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Tiling the Hyperbolic Plane with a Single Pentagonal Tile[1]

Maurice Margenstern
(Université de Metz, LITA, UFR MIM, Ile du Saulcy, 57045 METZ CEDEX, France
Email: margens@lita.univ-metz.fr)

Abstract: In this paper, we study the number of tilings of the hyperbolic plane that can be constructed, starting from a single pentagonal tile, the only permitted transformations on the basic tile being the replication by displacement along the lines of the pentagrid. We obtain that there is no such tiling with five colours, that there are exactly two of them with four colours and a single trivial tiling with one colour. For three colours, the number of solutions depends of the assortment of the colours. For half of them, there is a continuous number of such tilings, for one of them there are four solutions, for the two other ones, there is no such tiling. For two colours, there is always a continuous number of such tilings.
By contrast, there is no such analog in the euclidean plane with the similar constraints.
Key Words: Hyperbolic plane, tilings
Category: F.1, F.1.1, G.2

1 Introduction

The first paper about tilings of the hyperbolic plane in computer science is probably Robinson's paper in 1978, [Robinson 78], which is devoted to undecidable tiling problems in that setting. A few other papers appeared later in a combinatorial approach, in particular about aperiodic tilings of the hyperbolic plane, see for instance [Margulis and Moses 98].

This paper is an attempt to go on this work from the point of view of computer science. It starts from the simplest situation, as it was the case by Wang in [Wang 61] in the euclidean case.

In its spirit, this paper is a continuation of [Margenstern 00], which provides us with new algorithms to locate cells of a cellular automaton grounded on the pentagrid with right angles. As the basic features and references of what is needed of hyperbolic geometry are given in [Margenstern and Morita 01] and [Margenstern 00], we shall remind them very sketchily. We remind also the beginning of the proof of the existence of the pentagrid that is given in the just quoted papers. It is needed to understand the tool constituted by the Fibonacci trees that is there introduced. We shall also make use of the corresponding numbering that is introduced in [Margenstern 00] as a locating tool.

2 Representing the pentagrid by the Fibonacci tree

We take Poincaré's open unit disk as a model of the hyperbolic plane $I\!H^2$. Lines are diameters or intersection, within the open unit disk, of circles, that

[1] C. S. Calude, K. Salomaa, S. Yu (eds.). *Advances and Trends in Automata and Formal Languages. A Collection of Papers in Honour of the 60th Birthday of Helmut Jürgensen.*

are orthogonal to the unit circle. See Figure 1 for an illustrative representation. Reflections in lines are the same as in the euclidean case if the line is a diameter of the unit circle. Reflections in lines that are not diameters are the euclidean inversion with respect to the circle that defines the line.

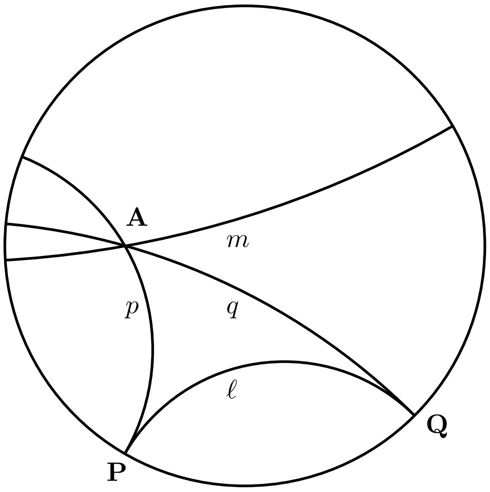


Figure 1: Lines p and q are parallel to line ℓ, with points at infinity **P** *and* **Q**. *h-line m is non-secant with ℓ.*

Let us remind that there are infinitely many tilings of $I\!H^2$ that are obtained from a regular polygon by a recursive reflection of the pentagon and its images in its sides and in the sides of the images. Such a tiling is said to be obtained by *tessellation* from the considered regular polygon. The infinite number of tessellations of $I\!H^2$ is a corollary of the following result:

Poincaré's Theorem, ([Poincaré 1882]) − *Any triangle with angles π/ℓ, π/m, π/n such that*

$$\frac{1}{\ell} + \frac{1}{m} + \frac{1}{n} < 1$$

generates a unique tiling by tessellation.

The theorem was first proved by Henri Poincaré, [Poincaré 1882], and other proofs were given later, for example in [Caratheodory 54].

In hyperbolic geometry, triangles are completely defined by their angles, up to isometries. Taking the triangle of $I\!H^2$ with angles $\frac{\pi}{5}$, $\frac{\pi}{2}$ and $\frac{\pi}{4}$, we obtain the tiling of $I\!H^2$ by the regular pentagon with right angles. We call that tiling the *pentagrid*.

In [Margenstern and Morita 01], we give another proof of the existence and uniqueness of the pentagrid which gives rise to a feasible algorithm in order to locate the tiles. We improved such an algorithm in [Margenstern 00] by constructing a new one, based on another principle. For this paper, we mainly need the tools of [Margenstern 00].

That proof is based on a bijection which is established between the tiling of a quarter of the hyperbolic plane with a special infinite tree: the *Fibonacci tree*. Here, we remind the main lines of that elementary proof, see also [Margenstern and Morita 01] and [Margenstern 00] for more details.

We split the unit disk into four quarters by taking two perpendicular diameters, one of which will be the *vertical* diameter and the other will be the *horizontal* diameter. We take one of the quarters defined by this way that we call the *south-western* quarter, denoted by $\mathcal{Q}$.

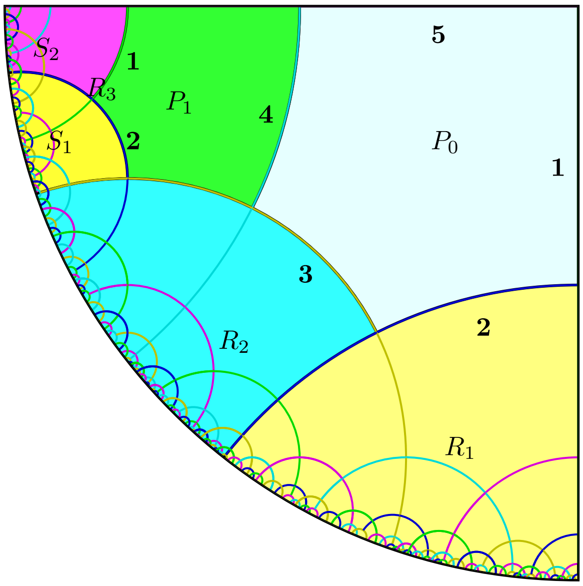


Figure 2: Splitting the quarter into four parts and then the fourth part into three sub-parts

We restrict ourselves to that quarter, and Figure 2, above, shows how the pentagrid looks for a quarter of $I\!H^2$ and how we can recursively split that quarter in order to obtain the tiling.

