Journal of Universal Computer Science, vol. 8, no. 2 (2002), 235-242
submitted: 15/9/01, accepted: 29/1/02, appeared: 28/2/02

Simply Normal Numbers to Different Bases[1]

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Abstract: Let $b \geq 2$ be an integer. A real number is called *simply normal to base b* if in its representation to base b every digit appears with the same asymptotic frequency. We answer the following question for arbitrary integers $a, b \geq 2$: if a real number is simply normal to base a, does this imply that it is also simply normal to base b? It turns out that the answer is different from the well–known answers to the corresponding questions for the related properties "normality", "disjunctiveness", and "randomness".

Key Words: Randomness, invariance properties
Category: F.m, G.2

1 Introduction and Statement of the Result

For any property of infinite sequences over a finite alphabet and any pair of integers $a, b \geq 2$ one can ask the following question: if the base a representation of a real number has this property, does this imply that also the base b representation has this property? In this paper we answer this question for the property "simple normality".

The answer is well–known for several other prominent properties of infinite sequences over a finite alphabet, which are related to simple normality. Let us introduce some notation. We denote by $\mathbb{N}$ the set of all nonnegative integers, that is, $\mathbb{N} = \{0, 1, 2, \ldots\}$. Furthermore, an *alphabet* is always a finite nonempty set. For an arbitrary alphabet Σ, we denote by Σ^* the set of all finite strings over Σ, and by $\Sigma^\omega := \{p \mid p : \mathbb{N} \to \Sigma\}$ the set of all one–way infinite sequences over Σ. Such sequences will often be called *ω–words*. For an alphabet Σ, a string $w = w(0) \ldots w(|w| - 1) = w_0 \ldots w_{|w|-1} \in \Sigma^*$ with $w(i) = w_i \in \Sigma$ and a string $v = v(0) \ldots v(|v| - 1) = v_0 \ldots v_{|v|-1} \in \Sigma^*$, we denote by $\#_v(w)$ the number of occurrences of the string v in w, that is, the number of i such that $w(i) \ldots w(i + |v| - 1) = v$. If for an ω-word $p = p_0p_1p_2 \ldots \in \Sigma^\omega$ and a string $v \in \Sigma^*$ the limit

$$A(v, p) := \lim_{n \longrightarrow \infty} \frac{\#_v(p_0 \ldots p_{n-1})}{n}$$

exists, we call it *the asymptotic frequency of v in p*.

Definition 1. Let Σ be an alphabet. An ω-word $p \in \Sigma^\omega$ is said to be

1. *disjunctive* or *rich* if every finite string in Σ^* appears as a substring in p,
2. *simply normal* if every digit in Σ appears with the asymptotic frequency $1/|\Sigma|$ in p,

[1] C. S. Calude, K. Salomaa, S. Yu (eds.). *Advances and Trends in Automata and Formal Languages. A Collection of Papers in Honour of the 60th Birthday of Helmut Jürgensen.*

3. *normal* if every finite string $w \in \Sigma^*$ appears with the asymptotic frequency $|\Sigma|^{-|w|}$ in p,
4. *random* if there is no randomness test $(U_n)_{n\in\mathbb{N}}$ with $p \in \bigcap_{n\in\mathbb{N}} U_n$. Here, a *randomness test* according to [Martin–Löf 1966] is a sequence $(U_n)_{n\in\mathbb{N}}$ of subsets U_n of Σ^ω with the following two properties: (1) $\mu(U_n) \leq 2^{-n}$ for all n, where μ is the usual product measure on Σ^ω, given by $\mu(w\Sigma^\omega) = 2^{-|w|}$ for $w \in \Sigma^*$, (2) there exists a computably enumerable set $A \subseteq \mathbb{N} \times \Sigma^*$ with $U_n = \bigcup_{(n,w)\in A} w\Sigma^\omega$, for all n.

