m a$ and $v_4 = aw^m$ are $\circ$-primitive.*

Proof. For $w \in X^+$, let $m = \prod_{b \in X, N_b(w) \neq 0} N_b(w)$. For any $a \in X$, suppose $w \notin a^+$ and let $v_1 \in \circ(w, w^{m-1}a)$, $v_2 \in \circ(aw^{m-1}, w)$, $v_3 = w^m a$ and $v_4 = aw^m$. If $b \neq a$ is a letter occurring in w, $N_a(v_1) = N_a(v_2) = N_a(v_3) = N_a(v_4) = mN_a(w) + 1$ whereas $N_b(v_1) = N_b(v_2) = N_b(v_3) = N_b(v_4) = mN_b(w)$. As the number of a's and b's in each v_i, $i = 1, 2, 3, 4$, are relatively prime, by Lemma 5, v_1, v_2, v_3 and v_4 are $\circ$-primitive words. □

4 $Q_\circ$: The set of $\circ$-primitive words

Let $Q_\circ(X)$ denote the set of all $\circ$-primitive words over X. A language $L \subseteq X^*$ is called *right $\circ$-dense* (resp. *left $\circ$-dense*) if for each $w \in X^+$, there exists $u \in X^*$ such that $\circ(w, u) \cap L \neq \emptyset$ (resp. $\circ(u, w) \cap L \neq \emptyset$). If $\circ$ is the catenation of words, then the right or left $\circ$-dense languages are called the right or left dense languages, respectively.

Proposition 7. *Let $\circ$ be plus-closed and propagating and let $|X| \geq 2$. Then $Q_\circ(X)$ is right and left $\circ$-dense.*

Proof. For each $w \in X^+$, since $|X| \geq 2$, there exists $a \in X$ such that $w \notin a^+$. As $\circ$ is plus-closed and propagating, by Proposition 6, there is $m \geq 1$ such that $\circ(w, w^{m-1}a) \subseteq Q_\circ(X)$ and $\circ(aw^{m-1}, w) \subseteq Q_\circ(X)$. Therefore, $Q_\circ(X)$ is right and left $\circ$-dense. □

Proposition 8. *Let $\circ$ be plus-closed and propagating and let $|X| \geq 2$. Then $Q_\circ(X)$ is right and left dense.*

Proof. Let $w \in X^+$. If $w = a^n$ for some $a \in X$, $n \geq 1$ and if $b \in X$, $b \neq a$, then by Lemma 5, $wb = a^n b \in Q_\circ(X)$ and $bw = ba^n \in Q_\circ(X)$. If $w \notin a^+$ then, according to Proposition 6, $w^m a \in Q_\circ(X)$ and $aw^m \in Q_\circ(X)$ for some $m \geq 1$. This proves that $Q_\circ(X)$ is right and left dense. □

Let $L^c = X^* \setminus L$ for any language L.

Proposition 9. *Let $\circ$ be plus-closed and propagating and $L \subseteq X^+$ a non-empty $\circ$-closed language such that L^c is also $\circ$-closed. Let $F(L)$ be the set of minimal words of L and $P_\circ(L) = L \cap Q_\circ(X)$. Then:*

(1) If $w \in L$ and if u is a $\circ$-root of w, then $u \in L$.

(2) If L' is a $\circ$-closed language containing $P_\circ(L)$ then $L \subseteq L'$.

(3) Every word $w \in F(L)$ is $\circ$-primitive.

Proof. (1) Since u is a $\circ$-root of w, $w \in u^{\circ(n)}$ for some $n \geq 1$. If $u \in L^c$, then, since L^c is $\circ$-closed, $u^{\circ(n)} = \circ^{n-1}(u, u) \subseteq L^c$ and $w \in L^c$, a contradiction. Hence $u \in L$.

(2) This follows from (1).

(3) Suppose w is not $\circ$-primitive. Then by Proposition 3, $w \in u^{\circ(n)}$ for some $\circ$-primitive word u and $n > 1$. By (1), $u \in L$. As $\circ$ is propagating, $\lg(w) = \sum_{a\in X} N_a(w) > \sum_{a\in X} N_a(u) = \lg(u)$. This contradicts the fact that w is one of the minimal words in L. □

From the definitions of $\circ$ and $Q_\circ(X)$, we have the following properties immediately.

Corollary 10. *(1) For any $A, B, C \subseteq X^+$, $\circ(A, C) \subseteq Q_\circ(X)$ and $\circ(B, C) \subseteq Q_\circ(X)$ imply $\circ(A \cup B, C) \subseteq Q_\circ(X)$.*

(2) For any $A \subseteq X^+$, $A \subseteq Q_\circ(X)$ implies ($B \subseteq A \Rightarrow B \subseteq Q_\circ(X)$).

(3) For any $A, B, C, D \subseteq X^+$, $\circ(A \cap B, C \cap D) \subseteq \circ(A, C) \cap \circ(B, D)$.

