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Synchronization and Stability of Finite Automata[1]

Jarkko Kari
(Department of Mathematics and Turku Centere for Computer Science, University of Turku, FIN-20014 Turku, Finland[2]
Email: jkari@utu.fi)

Abstract: Let $G = (V, E)$ be a strongly connected and aperiodic directed graph of uniform out-degree k. A deterministic finite automaton is obtained if the edges are colored with k colors in such a way that each vertex has one edge of each color leaving it. The automaton is called synchronized if there exists an input word that maps all vertices into the same fixed vertex. The road coloring conjecture asks whether there always exists a coloring such that the resulting automaton is synchronized. The conjecture has been proved for various types of graphs but the general problem remains open. In this work we investigate a related concept of stability, using techniques of linear algebra. We have proved in our earlier papers that the road coloring conjecture is equivalent to the conjecture that each strongly connected and aperiodic graph has a coloring where at least one pair of states is stable. In the present work we prove that stable pairs of states exist in all automata that are almost balanced in the sense that there is at most one state for each color where synchronization can take place.

Key Words: Finite automata, synchronization

Category: F.1.1

1 Introduction

The road-coloring problem is a challenging open problem concerning synchronization of finite automata. It was first stated as a problem in symbolic dynamics [1, 2]. The problem is to determine which directed graphs of uniform out-degree admit an edge labeling that makes them into a synchronized deterministic finite automaton (DFA), that is, a DFA in which an input word w and state s exist such that the input w moves the automaton into state s regardless of the start state. It is generally believed that such synchronizing labelings exist for all strongly connected and aperiodic graphs.

Synchronization allows simple error recovery in finite automata: if an error is detected, a synchronizing word can be used to reset the automaton into a known state. This is an old idea used in synchronizing codes to resume decoding after a transmission error. Another application of synchronized automata is leader identification in processor networks. If the network has a synchronized labeling, a synchronizing word takes a message from any processor to the leader vertex. This idea also indicates the origin of the problem name: if a road map has a synchronized coloring then it is impossible to get lost. By following a path labeled with the synchronizing word one always gets back to the original location.

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The road-coloring conjecture has been proved in various special cases. An early partial result states that a synchronizing coloring exists if the graph has a simple cycle of prime length, and there are no multiple edges [8]. Recently we proved the conjecture for graphs that have a uniform in-degree as well as out-degree [7]. Other partial results have been reported in [3, 4, 5, 6, 9].

In [4] we introduced the concept of stability. In a DFA, a pair s, t of states is called stable if for every input word u there exists a word w (which may depend on u) such that word uw synchronizes states s and t. In other words, states s and t are synchronizable, and they remain synchronizable no matter what input word u is applied to both of them. Clearly all pairs are stable if the automaton is synchronized, so the existence of stable pairs is a weaker property than synchronization. We proved in [4] that stability is an equivalence relation, and that the road-coloring conjecture is equivalent to the conjecture that labelings with non-trivial stable pairs exist.

In the present work we develop the idea of stability further. Our techniques are elementary and are based on linear algebra. We interpret the states of the automaton as basis vectors of a vector space. Transition functions are viewed as linear transformations. We show how the stability relation corresponds to a subspace, closed under the transition transformations. We show that if at least one pair of states can be synchronized then this subspace is non-trivial, i.e., contains some non-zero vectors. Unfortunately this does not always imply that a stable pair of states exist. However this is the case if the automaton has for each input letter a at most one state with more than one incoming edge with label a. As a corollary we see that a graph has a labeling with a stable pair if only only one state has an in-degree greater than the uniform out-degree of the graph.

2 Definitions

A directed graph $G = (V, E)$ is called admissible (k-admissible, to be precise) if all vertices have the same out-degree k. A deterministic finite automaton (DFA) without initial and final states is obtained if we color the edges of a k-admissible digraph with k colors in such a way that all k edges leaving any node have distinct colors. The vertices of the graph become the states of the automaton. We call G the underlying graph of the automaton.

Let $\Sigma = \{1, 2, \ldots, k\}$ be the labeling alphabet. As usual, Σ^* is the set of words over Σ. Every word $w \in \Sigma^*$ defines a forward state transition function $f_w : V \longrightarrow V$ on the vertex set V: the vertex $f_w(p)$ is the endpoint of the unique path that starts at p and whose labels read w. For a set $S \subseteq V$ we define

$$f_w(S) = \{f_w(p) \mid p \in S\}.$$

Word w is called synchronizing if $f_w(V)$ is a singleton set, and the automaton is called synchronized if a synchronizing word exists. A coloring of an admissible graph is synchronized if the corresponding automaton is synchronized.