The relation between these notions is given by the following diagram:

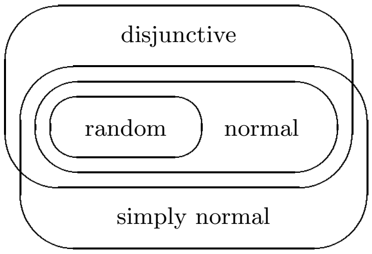

Figure 1: *Properties of ω–words.*

That means, any random sequence is normal but not vice versa, any normal sequence is disjunctive but not vice versa, any normal sequence is simply normal but not vice versa, there are simply normal sequences that are not disjunctive, and there are disjunctive sequences that are not simply normal. Furthermore, there are also sequences that are disjunctive and simply normal but not normal.

Normality and simple normality were introduced in [Borel 1909],[Borel 1914]. For an overview see [Kuipers and Niederreiter 1974]. Rich or disjunctive ω–words have been analyzed for example in [Compton 1983]. Disjunctive ω–words are also special cases of disjunctive ω–languages [Jürgensen, Shyr, Thierrin 1983]. Results concerning disjunctiveness of base representations of real numbers can be found e.g. in [Jürgensen and Thierrin 1988], [El-Zanati and Transue 1990], and in [Hertling 1996]. The notion of randomness above has been introduced in [Martin–Löf 1966]; for an overview see [Calude 1994] or [Li and Vitányi 1997].

Let $b \geq 2$ be an integer. The *expansion $\nu_b(x)$ to base b* of a real number x in the interval $[0,1)$ is the unique ω-word $p = p_0p_1p_2\ldots \in \Sigma_b^\omega$ over the alphabet $\Sigma_b := \{0,\ldots,b-1\}$ containing infinitely many digits $\neq b-1$ such that $x = \sum_{i=0}^{\infty} p_i \cdot b^{-(i+1)}$. A real number $x \in [0,1)$ is said to be *simply normal (disjunctive, normal, random) to base b* if $\nu_b(x) \in \Sigma_b^\omega$ is simply normal (disjunctive, normal, random). We come back to the question stated in the beginning. Assume that integers $a, b \geq 2$ and some property of ω–words —let us call it "Property A"— are given. Is $\{x \in [0,1) \mid \nu_a(x)$ has Property A$\} \subseteq \{x \in [0,1) \mid \nu_b(x)$ has Property A$\}$ true? Or, less formal: does Property A for $\nu_a(x)$ imply Property A for $\nu_b(x)$? For the properties "randomness", "normality", and "disjunctiveness" the answer is as follows.

– Randomness to base a implies randomness to base b, for any $a, b \geq 2$. That means, randomness is a base–invariant property of real numbers. This was first shown in [Calude and Jürgensen 1994]. Other proofs can be found in [Li and Vitányi 1997], p. 219., and in [Hertling and Weihrauch 1998].
– Normality to base a implies normality to base b if and only if a and b are equivalent, i.e., there are positive integers m and n such that $a^m = b^n$. This was shown in the famous paper [Schmidt 1960]. A special case had been obtained independently in [Cassels 1959].
– Disjunctiveness to base a implies disjunctiveness to base b if and only if a and b are equivalent; see e.g. [El-Zanati and Transue 1990] or [Hertling 1996]. The negative part of this statement, for nonequivalent a and b, is already contained in [Schmidt 1960].

In this paper we give the answer to the above question for the property "simple normality".

Theorem 2. *Let $a, b \geq 2$ be integers. If there is a positive integer n with $a = b^n$, then any number simply normal to base a is also simply normal to base b. If there is no such n, then the cardinality of the set of real numbers that are simply normal to base a but not to base b is equal to the cardinality of the continuum.*

Short:

> *simple normality to base a implies simple normality to base b if and only if a is a power of b.*

To the best of my knowledge this statement is not contained in the literature prior to [Hertling 1995]. In the following section we give a simplified presentation of the proof presented ibid.