Let $\mathcal{F} = 2^{Q_\circ(X)}$ and for $A \subseteq X^+$, let $\alpha_{\mathcal{F}}(A) = \{B \subseteq X^+ \mid \circ(A, B) \in \mathcal{F}\}$.

Proposition 11. *If for any $A, B \subseteq X^+$, $\circ(A, B) \subseteq Q_\circ(X) \iff \circ(B, A) \subseteq Q_\circ(X)$, then $\langle \alpha_{\mathcal{F}}(A), \subseteq, \cap, \cup \rangle$ forms a distributive lattice for any $A \subseteq X^+$.*

Proof. By Theorem 2.8 of [15] and Corollary 10, the assertion holds. □

5 $\circ$-Free languages

A non-empty language $L \subseteq X^+$ is called $\circ$-*free* if $\circ(L^{\circ(+)}, L) \cap L = \emptyset$. A non-empty language L is a *suffix code* (resp. *prefix code*) if $L \cap X^+L = \emptyset$ (resp. $L \cap LX^+ = \emptyset$). If $\circ$ is the catenation operation of words, suffix codes is an important class of $\circ$-free languages. For any non-empty language $L \subseteq X^+$, we define the following sets:

$$K_1 = L,$$
$$I_i = \{w \mid w \in K_i \text{ and } \lg(w) \leq \lg(y) \text{ for all } y \in K_i\} \text{ for all } i \geq 1,$$
$$K_i = L \setminus (\textstyle\bigcup_{1\leq j\leq i-1} I_j)^{\circ(+)} \text{ for all } i \geq 2,$$
$$\beta_\circ(L) = \textstyle\bigcup_{i\geq 1} I_i.$$

Fact 1. $\beta_\circ(L) \subseteq L$.

Proposition 12. *Let $\circ$ be plus-closed and length-increasing. Then $L \subseteq \beta_\circ(L)^{\circ(+)}$ and $\beta_\circ(L)$ is $\circ$-free.*

Proof. Let $w \in L$. Then $w \in (\bigcup_{1\leq j\leq \lg(w)} I_j)^{\circ(+)} \subseteq \beta_\circ(L)^{\circ(+)}$. Thus $L \subseteq \beta_\circ(L)^{\circ(+)}$. Next, we shall show that $\beta_\circ(L)$ is $\circ$-free. Assume that there exist $w \in L$, $u \in \beta_\circ(L)^{\circ(+)}$ and $v \in \beta_\circ(L)$ such that $w \in \circ(u, v)$. Then there exists a minimal number k such that $u \in (\bigcup_{1\leq j\leq k} I_j)^{\circ(+)}$ and $i \geq 1$ such that $v \in I_i$. If $k \leq i$, then $w \in (\bigcup_{1\leq j\leq i} I_j)^{\circ(+)}$. From the definition of K_{i+1}, one must have $w \notin K_{i+1}$. This yields $w \notin I_m$ for all $m \geq i + 1$. As $\lg(w) >$

$\lg(v) \geq \lg(y)$ for all $y \in \bigcup_{1\leq j\leq i} I_j$, $w \notin \bigcup_{1\leq j\leq i} I_j$. Thus $w \notin \beta_\circ(L)$. If $k > i$, then $w \in (\bigcup_{1\leq j\leq k} I_j)^{\circ(+)}$. From the definition of K_{i+1}, one must have that $w \notin K_{k+1}$. Hence, $w \notin I_m$ for all $m \geq k+1$. Since $\lg(w) > \lg(u) \geq \lg(y)$ for all $y \in \bigcup_{1\leq j\leq k} I_j$. This implies that $w \notin \bigcup_{1\leq j\leq k} I_j$. Thus $w \notin \beta_\circ(L)$. Therefore, for every $w \in \beta_\circ(L)$, there exist no $u \in \beta_\circ(L)^{\circ(+)}$ and $v \in \beta_\circ(L)$ such that $w \in \circ(u,v)$, i.e., $\beta_\circ(L)$ is $\circ$-free. □

Proposition 13. *Let $\circ$ be plus-closed and length-increasing. If $L \subseteq X^+$ is $\circ$-closed, then $L = \beta_\circ(L)^{\circ(+)}$.*

Proof. Since $\beta_\circ(L) \subseteq L$ and L is $\circ$-closed, $\beta_\circ(L)^{\circ(+)} \subseteq L$. By Proposition 12, $L \subseteq \beta_\circ(L)^{\circ(+)}$. Thus $L = \beta_\circ(L)^{\circ(+)}$. □

For $L \subseteq X^*$, if there exists a $\circ$-free language $B \subseteq L \setminus \{1\}$ such that $B^{\circ(+)} = (L \setminus \{1\})^{\circ(+)}$, then B is called a $\circ$-*base* of L. From the definitions of $\circ$-free and $\circ$-base, one must have that a language L having a $\circ$-base implies that $L \neq \emptyset$ and $L \neq \{1\}$.

Fact 2. Let $\circ$ be plus-closed and length-increasing and $L \subseteq X^*$ be a non-empty language with $L \neq \{1\}$. Then the following two statements hold true.