On that figure, P_0 is the pentagon of $\mathcal{Q}$ for which the centre of the unit disk is a vertex. We call P_0 the *leading* pentagon of the quarter. We number its sides clockwise by **1**, **2**, **3**, **4** and **5** as indicated on the figure. Lines **2** and **3** delimit two regions, R_1 and R_2 on the figure, that are in $\mathcal{Q}\backslash P_0$ and that are isometric to $\mathcal{Q}$. Indeed, the shift along **1** that transforms **5** into **2** transforms $\mathcal{Q}$ into R_1. Similarly, the shift along **4** that transforms **5** into **3** transforms $\mathcal{Q}$ into R_2. We define R_3 as the complement in $\mathcal{Q}$ of $P_0 \cup R_1 \cup R_2$.

By the indicated shifts, we can repeat the above splitting in R_1 and R_2 and the reflection of P_0 in **2** and in **3** are, respectively, the leading pentagons of R_1 and R_2.

Denote by P_1 the reflection of P_0 through **4**, and number the sides of P_1 also by **1**, **2**, **3**, **4** and **5** in such a way that the lines in common with P_0 share the same number. Say that P_1 is the leading pentagon of R_3.

Region R_3, is not isomorphic to a quarter, it is a *strip*. It can be split to its turn into three parts: P_1, the image of P_0 by the reflection through **4**, then S_1, delimited by **3** and **2** of P_1. S_1 is isometric to a quarter by the shift along **1** of P_1 that transforms **5** into **2** of P_1 and its leading pentagon is the reflection of P_1 in **2**. Denote by S_2 the complement in R_3 of $P_1 \cup S_1$. Now, S_2 is also a strip. It is the isometric image of R_3 under the shift along **5** that transforms **4** into **1** of P_1. The leading pentagon of S_2 is the reflection of P_1 in **1**.

Now we can see that the indicated shifts allow us to repeat the same splitting in R_1, R_2 and S_2.

This recursive process inductively defines a tree: its root is attached to P_0 and it has three sons that are respectively attached to the leading pentagons of R_3, R_2 and R_1, from left to right. We shall say that the root is a 3-node because it has three sons. We immediately see that the sons of the root that are attached to R_2 and R_1 are also 3-nodes. We see also that the node attached to R_3 has two sons, that are attached to the leading pentagons of S_2 and S_1, from left to right. We shall say that such a node is a 2-node. In [Margenstern and Morita 01], we proved that the number of nodes of the tree that are on the n^{th} level is f_{2n+1}, where $\{f_n\}_{n\in\mathbb{N}}$ is the Fibonacci sequence. This is why we call such a tree the *Fibonacci tree*. We proved in [Margenstern and Morita 01] that the tree is in bijection with the tiling of $\mathcal{Q}$.

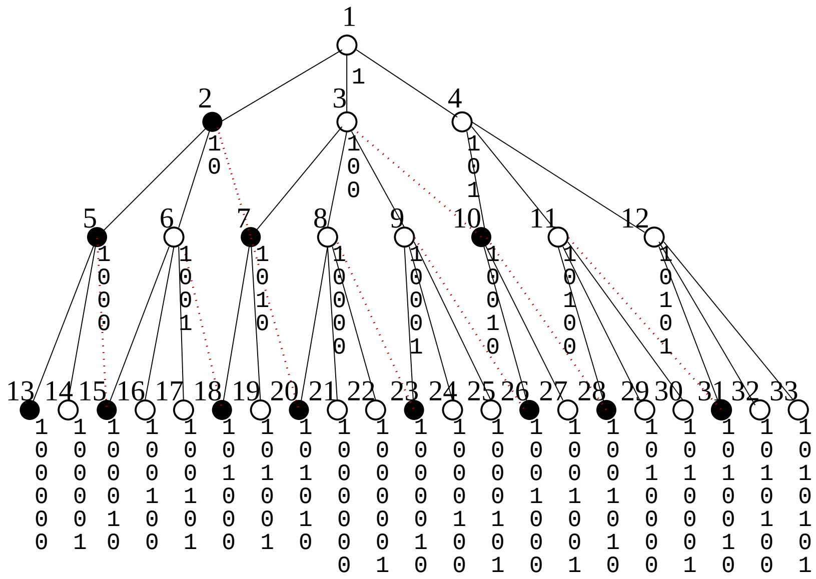


Figure 3: The standard Fibonacci tree and the standard Fibonacci representation of the node numbering.
In dotted red lines: the additional arcs to restore the dual graph of the pentagrid restricted to $\mathcal{Q}$.

The distribution of 2-nodes has nice properties that are studied in [Margenstern 00]. It is important for our purpose, to notice that the Fibonacci tree is a subgraph of the dual graph associated to the pentagrid. It is easy to recover the whole dual graph from the Fibonacci tree. This is obtained by adding 1 arc to 3-nodes and 2 arcs to 2-nodes. Figures 2 and 3 allows us to find the needed arcs, and there is an easy rule to determine that, see [Margenstern 00]: we always add such an arc from a node ν to the leftmost son of the next node that is on the same level as ν. For a node on the right border, the arc goes to the first node of the same level as ν in the other tree. The just defined *additional* arcs will be represented by dotted lines in the figures, namely in Figure 3. We call *bridge node* of node ν the node to which the additional arc from ν leads.

Now, remind that in [Margenstern 00], we numbered the nodes of the Fibonacci tree starting from 1 with the root and going on level after level, from left to right on a same level. We noticed there that if we associate to each number its maximal representation in the numbering basis defined by the Fibonacci sequence, many properties of the nodes can be derived from that representation by linear algorithms. In particular, if we know the number of a node, we can obtain the number of its five neighbours in that way, see [Margenstern 00].

Below, Figure 4 gives another view of the Fibonacci tree.