It is interesting that the situation in the case of simple normality differs from the situation in the case of normality and of disjunctiveness. In fact, the (negative) statement of Theorem 2 for non–equivalent a and b is already contained in [Schmidt 1960]. Thus, for the negative part of the statement of Theorem 2 we only have to treat the case that a and b are equivalent, but a is not a power of b. This is exactly the case in which the situation for simple normality is different from the situation for normality or disjunctiveness. In this case we are even able to construct *rational* numbers which are simply normal to base a but not to base b. Note that rational numbers cannot be disjunctive and, hence, also not be normal or random to any base. This is due to the fact that the base representations of rational numbers are exactly those ω–words which are ultimately periodic. Ultimately periodic ω–words can be considered as the simplest possible ω–words.

2 The Proof

In this section we prove Theorem 2. Let $a, b \geq 2$ be fixed integers. We distinguish the following three cases.

I The base a is a power of the base b, i.e., there is some positive integer n with $a = b^n$.

II The base a is not a power of b, but a and b are equivalent, as defined above.
III The bases a and b are not equivalent.

We shall not treat Case III, since the statement of Theorem 2 for Case III is already contained in [Schmidt 1960]. In fact, Schmidt's result even implies that for non–equivalent a and b there are continuum many real numbers in $[0, 1)$ which are normal to base a but neither disjunctive nor simply normal to base b.

We come to Case I. We fix integers $b \geq 2$ and $n \geq 1$. We wish to show that if a real number $x \in [0, 1)$ is simply normal to base b^n then it is also simply normal to base b.

We define a bijection $f : \Sigma_b^n \longrightarrow \Sigma_{b^n}$ by $f(b_0 \ldots b_{n-1}) := \sum_{i=0}^{n-1} b^i \cdot b_{n-1-i}$. It is easy to see that for a real number $x \in [0, 1)$ with $\nu_b(x) = b_0 b_1 b_2 \ldots$ and $\nu_{b^n}(x) = a_0 a_1 a_2 \ldots$ one has $a_k = f(b_{k \cdot n} \ldots b_{k \cdot n+n-1})$ for any k. Let us assume that x is simply normal to base b^n, i.e.

$$A(d', \nu_{b^n}(x)) = \frac{1}{b^n}$$

for all $d' \in \Sigma_{b^n}$. We obtain for any $d \in \Sigma_b$

$$\begin{aligned}
A(d, \nu_b(x)) &= \frac{1}{n} \sum_{d' \in \Sigma_{b^n}} A(d', \nu_{b^n}(x)) \cdot \#_d(f^{-1}(d')) \\
&= \frac{1}{n} \cdot \frac{1}{b^n} \cdot \sum_{d' \in \Sigma_{b^n}} \#_d(f^{-1}(d')) \\
&= \frac{1}{n} \cdot \frac{1}{b^n} \cdot \sum_{w \in \Sigma_b^n} \#_d(w) \\
&= \frac{1}{n} \cdot \frac{1}{b^n} \cdot n \cdot b^{n-1} \\
&= \frac{1}{b}.
\end{aligned}$$

Hence, x is simply normal to base b. This ends the proof of the statement of Theorem 2 in Case I.

It remains to treat Case II. We need the following simple lemma.

Lemma 3. *Two integers $a, b \geq 2$ are equivalent if and only if there are positive integers m, n with $(m, n) = 1$ and an integer $c \geq 2$ such that $a = c^n$ and $b = c^m$.*

Proof. The if–part is trivial. Assume that a and b are equivalent. Then there are positive integers $\widetilde{m}$ and $\widetilde{n}$ such that $a^{\widetilde{m}} = b^{\widetilde{n}}$. Let k be the gcd of $\widetilde{m}$ and $\widetilde{n}$, and set $m := \widetilde{m}/k$ and $n := \widetilde{n}/k$. Then $(a^m)^k = a^{\widetilde{m}} = b^{\widetilde{n}} = (b^n)^k$, hence, also $a^m = b^n$. Note that $(m, n) = 1$. There are integers x, y with $mx + ny = 1$. The rational number $c := a^y b^x$ satisfies $c^n = a^{ny} b^{nx} = a^{ny+mx} = a$ and $c^m = b$. Hence it must be an integer ≥ 2. □

Let $a, b \geq 2$ be equivalent integers. By Lemma 3 there are positive integers m and n with $(m, n) = 1$ and an integer $c \geq 2$ such that $a = c^n$ and $b = c^m$. Additionally we assume that a is not a power of b. That means $m \geq 2$.