(1) The set $\beta_\circ(L)$ defined by the above method for $L \setminus \{1\}$ is a $\circ$-base of L.

(2) If B is a $\circ$-base of L then L being $\circ$-closed implies that $L \setminus \{1\} = B^{\circ(+)}$.

In the following proposition, another construction of a $\circ$-base of a given language is proposed.

Proposition 14. *Let $\circ$ be plus-closed and length-increasing and $L \subseteq X^+$ be a non-empty language. Then $L^{\circ(+)} \setminus \circ(L^{\circ(+)}, L)$ is a $\circ$-base of L.*

Proof. Let $B = L^{\circ(+)} \setminus \circ(L^{\circ(+)}, L)$. Then $B^{\circ(+)} = L^{\circ(+)}$. From the definition, $L^{\circ(+)} = \circ(L^{\circ(+)}, L) \cup L$. Thus $B \subseteq L$. This yields that $\circ(B^{\circ(+)}, B) \cap B = \circ(L^{\circ(+)}, B) \cap \left(L^{\circ(+)} \setminus \circ(L^{\circ(+)}, L)\right) \subseteq \circ(L^{\circ(+)}, L) \cap \left(L^{\circ(+)} \setminus \circ(L^{\circ(+)}, L)\right) = \emptyset$. Hence, B is $\circ$-free. Therefore, B is a $\circ$-base of L. □

Consider two $\circ$-closed languages $S_1, S_2 \subseteq X^*$ such that the empty word in or not in both S_1 and S_2 simultaneously. Then we have the following property:

Proposition 15. *Let $\circ$ be $\circ$-power-left-inclusive and length-increasing and let S_1 and S_2 be two $\circ$-closed languages of X^* with $\circ$-bases B_1 and B_2, respectively. Then $S_1 = S_2 \iff B_1 = B_2$.*

Proof. For the necessity of the proof, we assume that there exists a $\circ$-closed language S of X^* with two distinct $\circ$-bases B_1 and B_2. Without loss of generality, let $B_1 \setminus B_2 \neq \emptyset$. By Proposition 13, $S \setminus \{1\} = B_1^{\circ(+)} = B_2^{\circ(+)}$. If $w \in B_1 \setminus B_2$, then $w \in B_1 \subseteq B_1^{\circ(+)} = B_2^{\circ(+)}$. That is, $w \in B_2^{\circ(n)} = \circ(B_2^{\circ(n-1)}, B_2)$ for some $n \geq 2$. Thus there exist $u \in B_2^{\circ(+)} = B_1^{\circ(+)}$ and $v \in B_2 \subseteq B_2^{\circ(+)} = B_1^{\circ(+)}$ such that $w \in \circ(u,v)$. Note that $u \in B_1^{\circ(i)}$ and $v \in B_1^{\circ(j)}$ for some $i, j \geq 1$. As $\circ$ is $\circ$-power-left-inclusive, $w \in \circ(u,v) \subseteq \circ(B_1^{\circ(i)}, B_1^{\circ(j)}) \subseteq \circ(B_1^{\circ(i+j-1)}, B_1)$, which contradicts the fact that $w \in B_1$ and B_1 is $\circ$-free. Therefore, $B_1 = B_2$. Conversely, it is immediate that $B_1 = B_2$ implies $S_1 = S_2$. □

The next result shows that if a language L has a $\circ$-base, then this $\circ$-base is unique and it is called *the $\circ$-base* of L.

Proposition 16. *Let $\circ$ be $\circ$-power-left-inclusive and length-increasing. The $\circ$-base of a language $L \subseteq X^*$ is unique.*

Proof. Suppose B_1, B_2 be two $\circ$-bases of L. Then $B_1^{\circ(+)} = (L \setminus \{1\})^{\circ(+)} = B_2^{\circ(+)}$. That is, B_1 and B_2 are $\circ$-bases of the same $\circ$-closed language $(L \setminus \{1\})^{\circ(+)}$. By Proposition 15, $B_1 = B_2$. □

By (1) of Fact 2, $\beta_\circ(L)$ is a $\circ$-base of L. By Proposition 16, the $\circ$-base of a language is unique. Thus, the notation $\beta_\circ(L)$ will be used to denote the $\circ$-base of a language L. The $\circ$-free sets and the $\circ$-bases are the so-called independent sets and bases in some sense, respectively. For properties concerning the theory of dependence in universal algebras, including the concepts of bases and the sets generated by bases, one is referred to [2] and [4]. It is known that if S is a set with a transitive dependence D then the properties of being a basis, a maximal independent subset, and a minimal spanning (or, generating) subset are equivalent.

6 The bi-catenation

In this section, we consider properties of the bi-catenation $\bullet$ of words and related languages. By definition, the bi-catenation $\bullet$ is propagating.