The road coloring conjecture states that a synchronized coloring exists for every admissible graph that is (i) strongly connected, and (ii) aperiodic. Aperiodicity means that there is no number $k > 1$ that divides the lengths of all cycles in the graph. Clearly, if there is such a common divisor $k > 1$ then no synchronizing coloring can exist. Throughout this work we assume that all given graphs are strongly connected and aperiodic.

Next we define two binary relations on the state set of a DFA [4]:

Definition 1 *A pair p, q of states is* reducible*, in symbols $p \sim q$, if there exists a word w such that $f_w(p) = f_w(q)$, i.e. word w takes p and q to the same state. Accordingly the DFA is called* (p, q)-synchronized*. We say that a state pair p, q is* stable*, denoted $p \equiv q$, if for every word u there exists a word w such that $f_{uw}(p) = f_{uw}(q)$.*

It follows immediately from the definition that the reducibility and the stability relations $\sim$ and $\equiv$ are symmetric and reflexive. The reducibility relation is not always transitive. In contrast, the stability relation is transitive, and hence an equivalence relation [4]. Moreover, $p \equiv q$ implies $f_w(p) \equiv f_w(q)$ for all $w \in \Sigma^*$, so that stability is a congruence of the automaton. Using this fact we proved in [4] that the road coloring conjecture is equivalent to the following conjecture:

Conjecture 1 *Every admissible, aperiodic, strongly connected graph has a coloring such that a pair p, q of states is stable for some $p \neq q$.*

Therefore we concentrate on the seemingly easier task of looking for colorings with non-trivial stability relations.

In this work we view the state transitions f_w as linear transformations. Let $n = |V|$ be the number of states. For each letter $a \in \Sigma$, the state transition function f_a has a representation as the adjacency matrix of the edges with color a, that is, the binary $n \times n$ matrix M_a whose entry $M_a(p, q)$ is 1 if $q = f_a(p)$, and 0 otherwise. The matrix defines a linear transformation of an n-dimensional vector space U. We can view each state $p \in V$ as a basis vector $\bar{p} \in U$ of this space. We call this the natural basis of U. Let $\bar{f}_a : U \longrightarrow U$ be the linear transformation of U whose matrix under the natural basis is M_a. In other words, for $p, q \in V$ we have $\bar{f}_a(\bar{p}) = \bar{q}$ in the vector space U if and only if $f_a(p) = q$ in the automaton.

Each subset $S \subseteq V$ of vertices is also viewed as a vector $\bar{S}$: it is the sum of the basis vectors that correspond to its elements:

$$\bar{S} = \sum_{p \in S} \bar{p}.$$

For any word $w = a_1 a_2 \ldots a_m$, the transformation $\bar{f}_w$ is the composition

$$\bar{f}_w = \bar{f}_{a_1} \circ \bar{f}_{a_2} \circ \ldots \circ \bar{f}_{a_m}.$$

We clearly have the property $\bar{f}_w(\bar{p}) = \bar{q}$ if and only if $f_w(p) = q$ in the automaton.

Elements of U are all linear combinations of the basis vectors $\bar{p}$, for $p \in V$. Now we can generalize the concepts of reducibility and stability to arbitrary vectors:

Definition 2 *A vector $\bar{x}$ is* reducible *if there exists a word w such that $\bar{f}_w(\bar{x}) = \bar{0}$, the zero vector. Vector $\bar{x}$ is stable if for every word u there exists a word w such that $\bar{f}_{uw}(\bar{x}) = \bar{0}$.*

Notice that a pair p, q of states is reducible (stable) according to Definition 1 if and only if the vector $\bar{p} - \bar{q}$ is reducible (stable, respectively) according to Definition 2. The fact that the stability relation is an equivalence relation has the following counter part in the linear spaces:

Proposition 1 *The set of stable vectors is a linear subspace.*

Proof. The zero vector is stable, so there always exist stable vectors. If $\bar{x}$ and $\bar{y}$ are stable vectors and a and b are arbitrary real numbers then $a\bar{x} + b\bar{y}$ is stable: For every $u \in \Sigma^*$ there exists word $v \in \Sigma^*$ such that $\bar{f}_{uv}(\bar{x}) = \bar{0}$, and a word $w \in \Sigma^*$ such that $\bar{f}_{uvw}(\bar{y}) = \bar{0}$. Hence

$$\bar{f}_{uvw}(a\bar{x} + b\bar{y}) = a\bar{f}_w(\bar{0}) + b\bar{0} = \bar{0}.$$

□

Let us call the subspace formed by the stable vectors the *stability space* of the automaton. In this work we investigate this space, and prove that it is the null space $\{\bar{0}\}$ if and only if all input letters specify a permutation of the state set. Let us call such DFA a *permutation automata*, that is, an automaton is a permutation automaton iff

$$f_a(p) = f_a(q) \Longrightarrow p = q$$

for every $a \in \Sigma$ and all $p, q \in V$. We have proved in [7] the road coloring conjecture for the underlying graphs of permutation automata, so the present work shows that in all remaining open cases there exist non-trivial stable vectors in every labeling of the graph.

3 The stability space

Let us recall a few definitions from [4]. We call a set $S \subseteq V$ of states *non-reducible* if $|f_w(S)| = |S|$ for every $w \in \Sigma^*$, where $|X|$ is the notation we use for the cardinality of set X. The non-reducible sets S that can be reached from the full state set V form the set

$$V_{\min} = \{f_w(V) \mid \ f_w(V) \text{ is non-reducible } \}.$$

It is easy to see that $V_{\min}$ is not empty: the cardinality of a state set can only be decreased a finite number of times, after which it becomes non-reducible. It is also clear that all elements of $V_{\min}$ have the same cardinality. Indeed, if $A = f_w(V)$ and $B = f_v(V)$ are two elements of $V_{\min}$ then $f_v(A) \subseteq B$. Because A is non-reducible, we have $|B| \geq |A|$. Symmetrically, $|A| \geq |B|$, so the cardinalities of A and B are the same.

Following [4] we call the cardinality of the elements of $V_{\min}$ the *synchronization degree* of the automaton.

Proposition 2 *If $S, T \in V_{\min}$ then $\bar{S} - \bar{T}$ is a stable vector.*

Proof. Let $u \in \Sigma^*$ be arbitrary, and let $w \in \Sigma^*$ be such that $f_w(V) \in V_{\min}$. Let us denote $R = f_w(V)$. We have $f_{uw}(S) = f_{uw}(T) = R$. Because S and T are non-reducible we have $\bar{f}_{uw}(\bar{S}) = \bar{R}$ and $\bar{f}_{uw}(\bar{T}) = \bar{R}$ in the vector space U. Therefore

$$\bar{f}_{uw}(\bar{S} - \bar{T}) = \bar{f}_{uw}(\bar{S}) - \bar{f}_{uw}(\bar{T}) = \bar{R} - \bar{R} = \bar{0}.$$

□

Corollary 1 *The stability space is $\{\bar{0}\}$ if and only if the automaton is a permutation automaton.*

Proof. If the automaton is not a permutation automaton then the synchronization degree of the automaton is less than $|V|$. Since the automaton is strongly connected, every state $p \in V$ belongs to some element of $V_{\min}$. Consequently there must be at least two different sets S and T in $V_{\min}$. According to Proposition 2 the difference $\bar{S} - \bar{T}$ is a non-zero stable vector.

Conversely, if each input letter specifies a permutation of the state set then it is clear that every non-zero vector is mapped into a non-zero vector by the linear transformations $\bar{f}_a$. Therefore there are no other stable vectors except $\bar{0}$. □

Example 1. In the automaton $\mathcal{A}$ of Figure 1

$$V_{\min} = \{\{1,4\},\{2,3\}\}.$$

Therefore vector $\bar{x} = \bar{1} + \bar{4} - \bar{2} - \bar{3}$ is stable. It is easy to see that all stable vectors are scalar multiples of $\bar{x}$, so the stability space has dimension one.

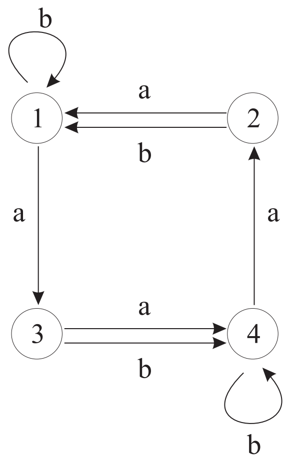

Figure 1: Automaton $\mathcal{A}$.