We define a bijective homomorphism $g : (\Sigma_c^n)^* \longrightarrow \Sigma_a^*$ by $g(c_0 \dots c_{n-1}) := \sum_{i=0}^{n-1} c^i \cdot c_{n-1-i}$, for $c_i \in \Sigma_c$, and by $g(w^{(1)} \dots w^{(k)}) := g(w^{(1)}) \dots g(w^{(k)})$ for $w^{(1)}, \dots, w^{(k)} \in \Sigma_c^n$, and analogously a bijective homomorphism $h : (\Sigma_c^m)^* \longrightarrow \Sigma_b^*$. Note that for a real number $x \in [0, 1)$ with

$$\nu_a(x) = a_0 a_1 a_2 \dots, \quad \nu_b(x) = b_0 b_1 b_2 \dots, \quad \nu_c(x) = c_0 c_1 c_2 \dots$$

one has

$$g(c_{kn} \dots c_{(k+1)n-1}) = a_k, \quad h(c_{km} \dots c_{(k+1)m-1}) = b_k,$$

and hence

$$hg^{-1}(a_{km} \dots a_{(k+1)m-1}) = b_{kn} \dots b_{(k+1)n-1}$$

for all $k \in \mathbb{N}$.

Lemma 4. *There is a set $M \subseteq \Sigma_a^m$ containing a elements such that*

$$\sum_{w \in M} \#_d(w) = m \quad \text{for each } d \in \Sigma_a \tag{1}$$

and

$$\sum_{w \in M} \#_0(hg^{-1}(w)) \neq n \cdot \frac{a}{b}\,. \tag{2}$$

Proof. We distinguish the cases $n < m$ and $n > m$.

For $n < m$ we set

$$M := \{0^m, 1^m, \dots, (a-1)^m\}.$$

Then, obviously $\sum_{w \in M} \#_d(w) = m$ for each $d \in \Sigma_a$, and

$$\sum_{w \in M} \#_0(hg^{-1}(w)) \geq \#_0(hg^{-1}(0^m)) = \#_0(0^n) = n > n \cdot c^{n-m} = n \cdot \frac{a}{b}\,.$$

Now let us assume $n > m$. We shall construct a set $M \subseteq \Sigma_a^m$ such that

$$\{w(0) \mid w \in M\} = \Sigma_a, \tag{3}$$

and such that for all $j \in \{1, \dots, m-1\}$ the following two conditions are satisfied:

$$\{w(j) \mid w \in M\} = \Sigma_a, \tag{4}$$

and

for all $w \in M$,
the rightmost digit of $g^{-1}(w(j-1))$ is not equal to zero or (5)
the leftmost digit of $g^{-1}(w(j))$ is not equal to zero.