The Fibonacci tree can be defined in a formal way by the following two rules:

- a 3-node has three sons: on the left a 2-node and, in the middle and on the right, in both cases, 3-nodes;
- a 2-node has 2 sons: the left one is again a 2-node and the right one a 3-node;

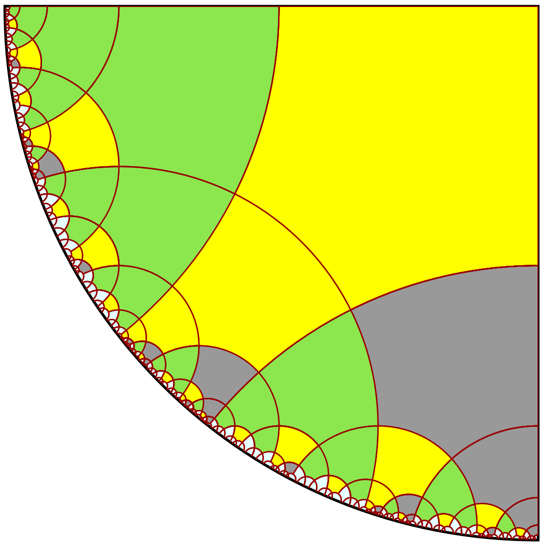


Figure 4: Another view of the standard Fibonacci tree: 2*-nodes are coloured with green,* 3*-nodes which are the right neighbour of a* 2*-node are coloured with yellow;* 3*-nodes that have a* 3*-node as a left neighbour are coloured with light purple.*

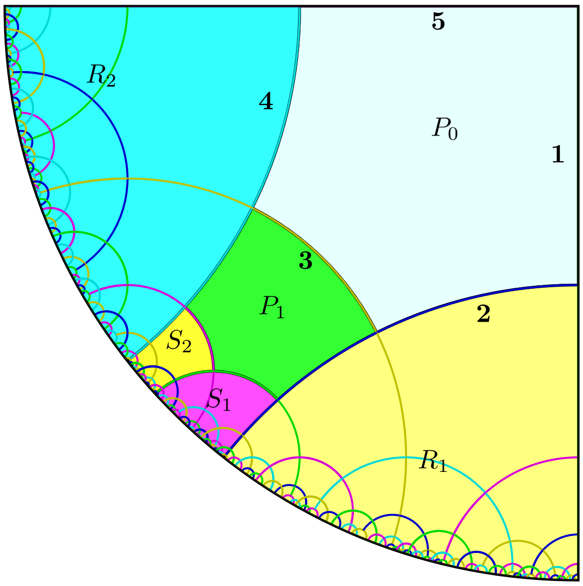


Figure 5: Splitting the quarter into four parts in another way: Region R_3 consists of P_1, S_1 and S_2

We can denote these two rules by **2** → **23** and **3** → **233**. The axiom can be denoted by **3**, which tells us that the root is a 3-node.

The properties of the Fibonacci tree are indicated in [Margenstern and Morita 99] and [Margenstern and Morita 01], and they are thoroughly proved in [Margenstern 99]. We shall not remind all of them here, where our attention is focused on the *location* of the elements of the pentagrid.

Let us just mention a property that is thoroughly studied in [Margenstern 00].

Indeed, it is possible to split the quarter $\mathcal{Q}$ in another way, as this is illustrated by Figure 5, above. This defines another rule, namely **3** → **323**.

This can be easily generalised: there are two possible rules for 2-nodes, namely **2** → **23** and **2** → **32**; there are three possible ones for 3-nodes, namely the rules: **3** → **233**, **3** → **323** and **3** → **332**. Accordingly, it is easy to conclude that there is a continuous family of trees that can be generated by a random construction of the tree: for each node, chose at random the appropriate rule to be applied to that kind of node. Paper [Margenstern 00] establishes several properties in order to compare the trees that are obtained in that way. It appears that the nice properties of the standard Fibonacci tree are not all true for all trees that are generated in that way. However, a continuous sub-family of those trees do possess some of these properties. See [Margenstern 00] for more details.

3 Basic feature of the pentagonal tiles

In the case of tilings of the euclidean plane, a unit being defined for right angle coordinates, *à la* Wang tiles are simply defined as squares with sides parallel to the coordinate axes and with length equal to the unit, the sides of the square being endowed with a *colouring*. In that new setting, *tilings* are first tilings according to the previous definition. They must observe another condition, called the *matching* condition: the colours that are attached to the sides of each tile must be the same on sides that are shared by neighbouring tiles.

A lot of problems are studied in that case, depending on the set of *initial* tiles that are at our disposal and on the way it is allowed to replicate tiles from the initial set.

We shall consider here the simplest case of analogous tilings in the *hyperbolic* plane. This means that the analogue of the square grid will be the pentagrid.

Here, the set of initial tiles consists of a *single* pentagonal tile. Colours are attached to its sides and for the tiling, the matching condition is defined as above. We insist on the constraint that is set on the replication of the original tile: copies must be obtained from the initial tile only by displacements along the lines that are defined by the sides of the pentagons that belong to the pentagrid.

In order to state the result, we need to establish the following basic property:

Rotation lemma – *Any rotation of the initial tile can be obtained by a finite number of shifts along the sides of the pentagons of the pentagrid.*

Proof. Consider figure 6. Denote by a the shift along a_3a_2 that transforms the pentagon numbered 1 into the one numbered 4. Similarly, denote by b the shift along b_4b_3 that transforms pentagon 4 into pentagon 10. Then $aba^{-1}b^{-1}$ transforms 1 into the rotated image of 1 by $\dfrac{2\pi}{5}$: see the successive images of sides a_0 and b_0 around A. The generalization is easy. □

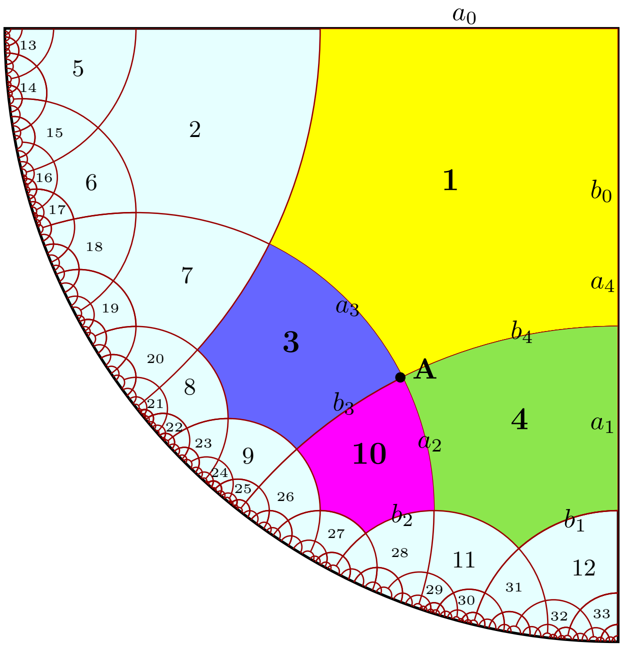


Figure 6: Rotating a cell by displacements

4 The result and its proof

We can now consider the assortments of colours, up to rotation. In the statement of our main result, we use the following format to describe an assortment of colours: $\boxed{a_0\ a_1\ a_2\ a_3\ a_4}$, where $a_i \in \{1, 2, 3, 4, 5\}$. This corresponds to a numbering of the sides, say clockwise, and it is defined up to any rotation of $\dfrac{2\pi}{5}$ thanks to Rotation lemma.