As an application of Corollary 1 we can prove that an automaton has non-trivial stable pairs of states (as defined in Definition 1) if for each label a there is at most one state $p_a \in V$ where synchronization with input letter a can occur.

Proposition 3 *Let A be a non-permutation automaton with at least two states. Assume that for every input letter $a \in \Sigma$ there exists a state $p_a \in V$ such that*

$$f_a(p) = f_a(q), p \neq q \Longrightarrow f_a(p) = f_a(q) = p_a.$$

(In other words, for every $a \in \Sigma$ there is at most one state with more than one incoming edge of label a.) Then the automaton has a stable pair p, q of states, where $p \neq q$.

Proof. Let $\bar{x} \in U$ be a vector, and let

$$\bar{x} = \sum_{p \in V} a_p \bar{p}$$

be its representation as a linear combination of the basis vectors $\bar{p}$, for $p \in V$. The support of $\bar{x}$ is the set

$$\operatorname{supp}(\bar{x}) = \{p \in V \mid a_p \neq 0\}$$

of states that correspond to non-zero coefficients.

Let $\bar{x} \neq \bar{0}$ be a stable vector whose support $\operatorname{supp}(\bar{x})$ has minimal cardinality among all non-zero stable vectors. According to Corollary 1 such vector $\bar{x}$ exists. Basis vectors $\bar{p}$ are not stable so the support of $\bar{x}$ must contain at least two states.

Let us prove that all $p, q \in \operatorname{supp}(\bar{x})$ form stable pairs. Because $\bar{x}$ is stable it is enough to prove that $f_w(p) = f_w(q)$ with every w such that $\bar{f}_w(\bar{x}) = \bar{0}$.

Let $w \in \Sigma^*$ be any word such that $\bar{f}_w(\bar{x}) = \bar{0}$, and let u be the longest prefix of w such that $\bar{f}_u(\bar{x}) \neq \bar{0}$. Let $\bar{y} = \bar{f}_u(\bar{x})$, and let $a \in \Sigma$ be the letter that follows the prefix u in w. Because $\bar{y}$ is stable its support cannot be smaller than the support of vector $\bar{x}$. Therefore, no states of $\operatorname{supp}(x)$ are synchronized by word u. On the other hand, because $\bar{f}_{ua} = \bar{0}$ each state in $\operatorname{supp}(x)$ is synchronized with another one by word ua. Because there is only one state $p_a \in V$ with more than one incoming edge labeled a, we must have $f_{ua}(p) = p_a$ with all $p \in \operatorname{supp}(x)$. This means that $f_w(p) = f_w(q)$ for all $p, q \in \operatorname{supp}(\bar{x})$. $\square$

Example 2. Consider the automaton $\mathcal{A}$ of Example 1. If we exchange the labels of the edges that go out of state 1, we get an automaton where states 1 and 4 are the only states with more than one incoming edge with labels a and b, respectively. According to the previous proposition the automaton has a stable pair of states. (This particular automaton is synchronized, so all pairs of states are stable.)

In [7] we proved the road coloring conjecture for graphs that have the same uniform in- and out-degree k. We also proved that such balanced graphs can be labeled in a fully non-synchronizing way: there is a labeling that creates a permutation automaton. From the Corollary 1 we know that this is the only case when there are no non-zero stable vectors.

Consider then a graph G that is almost balanced in the sense that there is only one vertex whose in-degree is greater than k, the uniform out-degree. Let us prove that G can be labeled in such a way that Proposition 3 can be applied:

Proposition 4 *Let $G = (V, E)$ be a k-admissible graph with at least two vertices. If at most one vertex has an in-degree greater than the common out-degree k then there exist a labeling with a non-trivial stable pair of vertices.*

Proof. If all vertices have the same in-degree k then the result was proved in [7]. Assume then that p is the only vertex with in-degree $m > k$. Construct another directed graph that is identical to G except that any $m - k$ edges that enter p in G have been changed in such a way that instead of entering p they enter some other vertices whose in-degree in G is less than k. Clearly they can be redistributed in such a way that in the new graph $G' = (V, E')$ all vertices have the same in-degree k.