Example. Let $X = \{a, b, c\}$, $L_1 = \{a\}$, $L_2 = \{b\}$ and $L_3 = \{c\}$. Then $\bullet(L_1, L_2) = \{ab, ba\}$, $\bullet(L_2, L_3) = \{bc, cb\}$, $\bullet\big(\bullet(L_1, L_2), L_3\big) = \{abc, cab, bac, cba\}$ and $\bullet\big(L_1, \bullet(L_2, L_3)\big) = \{abc, bca, acb, cba\}$. Clearly, $\bullet$ is not left-inclusive.

Lemma 17. *([13]) For $u, v \in X^+$, $uv = vu$ implies that u and v are powers of a common word.*

Proposition 18. *For $u, v, w \in X^*$, $\bullet\big(\bullet(u, v), w\big) = \bullet\big(u, \bullet(v, w)\big)$ if and only if u and w are powers of a common word.*

Proof. For $u, v, w \in X^*$, $\bullet\big(\bullet(u, v), w\big) = \{uvw, wuv, vuw, wvu\}$ and $\bullet\big(u, \bullet(v, w)\big)$ $= \{uvw, vwu, uwv, wvu\}$. Suppose $\bullet\big(\bullet(u, v), w\big) = \bullet\big(u, \bullet(v, w)\big)$. Then $(wuv = vwu$ and $vuw = uwv)$ or $(wuv = uwv$ and $vuw = vwu)$. Consider the following two cases:

(1) $wuv = vwu$ and $vuw = uwv$. By Lemma 17, wu, v and uw are powers of a common word. Since $\lg(wu) = \lg(uw)$ and they are powers of a common word, $wu = uw$. In view of Lemma 17, u, w and wu are powers of a common word.

(2) $wuv = uwv$ and $vuw = vwu$. Then $wu = uw$. By Lemma 17, u and w are powers of a common word.

Conversely, let $u = p^i$ and $w = p^j$. Then $\bullet\big(\bullet(u, v), w\big) = \{uvw, p^{i+j}v, vp^{i+j}, wvu\} = \bullet\big(u, \bullet(v, w)\big)$. □

Proposition 19. *For any non-empty language L, $L^{\bullet(n)} = L^n$ for any $n \geq 1$.*

Proof. Let L be a non-empty language. Then $L^{\bullet(1)} = L$. Suppose that $L^{\bullet(k)} = L^k$ for some $k \geq 1$. From definitions of $L^{\circ(n)}$ and $\bullet$, we have $L^{\bullet(k+1)} = \bullet(L^{\bullet(k)}, L) = \bullet(L^k, L) = L^{k+1}$. By induction on n, we have $L^{\bullet(n)} = L^n$ for any $n \geq 1$. □

Lemma 20. *For any non-empty language L, $\bullet(L^{\bullet(m)}, L^{\bullet(n)}) = L^{\bullet(m+n)}$ for $m, n \geq 1$.*

Proof. By Proposition 19, $L^{\bullet(m)} = L^m$ and $L^{\bullet(n)} = L^n$. From the definition of $\bullet$, we have $\bullet(L^m, L^n) = L^m L^n = L^n L^m = L^{m+n} = L^{\bullet(m+n)}$. □

In view of Proposition 19 and Lemma 20, we have that $\bullet(L^{\bullet(m)}, L^{\bullet(n)}) = L^{m+n} = \bullet(L^{\bullet(m+n-1)}, L)$, i.e., the bi-catenation $\bullet$ is $\bullet$-power-left-inclusive.

Corollary 21. *The bi-catenation of words is plus-closed.*

From Proposition 19, we have $L^{\bullet(+)} = L^+$. This in conjunction with the definition of $\bullet$-primitive words yields that a word is $\bullet$-primitive if and only if it is primitive (related to the catenation of words). However, the $\bullet$-bases of $\bullet$-closed non-empty languages and the bases of catenation-closed non-empty languages have the following difference: If $\circ$ is the catenation of words, then for any finite non-empty language $L \subseteq X^+$, the $\circ$-closed set $L^{\circ(+)}$ is called an *F-semigroup.* If S is an F-semigroup such that $S \cup \{1\}$ is $\triangleleft_\circ$-closed, then S is often called *right unitary.* It is known that the $\circ$-base $\beta_\circ(S)$ of any right unitary F-semigroup S is a prefix code (see [1]). A *bifix code* L is a prefix code and also a suffix code.

Proposition 22. *Let L be a $\bullet$-closed non-empty language with $L \neq \{1\}$. Then $L \cup \{1\}$ is $\triangleleft_\bullet$-closed if and only if the $\bullet$-base $\beta_\bullet(L)$ of L is a bifix code.*