This allows us to define the notion of *assortment* of colours: it lists the colours up to circular permutations and up to renamings of the colours.

It is easy to see that there is a single assortment for 5 colours, namely $\boxed{1\ 2\ 3\ 4\ 5}$. With four colours, a single colour occurs twice, call it 1. Hence, there are two possible assignments, depending on whether the occurrences of that colour are contiguous or not: $\boxed{1\ 1\ 2\ 3\ 4}$ and $\boxed{1\ 2\ 1\ 3\ 4}$. With three colours, we obtain at least 3 assortments by replacing successively the non-1 colours of the 4-assortments by colour 1: $\boxed{1\ 1\ 1\ 3\ 4}$, $\boxed{1\ 1\ 2\ 1\ 4}$ and $\boxed{1\ 1\ 2\ 3\ 1}$. However, that last assortment is obtained from the first one by a circular permutation and by renaming colours. In the case of three colours, two colours may also occur twice, let us call them 1 and 2. This gives rise to new cases: $\boxed{1\ 1\ 2\ 2\ 3}$, $\boxed{1\ 1\ 2\ 3\ 2}$, $\boxed{1\ 2\ 3\ 1\ 2}$ and $\boxed{1\ 2\ 3\ 1\ 3}$. With two colours, we may again replace one non-1 colour by colour 1 in the 3-assortments. It can easily be checked that by renaming and circular permutations, this leads to the following three cases: $\boxed{1\ 1\ 1\ 1\ 2}$, $\boxed{1\ 1\ 1\ 2\ 2}$ and $\boxed{1\ 1\ 2\ 1\ 2}$. It can be checked that there are no other cases.

We may now state the result which consists in classifying all assortments according to the number of possible tilings that exist for the assortment under consideration. It appears that we have either no solution, or finitely many ones (1, 2 or 4) or continuously many ones:

Theorem – *The number of possible tilings with a single pentagonal tile on the pentagrid with the assortments of j colours, $1 \leq j \leq 5$ and satisfying the condition of displacements only along the lines of the pentagrid are given by the following table* 1*:*

5	1 2 3 4 5	no solution
4	1 1 2 3 4	2 solutions
	1 2 1 3 4	no solution
3	1 1 1 2 3	$2^{\aleph_0}$ solutions
	1 1 2 1 3	$2^{\aleph_0}$ solutions
	1 1 2 2 3	4 solutions
	1 1 2 3 2	$2^{\aleph_0}$ solutions
	1 2 3 1 2	no solution
	1 2 3 1 3	no solution
2	1 1 1 1 2	$2^{\aleph_0}$ solutions
	1 1 1 2 2	$2^{\aleph_0}$ solutions
	1 1 2 1 2	$2^{\aleph_0}$ solutions
1	1 1 1 1 1	1 solution

Table 1. Table of the results of theorem 1

The rest of our paper is devoted to sketch the proof of the theorem, and in the next section, we give two applications of the result. For a thoroughly detailed proof, we refer the reader to [Margenstern 01]. We split the presentation of the proof into two parts: the case when there is a finite number of solutions, possibly no solution, and the case when there are infinitely many solutions. In that latter case, it turns out that there is a continuous family of solutions.

4.1 Finitely many solutions

Consider one of the assortments that are listed in the statement of the theorem.

Call *vertex pattern* any list ℓ of four colours among the colours of the assortment such that there is a vertex v of the tiling for which the list of colours of the edges that meet in v taken in clockwise order gives ℓ, possibly after a circular permutation. Formally, vertex patterns are the classes of equivalence induced on these lists by circular permutations.

Now it is easy to check that no vertex pattern can be associated to the assortments for which the theorem states that there is no solution. For the other assortments, vertex patterns do exist.

Consider now that we have an assortment for which vertex patterns exist. Consider a node ν and the set of its five neighbours in the pentagrid, *i.e.* the nodes that can be reached from ν by a reflection in the side shared with ν. Consider the colours of these sides that constitute the edges of the dual graph of the pentagrid. By considering the configuration of the colours for the neighbouring nodes, we can infer *rules* that indicate which colours are possible for the sons and the bridge node of the node if its own colours are given. We call *colouring* such a rule. The rules will be deduced by an argument of induction on the structure of the tree and by a *confluent* principle: if the tiling exists, then applying the rules to two neighbouring nodes ν_1 and ν_2 must induce the same colour on the bridge node of ν_1 which, at the same time, is a son of ν_2. It will then be possible to combine these rules into a procedure that we call the *continuation algorithm.*

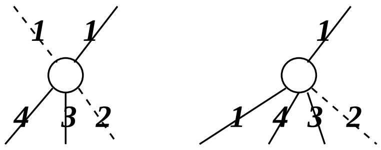
The general characteristic of the finite solution cases is that the compatibility tables define either a single set of rules, or two sets that are incompatible: if we start with a rule that belongs to one set of rules, we must go on with rules of the same set.

We shall first proceed with the four-colour assortment from which the next cases will proceed.

Assortment $\boxed{1\ 1\ 2\ 3\ 4}$

First, let us fix a notation for representing the assortments that we shall use later in order to formulate the colourings. Below, Figure 7 represents the *pattern colour* of a node that is defined by the colouring. We give also a schematic representation of that pattern in the right part of the figure.