In [7] we proved the simple fact that graph G' has a labeling into a permutation automaton. Notice that G' is not necessarily strongly connected or aperiodic, but this result holds for all admissible graphs of uniform in-degree.

If we transfer the labeling back to graph G, we get a labeling such that all vertices except p have at most one incoming edge of each color. The result now follows from Proposition 3. □

Notice that Proposition 4 is not enough to prove that "almost balanced" graphs with only one vertex of large in-degree are synchronized. It only proves that the stability relation is non-trivial. This means that the quotient automaton $A/\equiv$ is smaller than the original automaton A. But in order to continue reducing the size of $A/\equiv$ further, as we did in [4, 7], we would need to know that the quotient automaton has the same property of being almost balanced. This we are able to show for a smaller class of automata only:

Proposition 5 *Let A be a DFA whose underlying graph is strongly connected and aperiodic. If A has the property that for every input letter $a \in \Sigma$ there is at most one state that does not have an edge labeled with a entering it then A can be recolored in such a way that it becomes synchronized.*

Proof. Let us use mathematical induction on n, the number of states in the automaton. If $n = 1$ the claim is trivial. Next, let $n > 1$ and assume that the claim has been proved for automata with fewer states.

If A is a permutation automaton then the claim has been proved in [7]. Assume then that A is not a permutation automaton. For every input letter a, the number of edges with label a is the same as the number of states. So if there would be two states with more than one incoming edge of label a then there would necessarily be at least two states with no incoming edges of label a. This is not the case, so A satisfies the assumptions made in Proposition 3. According to that proposition A has a stable pair of states. This means that the quotient automaton $A/\equiv$ has fewer states than A has.

Let us prove that $A/\equiv$ satisfies the conditions of the proposition so that we can use the inductive hypothesis. We know from [4] that $A/\equiv$ is strongly connected and aperiodic since A has these properties. States of $A/\equiv$ are equivalence classes of A under the stability relation $\equiv$. If there is no incoming edge with label a into an equivalence class it means that there is no incoming edge of label a to any state that belongs to that equivalence class. Since there exists at most one such state, we conclude that there can exist at most one such equivalence class.

The quotient automaton $A/\equiv$ satisfies the conditions of the proposition. According to the inductive hypothesis there exists a recoloring of $A/\equiv$ into a synchronized automaton. This recoloring can be lifted to A as described in [4]. The result is a synchronized recoloring of A. □

4 Conclusions

We have introduced and studied the stability space of finite automata, with the ultimate goal of proving that any k-admissible, strongly connected and aperiodic graph has a coloring with some stable pairs of vertices. We proved that in all non-permutation automata there are non-trivial stable vectors. In some instances this implies the existence of stable pairs. In particular this is the case if the automaton has the property that there is at most one state for each color where synchronization can take place. In the future we hope to establish other situations where stable pairs of states can be found based on the stability space of the automaton.

References

1. R.L.Adler, L.W.Goodwyn, B.Weiss. Equivalence of topological Markov shifts. *Israel Journal of Mathematics* **27** (1977), 49–63.
2. R.L.Adler, B.Weiss. Similarity of automorphisms of the torus. *Memoires of the American Mathematical Society*, n.98. 1970.
3. A.Carbone. Cycles of relatively prime length and the Road Coloring Problem. *Israel Journal of Mathematics*, To appear.
4. K.Culik, J.Karhumäki, J.Kari. Synchronized automata and the road coloring problem. TUCS technical report 323. Turku Centere for Computer Science, University of Turku, 1999.
5. J.Friedman. On the road coloring problem. *Proceedings of the American Mathematical Society* **110** (1990), 1133–1135.
6. N.Jonoska, S.Suen. Monocyclic decomposition of graphs and the road coloring problem. *Congressum Numerantium* **110** (1995), 201–209.
7. J. Kari, *Synchronizing finite automata on Eulerian digraphs*, Proceedings of Mathematical Foundations of Computer Science 2001, LNCS 2136, 432-438, (2001).
8. G.L.O'Brien. The road coloring problem. *Israel Journal of Mathematics* **39** (1981), 145–154.
9. D.Perrin, M.P.Schützenberger. Synchronizing prefix codes and automata, and the road coloring problem. In *Symbolic Dynamics and Applications*, Contemporary Mathematics, Amer.Math.Soc. **135** (1992), 295–318.