Proof. Let $L \cup \{1\}$ be $\bullet$-closed and $\triangleleft_\bullet$-closed. By definition, $\beta_\bullet(L) \subseteq L$. Suppose on the contrary that $\beta_\bullet(L)$ is not a bifix code, i.e., there exist $u, v \in \beta_\bullet(L)$ such that $u = vw$ or $u = wv$ for some non-empty word w. Since L is $\triangleleft_\bullet$-closed, $w \in L$. Thus $u \in \bullet(v, w)$. By definition, $\lg(w) < \lg(u)$. By definitions of I_i, K_k and $\beta_\bullet(L)$, $u \notin \beta_\bullet(L)$, a contradiction. Therefore, $\beta_\bullet(L)$ is a bifix code. Conversely, let $\beta_\bullet(L)$ be a bifix code. In view of Proposition 13, we have $L \setminus \{1\} = \beta_\bullet(L)^{\bullet(+)}$. Suppose there exist $u \in L \setminus \{1\}$ and $w \in X^+$ such that $\bullet(u, w) = \{uw, wu\} \cap L \neq \emptyset$. Then $u \in \beta_\bullet(L)^{\bullet(+)}$ and $uw \in \beta_\bullet(L)^{\bullet(+)}$ or $wu \in \beta_\bullet(L)^{\bullet(+)}$. Since $\beta_\bullet(L)$ is a bifix code, $w \in \beta_\bullet(L)^{\bullet(+)} = L \setminus \{1\}$. Thus $L \cup \{1\} = \beta_\bullet(L)^{\bullet(+)} \cup \{1\}$ is $\triangleleft_\bullet$-closed. □

7 Right $\circ$-residuals

When considering the right $\circ$-residuals of languages, we can find another difference between the bi-catenation and the catenation of words. Let $X = \{a, b\}$ and $L = ab^+$. Then $\rho_\bullet(L) = \{1\}$. But if $\circ$ is the catenation of words then $\rho_\circ(L) = b^*$. In this section, we investigate some properties concerning the set $\rho_\circ(L)$ of right $\circ$-residuals of languages L. We will give a characterization of the set of right $\circ$-residuals of languages. First, we show that the set of right $\circ$-residuals of any non-empty language is a $\circ$-closed language.

Proposition 23. *Let $\circ$ be left-inclusive and L a non-empty language. Then $1 \in \rho_\circ(L)$ and $\rho_\circ(L)$ is $\circ$-closed.*

Proof. As $\circ(L,1) = L$, $1 \in \rho_\circ(L)$. Take $u, v \in \rho_\circ(L)$. Then $\circ(L,u) \subseteq L$ and $\circ(L,v) \subseteq L$. Since $\circ$ is left-inclusive, $\circ\big(L, \circ(u,v)\big) \subseteq \circ\big(\circ(L,u), v\big) \subseteq \circ(L,v) \subseteq L$. That is, $\circ(u,v) \subseteq \rho_\circ(L)$. □

A bw-operation $\circ$ satisfies the *left-identity* condition if $\circ(1,L) = L$ for any language L.

Proposition 24. *Let $\circ$ satisfy the left-identity condition. If a non-empty language L is $\circ$-closed and $1 \in L$, then $L = \rho_\circ(L)$.*

Proof. Suppose L is $\circ$-closed with $1 \in L$. For $u, v \in L$, since L is $\circ$-closed, $\circ(u,v) \subseteq L$. Thus $\circ(L,v) \subseteq L$ for every $v \in L$, i.e., $L \subseteq \rho_\circ(L)$. For $w \in \rho_\circ(L)$, as $1 \in L$ and $\circ$ satisfies the left-identity condition, $\rho_\circ(L) = \circ\big(1, \rho_\circ(L)\big) \subseteq L$. Hence, $L = \rho_\circ(L)$. □

From Propositions 23 and 24, we have the following property of $\circ$-closed languages immediately.

Corollary 25. *Let $\circ$ be left-inclusive and satisfy the left-identity condition. Then a non-empty language L is $\circ$-closed and contains 1 if and only if $L = \rho_\circ(L)$.*

Proposition 26. *Let L_1 and L_2 be two non-empty languages. Then $\rho_\circ(L_1) \cap \rho_\circ(L_2) \subseteq \rho_\circ(L_1 \cap L_2) \cap \rho_\circ(L_1 \cup L_2)$.*

Proof. Let $w \in \rho_\circ(L_1) \cap \rho_\circ(L_2)$. Then $\circ(L_1, w) \subseteq L_1$ and $\circ(L_2, w) \subseteq L_2$. This implies that $\circ(L_1 \cap L_2, w) \subseteq \circ(L_1, w) \cap \circ(L_2, w) \subseteq L_1 \cap L_2$. Moreover, by the definition of $\circ$, $\circ(L_1 \cup L_2, w) = \circ(L_1, w) \cup \circ(L_2, w) \subseteq L_1 \cup L_2$. Thus $w \in \rho_\circ(L_1 \cap L_2) \cap \rho_\circ(L_1 \cup L_2)$. □

A bw-operation $\circ$ is called *right-inclusive* if for any three words u, v, w, $\circ\big(\circ(u,v), w\big) \subseteq \circ\big(u, \circ(v,w)\big)$. Every associative bw-operation is right-inclusive.