Next, Figure 8 displays all the possible colour patterns that are associated with the assortment $\boxed{1\ 1\ 2\ 3\ 4}$. There are ten of them.



a	a'	a	a'
		1 1 4 3 2	1 1 4 3 2

Figure 7: Representation of the colour patterns

1 1 4 3 2	4 1 3 2 1	3 4 2 1 1	2 3 1 1 4	1 2 1 4 3
a	b	c	d	e
1 1 4 3 2	1 4 3 2 1	4 3 2 1 1	3 2 1 1 4	2 1 1 4 3
a'	b'	c'	d'	e'

Figure 8: The ten patterns defined by the assortment 1 1 2 3 4

Now, it is possible to define the vertex patterns that are associated with the considered assortment of colours. There are basically two cases: either all colours are the same and colour 1 is the single possibility; or all colours are different, and they have a fix order: 1, 2, 3, 4 if we display them anti-clockwise, see Figure 9, below. Indeed, it is easy to check that the second case is necessary if a colour different from 1 is present on one of the edges that meet on the vertex.

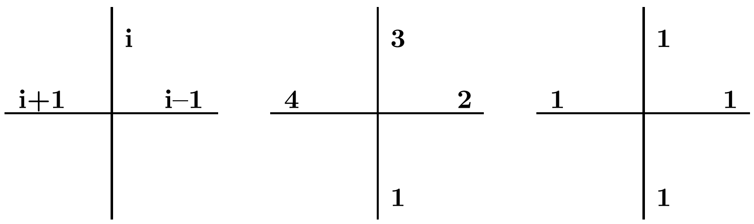


Figure 9: The vertex patterns defined by the assortment 1 1 2 3 4

In order to illustrate the construction of the rules, consider Figure 10 which displays the configuration of a node and its sons. The figure displays the two sub-cases: on left, when the node is a 2-node, on right, when it is a 3-node.

We get the following rules:

$$a \rightarrow cd' + d \qquad a' \rightarrow ac'd' + d$$

where the right argument of + is the pattern of the bridge node.

The other rules can be obtained by a similar argument, starting from the considered colour pattern of the node. We find a unique set of rules:

$$\begin{array}{ll}
a \to cd' + d & a' \to ac'd' + d \\
b \to de' + a & b' \to cd'e' + a \\
c \to eb' + e & c' \to de'b' + e \\
d \to ba' + b & d' \to eb'a' + b \\
e \to ac' + c & e' \to ba'c' + c
\end{array}$$

Notice that we may only display the rules that are associated to a node x'. Indeed, if the rule for x' is $x' \to uy'z' + v$, then the rule for x is $x \to yz' + v$. We shall later refer to this property when we shall invoke the *reduction principle*. Indeed, such a principle can easily be derived from the structure of the tree. It should be stressed that the specific configuration of the *standard* Fibonacci tree is used for that latter property.

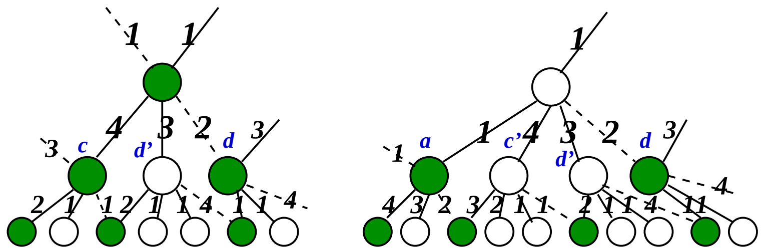


Figure 10: Determining the rules associated to patterns a *and* a'

The rules have to be applied from a starting point: the root of the tree.

Again, applying an argument that is similar to what we did with figure 10, we obtain that there are five possible starting rules that we call *axioms*:

$$a \to cd'e' \qquad b \to de'b' \qquad c \to eb'a' \qquad d \to ba'c' \qquad e \to ac'd'$$

Notice that here, we wrote x for the root instead of x'. Indeed, the root is a 3-node, but its connections look like those of a 2-node: there are two dotted arcs. However, as the root is a 3-node, the right part of the rule indicates three colours.

Now, in order to tile the plane, we have to see how we can *glue* four quarters when each one has been tiled according to the above rules and according to the above axioms.

As we have a single set of rules, it is easy to see, by induction on the structure of the tree, that once the rule of the root is fixed, the tiling of the quarter is uniquely determined.

Consider now four trees, as it is displayed in Figure 11.

It can be checked with no difficulty that the colouring of the nodes on the border branches of the tree is periodic. The periods are 2 on one border and 3 on the other one. And so, to establish that the borders of two trees are compatible, it is enough to check the compatibility on two or three periods, starting from the root, as far as the sequence along the borders has no aperiodic beginning.

As it is indicated in [Margenstern 01], there are two possible associations of the colours of the roots: either all roots are coloured with a, or they have all different colours, namely b, c, d, e if they are listed clockwise. This gives rise to the two solutions that are indicated by the theorem.

Assortment $\boxed{1\ 1\ 2\ 2\ 3}$

First, we notice that if we replace colour 3 by colour 2 in the 4-assortments and then we rename the colours, we get the assortment $\boxed{1\ 1\ 2\ 2\ 3}$.

And so, the two solutions that were found for the 4-assortment case provide us with two solutions for the present case by applying the indicated transformation on the colour patterns. Accordingly, the same rules and axioms can be applied. This gives us the *standard* solutions.

However, if we repeat the above analysis, we find that, already from the study of the vertex patterns, there can be other solutions.

The reader may find without difficulty that the vertex patterns are now as displayed by Figure 12, below.

As there are more patterns here than in the previous case with four colours, the patterns are not enough to fix the colourings immediately. We have to refine the technique of the bridge node that we already used by applying it not only to the sons of the considered node, but also to its grandsons. Below, Figure 13 is an illustrative example of this technique applied to the rules for a', which is the most complex case for assortment $\boxed{1\ 1\ 2\ 2\ 3}$.

From Figure 13, it can be checked that two rules can be derived: $a'_1 \to ac'd'+d$ and $a'_2 \to bc'e'+c$. This depends on the pattern that is used for the node that is denoted by a': two possibilities are left by the colouring of the node. Once one pattern is fixed, the whole configuration is fixed, going from left to right by the bridge nodes of the grandsons and applying the continuation algorithm. Using the same technique, the reader can check that we obtain the following rules:

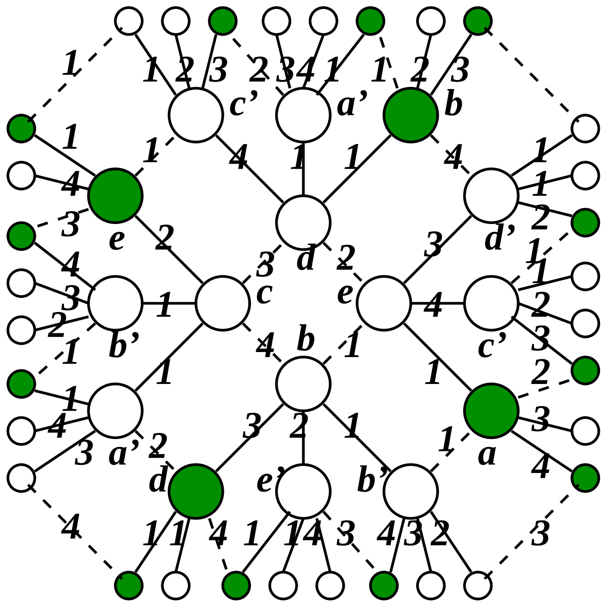


Figure 11: Continuation of the colouring, starting from the south-west quarter

$a'_1 \to ac'd' + d$	$a'_2 \to bc'e' + c$
$b'_1 \to cd'e' + a$	$b'_2 \to ce'd' + e$
$c'_1 \to de'b' + e$	$c'_2 \to ed'a' + a$
$d'_1 \to eb'a' + b$	$d'_2 \to da'b' + b$
$e'_1 \to ba'c' + c$	$e'_2 \to ab'c' + d$

The question is, if we consider a rule of one set, do the other rules that occur on the right side of the arrow belong to the same set of rules or to the other one, and possibly to which extent?

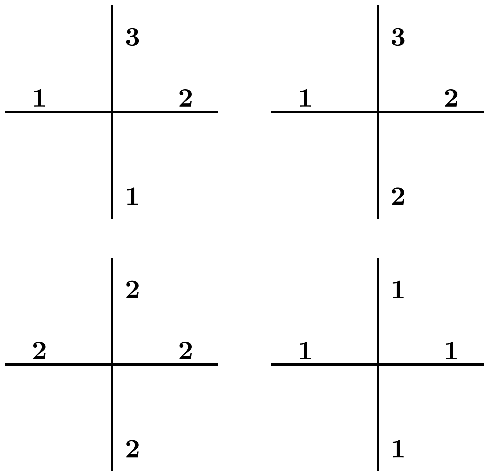


Figure 12: The vertex patterns defined by the assortment 1 1 2 2 3

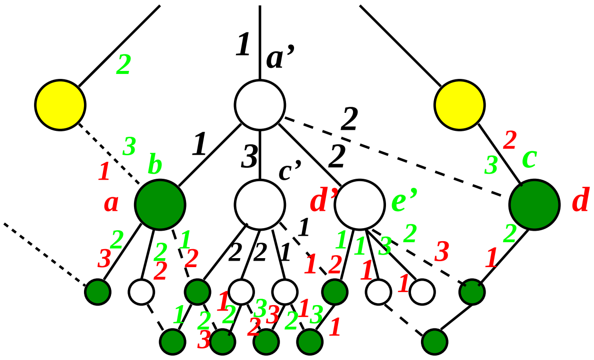


Figure 13: Defining two possible rules for colouring a'

The answer to the question is simple in the present case. In all rules indexed with 2, the rules that occur in the right part of the rules must also belong to the same set of rules. This is fixed by the bridge son between y' and z' in the rules $x' \to uy'z' + v$. In the set of rules indexed with 1, the situation is the same, except for one rule: the rule for b' accepts any rule in a for the colouring of the right neighbour. But, as such a node is the leftmost son of another node, its father imposes a rule where the leftmost son must have colouring a. Now, in the set of rules indexed with 1, there is a single rule with that property, namely the rule for a'_1. It can easily be checked that the rule needs a_1 in its right part. And so, the above sets of rules are separated.

4.2 Continuously many solutions

In the other cases, we have also at least three vertex patterns. There is also an alternative set of rules to the standard one but, now, the two sets can be mixed.

Consider, for instance the case of the colouring $\boxed{1\ 1\ 1\ 2\ 3}$. Besides the basic set of rules, that we obtain by identifying 4 with 1 and then by renaming, it is possible to determine another one: it is enough to apply the technique that was indicated for the assortment $\boxed{1\ 1\ 2\ 2\ 3}$. Precisely, we obtain the following two sets of rules:

$$
\begin{array}{ll}
a'_1 \rightarrow ac'_1d'_\bullet + d_\bullet & \qquad a'_2 \rightarrow c_2b'_2d'_\bullet + d_\bullet \\
b'_1 \rightarrow c_1d'_\bullet e'_\bullet + a_1 & \qquad b'_2 \rightarrow b_2d'_\bullet e'_\bullet + e_2 \\
c'_1 \rightarrow d_\bullet e'_\bullet b'_1 + e_1 & \qquad c'_2 \rightarrow d_\bullet e'_\bullet a'_2 + b \\
d'_1 \rightarrow eb'_1a'_1 + b & \qquad d'_2 \rightarrow ea'_2c'_2 + a_2 \\
e'_1 \rightarrow b_\bullet a'_\bullet c'_1 + c & \qquad e'_2 \rightarrow ac'_2b'_2 + c
\end{array}
$$

If we apply rules of one of the above sets only, this gives rise to a tiling. This time, it is possible to start with some rules of one set and to go on with rules of the other set. The above display of the rules shows to which extent this is possible: a letter indexed by the symbol $\bullet$ tells us that for this letter we can later apply the rule indexed with 1 or 2 indifferently. There is no incompatibility at this level, but we have to check whether this will go on or not.

It is indeed possible to mix the above set of rules up to a certain extent. Below, Figure 14 illustrates a technique which indicates how to proceed. The idea is to introduce a *perturbation* in the tree as follows: we assume that we have a tiling with the second set of rules. Starting from a node b'_2, we apply the rules 1 for its sons d' and e' instead of the rules 2. From this new tree, rooted in b'_2, we apply the rules of set 1 and we check that, on the borders, *i.e.* for the leftmost sons on the left border and for the bridge nodes on the right one, the colouring induced by the rule 1 is compatible with the colouring induced by the rule 2 applied to the father node which is in the untouched rule 2 part of the tiling. As we noticed in the case of gluing trees in section 4.1., we notice that the succession of nodes on the border is periodic. Accordingly, it is enough to check the compatibility of the colouring for a period. Figure 14 illustrates this point also.