Proposition 27. *If $\circ$ is right-inclusive, then for any two non-empty languages L_1 and L_2, $\rho_\circ(L_2) \subseteq \rho_\circ\big(\circ(L_1, L_2)\big)$.*

Proof. Let $w \in \rho_\circ(L_2)$. Since $\circ$ is right-inclusive, $\circ\big(\circ(L_1, L_2), w\big) \subseteq \circ\big(L_1, \circ(L_2, w)\big) \subseteq \circ(L_1, L_2)$. This yields $w \in \rho_\circ\big(\circ(L_1, L_2)\big)$. Thus $\rho_\circ(L_2) \subseteq \rho_\circ\big(\circ(L_1, L_2)\big)$. □

The shuffle operation of words is right-commutative and satisfies the left-identity condition.

Proposition 28. *Let $\circ$ be right-commutative and satisfy the left-identity condition. Then for any two languages L_1 and L_2,*
(1) $\circ(L_1, L_2) = \circ(L_2, L_1)$,
(2) $\circ\big(\rho_\circ(L_1), \rho_\circ(L_2)\big) \subseteq \rho_\circ\big(\circ(L_1, L_2)\big)$.

Proof. (1) If L_1 or L_2 is empty, then $\circ(L_1, L_2) = \emptyset = \circ(L_2, L_1)$. Let L_1 and L_2 be two non-empty languages. Since $\circ$ satisfies the left-identity condition, $\circ(1, L) = L$ for any L. If $\circ$ is right-commutative then $\circ(L_1, L_2) = \circ\big(\circ(1, L_1), L_2\big) = \circ\big(\circ(1, L_2), L_1\big) = \circ(L_2, L_1)$.

(2) If L_1 or L_2 is empty, then $\rho_\circ(L_1)$ or $\rho_\circ(L_2)$ is empty. By definition, $\circ(L_1, L_2)$, $\rho_\circ\big(\circ(L_1, L_2)\big)$ and $\circ\big(\rho_\circ(L_1), \rho_\circ(L_2)\big)$ are empty. The assertion holds. Now, let L_1 and L_2 be non-empty. Take $u \in \rho_\circ(L_1)$ and $v \in \rho_\circ(L_2)$. Since $\circ$ is right-commutative, $\circ\big(\circ(L_1, L_2), \circ(u, v)\big) = \circ\Big(\circ\big(L_1, \circ(u, v)\big), L_2\Big)$. By (1), $\circ\big(L_1, \circ(u, v)\big) = \circ\big(\circ(u, v), L_1\big)$. Since $\circ$ is right-commutative, $\circ\big(\circ(u, v), L_1\big) = \circ\big(\circ(u, L_1), v\big)$. By (1), $\circ\Big(\circ\big(\circ(u, L_1), v\big), L_2\Big) = \circ\Big(\circ\big(v, \circ(L_1, u)\big), L_2\Big)$. As $u \in \rho_\circ(L_1)$ and $v \in \rho_\circ(L_2)$, $\circ(L_1, u) \subseteq L_1$ and $\circ(L_2, v) \subseteq L_2$. Since $\circ$ is right-commutative, $\circ\Big(\circ\big(v, \circ(L_1, u)\big), L_2\Big) = \circ\big(\circ(v, L_2), \circ(L_1, u)\big) = \circ\big(\circ(L_1, u), \circ(L_2, v)\big) \subseteq \circ(L_1, L_2)$. Thus $\circ(u, v) \subseteq \rho_\circ\big(\circ(L_1, L_2)\big)$. This yields that $\circ\big(\rho_\circ(L_1), \rho_\circ(L_2)\big) \subseteq \rho_\circ\big(\circ(L_1, L_2)\big)$. □

For a word $w = a_1a_2 \cdots a_n \in X^*$, $k \geq 0$, we define

$$com(w) = \{a_{s(1)}a_{s(2)} \cdots a_{s(n)} \mid s \text{ a permutation of } \{1, \ldots, n\}\}.$$

For any language $L \subseteq X^*$, let $com(L) = \bigcup_{w \in L} com(w)$. A language L is *commutative* if $L = com(L)$. In [8], it is shown that every shuffle-closed and shuffle-left-quotient-closed language is a commutative language. Recently, properties concerning shuffle closures of regular languages are investigated by Imreh, Ito and Katsura in [6]. In the following, we consider a property of the relations between commutative languages and their right $\circ$-residuals with respect to a propagating bw-operation $\circ$.

Proposition 29. *If $\circ$ is propagating, then a language L being commutative implies that $\rho_\circ(L)$ is commutative.*

Proof. Let $w \in \rho_\circ(L)$ and $u \in L$. Then $\circ(u, w) \subseteq L$. Since L is commutative, $com\big(\circ(u, w)\big) \subseteq L$. As $\circ$ is propagating, $\circ\big(u, com(w)\big) \subseteq com\big(\circ(u, w)\big) \subseteq L$. Thus $com(w) \subseteq \rho_\circ(L)$. Therefore, $\rho_\circ(L)$ is commutative. □

The following property is a construction of $\rho_\circ(L)$ for a given language L.