Next, we follow a trick that is used in [Margenstern 00] as far as the perturbation remains inside a subtree $\mathcal{T}$ of the Fibonacci tree. Now, we can find other sites, outside $\mathcal{T}$, from which the same perturbation can be introduced. If we can manage a sequence of subtrees $\mathcal{T}_n$ that are pairwise not intersecting, we can decide for each root of $\mathcal{T}_n$ whether we introduce or not the perturbation. Below, Figure 15 shows how this can be managed. Accordingly, we obtain continuously many different tilings.

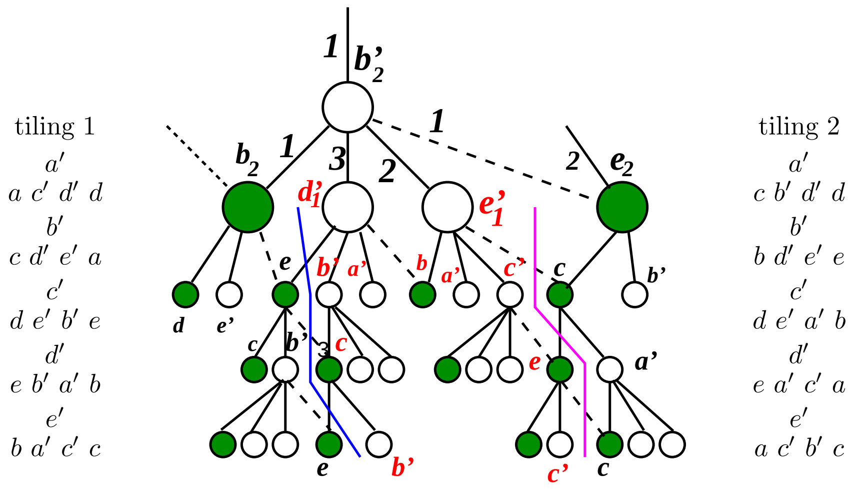


Figure 14: Inserting a zone of tiling 1 into tiling 2. Notice the use of the reduction principle.

As it is checked in detail in [Margenstern 01], the same technique can be performed in the remaining assortments. And so, as it is stated in the theorem, in each one of these cases, we obtain a continuous family of different tilings. □

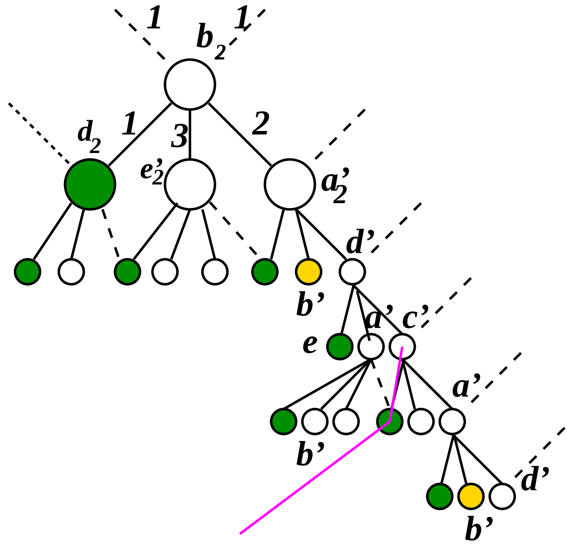


Figure 15: Separating two consecutive subtrees

Notice that in the case of the assortment $\boxed{1\ 1\ 1\ 1\ 2}$, we can prove the same result with a very different technique. In that case, we can use the argument with which we proved that there are continuously many Fibonacci trees, each one corresponding to a way to split a quarter and to split a strip.

Consider a fixed tree that is constituted by a choice at random of the rule to be applied. For the root, colour with 2 the arc that leads to the 2-node. Next, for any other node ν: if the arc to the father is coloured with 1, colour with 2 the arc to the 2-node son of ν. It can be checked by induction on the level and, on each level, by induction from left to right that the application of that rule gives a tiling. See an illustration on Figure 16 with the standard Fibonacci tree and on Figure 17 with a Fibonacci tree associated to a random choice.

To complete this proof, we can consider a sub-family of the just described family, provided that the sub-family has still a continuous cardinality. The restriction is the following: for 2-nodes, we decide that the leftmost son is always a 2-node and for 3-nodes, we decide that the rightmost son is always a 3-node. The random choice is restricted to the choice between the leftmost son and the middle one: this is enough to obtain continuously many trees. It is also plain that different trees induce different tilings. These tilings have an additional property: for any vertex, two contiguous side cannot be coloured with 2. This is the case in Figure 17.

5 Two applications

We give two applications of the above technique: one for a property of some Caley graphs and another one for undecidable problems.

5.1 An application to Cayley graphs

It is well known in the folklore, that the pentagrid is not the Cayley graph of a subgroup of the group of isometries of the hyperbolic plane.

However, the pentagrid *is* the Cayley graph of some abstract group.

In [Margenstern 01], we prove these two facts in full details. Here, we only give a sketchy proof.

The pentagrid is not the Cayley graph of a subgroup of the isometries of $\mathbb{H}^2$**:**

Assuming that the pentagrid is the Cayley graph of a subgroup of the isometries of $\mathbb{H}^2$, we consider all the possibilities for the set of generators that meet in a vertex, say g_1, g_2, g_3, g_4, all distinct from 1. We next consider a product of these generators that label the sides of a pentagon, say $g_{i_1}, g_{i_2}, g_{i_3}, gi_4, gi_5$. We have $g_{i_1}.g_{i_2}.g_{i_3}.gi_4.gi_5 = 1$, from which we conclude that at least one g_i is a product of two reflections, and it is not difficult to prove that it must the shift along the side of the pentagon that it labels, see [Margenstern 01]. Next, the technique of the Rotation lemma shows that all g_i's cannot be shifts. From that, as we have a product that gives 1, there are exactly two shifts and two odd products of reflections as generators. Odd products of reflections are either single reflections or products of three reflections, which are called *glides*. Now there are six cases to be examined, depending on whether these odd products are reflections or glides, and depending on the relative position of the shifts. In all cases, the product of five elements that label the sides of a pentagon cannot be 1.

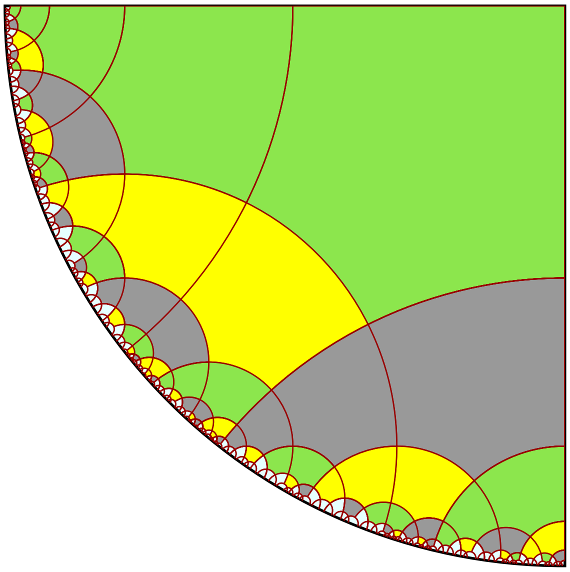


Figure 16: An illustrative representation of colouring $\boxed{1\ 1\ 1\ 1\ 2}$ *that makes use of the standard Fibonacci tree.*

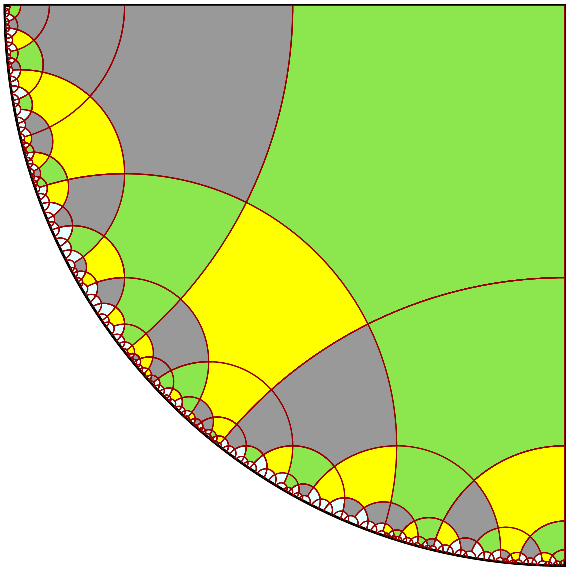


Figure 17: Another illustrative representation of colouring $\boxed{1\ 1\ 1\ 1\ 2}$*. The colouring is based on the same principle as in Figure 16, but here, we do not consider the standard Fibonacci tree: 2-nodes are chosen randomly.*

The pentagrid is the Cayley graph of some abstract group:

First, the existence of the group can be reduced to the existence of a tiling. Indeed, if we impose the condition $g^2 = 1$ on the generators, the Cayley graph of the group becomes undirected. The construction of the group boils down to define a tiling with four colours such that for any vertex, the vertex pattern is always a list of the four colours.

We fix the vertex pattern associated to the roots of four Fibonacci trees. Then, level after level, we start from the leftmost node of the south-western tree and then, using a simple combinatorial argument, we show how to fix the patterns, vertex after vertex, going from one tree to the next one, until we find again the starting node.

Below, Figure 18 illustrates the method in the case of a 2-node and in the case of a 3-node. In both cases, a, b, c, d is fixed by the induction hypothesis: it is the just fixed pattern. Also, e and f are fixed as elements of patterns fixed at the previous level. We have to fix x or x_1, x_2, and y, z that concern the bridge node of $\mathbf{k{+}1}$. We always may choose $y \neq b$, which fixes z, and then x or $x2$ can be set to b so that we can fix $x_1 = a$. The case of the leftmost node, of crossing the border of a tree and the return to the leftmost node are fixed in a similar way. This is done in [Margenstern 01].

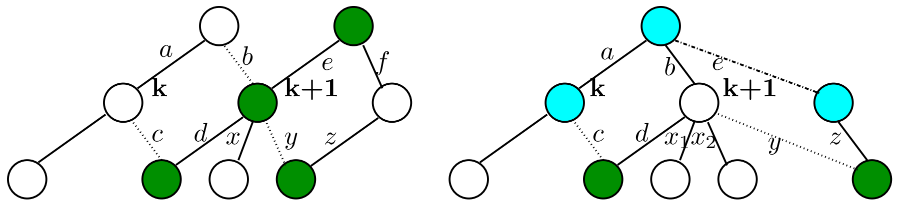


Figure 18*: Fixing the vertex patterns: on the left, the case of a 2-node, on the right, the case of a 3-node*

And so, in that second part, we constructed an explicit representation of the abstract group by applying the technique that we developed in this paper.

5.2 An application to undecidable problems

As we have in many cases a continuous family of solutions, it is plain that most of those tilings are neither recursive nor recursively enumerable. Consequently, it is possible to show undecidable problems for these tilings.

In order to illustrate that point, consider the case of colouring $\boxed{1\ 1\ 1\ 2\ 3}$ that is illustrated by Figure 14. Now, consider the following problem: is a tiling of the family identical to tiling 2, defined in Figure 14? This problem is undecidable. Indeed, consider the sub-trees $\mathcal{T}_n$ that are associated to the mixing process in section 4.2 for the indicated assortment. Put $t(n) = 0$ if we go on to apply tiling 2 on $\mathcal{T}_n$ and put $t(n) = 1$ if we apply tiling 1. Our problem is equivalent to the following one on the sequence $\{t(n)\}_{n \in I\!N}$ of 0's and 1's: are all the terms of the sequence equal to 0? This problem is known to be undecidable, for instance apply the argument of Rice theorem, see [Oddifredi 89].

6 Conclusion

It is trivial that in the euclidean case, with the same constraints of displacements along the lines of the grid only, the situation is drastically different and much more trivial. We leave that point to the reader.

A large set of problems remain open that continues this line of researches.

A first direction concerns tilings: what can be said for two initial tiles? We don't know. What can be said for grids based on other regular polygons? We only know that the same tree-technique works for regular rectangular n-gons, see [Margenstern and Skordev 00], and that there should be at most four colours, because of the constraints on the vertex patterns.

A second direction concerns undecidable problems. We gave an easy example on that line. There are much more difficult problems. An example is the general tiling problem. It is known to be undecidable for the euclidean plane, as it was proved by Berger, see [Berger 66]. Nothing is known for that problem in the hyperbolic plane. So far, the single known result for $I\!H^2$ is Robinson's theorem on the undecidability of the constrained tiling problem, see [Robinson 78].

Accordingly, it seems to us that the tools that we used to solve the problem raised in this paper could be of some interest to solve other problems in the hyperbolic plane, on both suggested directions.

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