Proposition 30. *For any language $L \subseteq X^*$, $\rho_\circ(L) = \big(\lhd_\circ(L^c, L)\big)^c$.*

Proof. Assume that there exists $w \in \rho_\circ(L) \setminus \big(\lhd_\circ(L^c, L)\big)^c$. Then $w \in \lhd_\circ(L^c, L)$. This means that $v \in \circ(u, w)$ for some $v \in L^c$ and $u \in L$, which contradicts the fact that $w \in \rho_\circ(L)$ and $u \in L$. Now, suppose that there exists $w \in \big(\lhd_\circ(L^c, L)\big)^c \setminus \rho_\circ(L)$. Then there exist $u \in L$ such that $\circ(u, w) \cap L^c \neq \emptyset$. Let $v \in \circ(u, w) \cap L^c$. Then $w \in \lhd_\circ(v, u) \subseteq \lhd_\circ(L^c, L)$, which contradicts the fact that $w \in \big(\lhd_\circ(L^c, L)\big)^c$. □

8 $\lhd_\circ$-Closed languages

In this section, we are going to study some general properties concerning the $\circ$-left-quotient $\lhd_\circ$ and $\lhd_\circ$-closed languages.

Proposition 31. *If $\circ$ is right-commutative and satisfies the left-identity condition then $\lhd_\circ$ is right-commutative.*

Proof. Take $w \in \lhd_\circ\big(\lhd_\circ(u_1, u_2), u_3\big)$ for some $u_1, u_2, u_3 \in X^*$. This means that $w \in \lhd_\circ(v_1, u_3)$ for some $v_1 \in \lhd_\circ(u_1, u_2)$, i.e., $u_1 \in \circ(u_2, v_1)$ and $v_1 \in \circ(u_3, w)$. Hence, $u_1 \in \circ(u_2, v_1) \subseteq \circ\big(u_2, \circ(u_3, w)\big)$. By (1) of Proposition 28, $\circ\big(u_2, \circ(u_3, w)\big) = \circ\big(\circ(u_3, w), u_2\big)$. As $\circ$ is right-commutative, $\circ\big(\circ(u_3, w), u_2\big) = \circ\big(\circ(u_3, u_2), w\big)$. Similarly, $\circ\big(\circ(u_3, u_2), w\big) = \circ\big(\circ(u_2, u_3), w\big) = \circ\big(\circ(u_2, w), u_3\big)$, which implies $u_1 \in \circ\big(\circ(u_2, w), u_3\big)$. This in conjunction with (1) of Proposition 28 again yields $u_1 \in \circ(v_2, u_3) = \circ(u_3, v_2)$ for some $v_2 \in \circ(u_2, w)$. Hence $v_2 \in \lhd_\circ(u_1, u_3)$. It follows that $w \in \lhd_\circ(v_2, u_2) \subseteq \lhd_\circ\big(\lhd_\circ(u_1, u_3), u_2\big)$. That is, $\lhd_\circ\big(\lhd_\circ(u_1, u_2), u_3\big) \subseteq \lhd_\circ\big(\lhd_\circ(u_1, u_3), u_2\big)$. By changing u_2 and u_3, we have $\lhd_\circ\big(\lhd_\circ(u_1, u_3), u_2\big) \subseteq \lhd_\circ\big(\lhd_\circ(u_1, u_2), u_3\big)$. Therefore, $\lhd_\circ\big(\lhd_\circ(u_1, u_2), u_3\big) = \lhd_\circ\big(\lhd_\circ(u_1, u_3), u_2\big)$. □

Proposition 32. *Let $\circ$ be plus-closed and length-increasing. Then it is true that $\lhd_\circ\big(\beta_\circ(L), \beta_\circ(L)\big) \subseteq \lhd_\circ(L, L) \setminus (L \setminus \{1\})$ for any non-empty language $L \subseteq X^*$.*

Proof. Take $w \in \lhd_\circ\big(\beta_\circ(L), \beta_\circ(L)\big)$. Then there exist $u, v \in \beta_\circ(L)$ such that $u \in \circ(v, w)$. In view of Proposition 14 and Proposition 16, we have $\beta_\circ(L) = L^{\circ(+)} \setminus \circ(L^{\circ(+)}, L)$. Thus $w \notin L$ or $w = 1$. As $\beta_\circ(L) \subseteq L$, $w \in \lhd_\circ(L, L)$. Thus $w \in \lhd_\circ(L, L) \setminus (L \setminus \{1\})$, i.e., $\lhd_\circ\big(\beta_\circ(L), \beta_\circ(L)\big) \subseteq \lhd_\circ(L, L) \setminus (L \setminus \{1\})$. □

Proposition 33. *Let $L \subseteq X^*$ be a non-empty $\circ$-closed and $\lhd_\circ$-closed language. Then $1 \in L$ and $\lhd_\circ(L, L) = L$.*

Proof. As $\circ(L, 1) = L$, $1 \in \lhd_\circ(L, L)$. The inclusion $\lhd_\circ(L, L) \subseteq L$ follows immediately from the fact that L is $\lhd_\circ$-closed. Now, let $w \in L$. Since L is $\circ$-closed, $\circ(L, w) \subseteq L$. This yields $w \in \lhd_\circ(L, L)$. □

For any language L, the $\circ$-*leftover* of L is defined as $\lambda_\circ(L) = \{u \in X^* \mid \circ(X^*, u) \cap L = \emptyset\}$. The $\circ$-leftover is also called the (left) residue of a language (see [9]). A non-empty language L is a *left* $\circ$-*ideal* if $\circ(X^*, L) \subseteq L$. Clearly, we have that $1 \notin \lambda_\circ(L)$ whenever $L \neq \emptyset$.

Remark. If $\circ$ is star-left-inclusive then every non-empty $\circ$-leftover of a language is a left $\circ$-ideal.

A language L is left $\circ$-dense if and only if $\lambda_\circ(L) = \emptyset$. Let $|X| \geq 2$. Recall from Proposition 7 that $Q_\circ(X)$ is left $\circ$-dense whenever $\circ$ is plus-closed and propagating.

Proposition 34. *Let L be a $\circ$-closed non-empty language such that $\lambda_\circ(L) \neq \emptyset$. Then*
(1) $L \subseteq \big(\lambda_\circ(L)\big)^c$,
(2) If $\lambda_\circ(L) = L^c$, then L is $\lhd_\circ$-closed.

Proof. (1) For $u \in L$, since L is $\circ$-closed, $\circ(L, u) \subseteq L$. Thus $u \notin \lambda_\circ(L)$, i.e., $u \in \big(\lambda_\circ(L)\big)^c$ and $L \subseteq \big(\lambda_\circ(L)\big)^c$.

(2) As $\lambda_\circ(L) = L^c$, $L = \{u \in X^* \mid \circ(X^*, u) \cap L \neq \emptyset\}$. For $w \in L$ and $u \in X^*$, $\circ(w, u) \cap L \neq \emptyset$ implies that $u \in L$. That is, $\lhd_\circ(L, w) = \{u \in X^* \mid \circ(w, u) \cap L \neq \emptyset\} \subseteq L$. Thus L is $\lhd_\circ$-closed. □

Lemma 35. *Let L be a $\circ$-closed and $\lhd_\circ$-closed non-empty language. Then it is true that $\circ(L, L^c) \subseteq L^c$.*

Proof. If $L = X^*$, then $L^c = \emptyset$ and $\circ(X^*, \emptyset) = \emptyset \subseteq \emptyset = L^c$. Now, let $L \neq X^*$. If there exists $w \in \circ(L, v) \cap L$ for some $v \in L^c$, then since L is $\triangleleft_\circ$-closed, $v \in L$, a contradiction. Thus $\circ(L, L^c) \subseteq L^c$. □

Proposition 36. *Let L be a $\circ$-closed and $\triangleleft_\circ$-closed non-empty language with $L \neq X^*$. Then the following three statements are equivalent:*
(1) L^c is $\circ$-closed,
(2) L^c is a left $\circ$-ideal,
(3) $L^c = \lambda_\circ(L)$.

Proof. (1) $\Rightarrow$ (2): Suppose that L^c is $\circ$-closed. Then $\circ(L^c, L^c) \subseteq L^c$. By Lemma 35, $\circ(L, L^c) \subseteq L^c$. Thus $\circ(X^*, L^c) \subseteq L^c$.

(2) $\Rightarrow$ (3): From (1) of Proposition 34, we have $\lambda_\circ(L) \subseteq L^c$. By (2), L^c is a left $\circ$-ideal, i.e., $\circ(X^*, L^c) \subseteq L^c$. Thus $L^c \subseteq \lambda_\circ(L)$.

(3) $\Rightarrow$ (1): From the definition of $\lambda_\circ(L)$, we have that $\lambda_\circ(L) = L^c$ is $\circ$-closed. □

Now, we consider languages defined on X^+ instead of X^*, i.e., $L \subseteq X^+$, $\triangleleft_\circ(L_1, L_2) = \{w \in X^+ \mid \circ(L_2, w) \cap L_1 \neq \emptyset\}$ and $\lambda_\circ(L) = \{u \in X^+ \mid \circ(X^+, u) \cap L = \emptyset\}$. Then Proposition 33 will become the following case:

Corollary 37. *Let $L \subseteq X^+$ be a non-empty $\circ$-closed and $\triangleleft_\circ$-closed language. Then $\triangleleft_\circ(L, L) = L$.*

From Proposition 36, we have the following result immediately.

Corollary 38. *If $L \subseteq X^+$ is a $\circ$-closed and $\triangleleft_\circ$-closed non-empty language such that $L' = X^+ \setminus L \neq \emptyset$ is $\circ$-closed and $\triangleleft_\circ$-closed, then $L' = \lambda_\circ(L)$ and $L = \lambda_\circ\big(\lambda_\circ(L)\big)$.*

For example: let $X = \{a, b\}$, $L = X^*a$ and $\circ$ be the catenation operation of words. Then $L' = \lambda_\circ(L) = X^*b$ and $L = \lambda_\circ(L')$. Note that in this case, $\circ(L_1, L_2) \neq \circ(L_2, L_1)$ for some non-empty languages L_1 and L_2.

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