We construct the a strings $w^{(0)}, \dots, w^{(a-1)}$ in M in parallel, digit by digit, from the left. First, we set $w^{(i)}(0) := i$, for all $i \in \Sigma_a$. Then, clearly, Condition (3) is satisfied. Now, we assume that $w^{(i)}(j-1)$ is defined for some $j \in \{1, \dots, m-1\}$ and all $i \in \Sigma_a$, satisfying $\{w(j-1) \mid w \in M\} = \Sigma_a$. We wish to define the digits $w^{(i)}(j) \in \Sigma_a$ for all $i \in \Sigma_a$ in such a way that (4) and (5) are satisfied. Thus, we have to distribute the digits in Σ_a onto the $w^{(i)}(j)$ in such a way that whenever the rightmost digit of $g^{-1}(w^{(i)}(j-1))$ is equal to zero, the leftmost digit of $g^{-1}(w^{(i)}(j))$ is not equal to zero. But, there are exactly $c^n - c^{n-1}$ strings in Σ_c^n whose rightmost digit is not equal to zero, hence, there are exactly $c^n - c^{n-1}$ indices $i \in \Sigma_a$ such that the rightmost digit of $g^{-1}(w^{(i)}(j-1))$ is not equal to zero. On the other hand, there are exactly c^{n-1} strings in Σ_c^n whose leftmost digit is equal to zero. Since g is a bijection between Σ_c^n and Σ_a, and since $c^{n-1} \leq c^n - c^{n-1}$, we can choose the digits $w^{(i)}(j)$ for $i \in \Sigma_a$ appropriately. Thus, we can construct a set $M \subseteq \Sigma_a^m$ satisfying (3), and (4) and (5) for all $j \in \{1, \ldots, m-1\}$.

We still have to show that M satisfies also (1) und (2). Indeed, (1) follows directly from (3) and the validity of (4) for all $j \in \{1, \ldots, m-1\}$. For deriving (2), let us consider some string $w \in \Sigma_a^m$. Via the bijection $hg^{-1}|_{\Sigma_a^m}$ from Σ_a^m to Σ_b^n, the string w corresponds to some string $v := hg^{-1}(w) \in \Sigma_b^n$. Each digit in v corresponds via h^{-1} to a substring of length m of $g^{-1}(w) \in \Sigma_c^{mn}$. The digits of v fall into two classes:

1. the class of all digits in v that correspond to a substring of the string $g^{-1}(w(j))$ corresponding to the digit $w(j)$ of w, for some j,
2. the class of all digits in v that correspond to a substring of $g^{-1}(w)$ which contains digits of two different substrings $g^{-1}(w(j-1))$ and $g^{-1}(w(j))$, for some $j \in \{1, \ldots, m-1\}$.

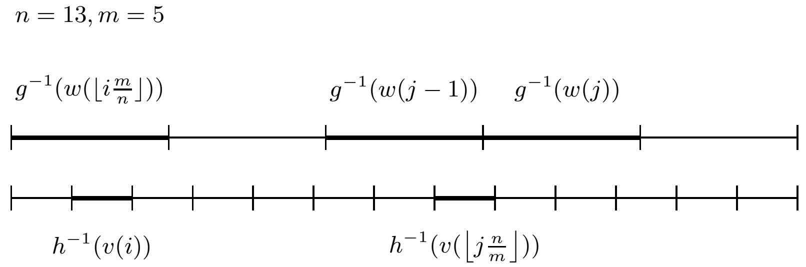

Figure 2: *Substrings corresponding to digits in base c^{13} and in base c^5.*

In fact, a simple calculation shows that for any $i \in \{0, \ldots, n-1\} \setminus I$ with

$$I := \left\{ \left\lfloor j\frac{n}{m} \right\rfloor \;\middle|\; j \in \{1, \ldots m-1\} \right\}.$$

the string $h^{-1}(v(i))$ corresponding to the digit $v(i)$ is a substring of the string $g^{-1}(w(\lfloor i\frac{m}{n}\rfloor))$ corresponding to the digit $w(\lfloor i\frac{m}{n}\rfloor)$, that is, the digit $v(i)$ belongs into the first class. All digits $v(i)$ for $i \in I$ belong into the second class.

Claim 5. *For any digit $d \in \Sigma_b$ and any $i \in \{0, \ldots, n-1\} \setminus I$,*

$$|\{w \in M \,|\, (hg^{-1}(w))(i) = d\}| = \frac{a}{b}.$$

That means: every digit in $hg^{-1}(w)$ in the first class runs through all digits in Σ_b the same number of times if w runs through all strings in M.

Proof. Fix an $i \in \{0,\dots,n-1\} \setminus I$ and let w run through M. Then by Condition (3) or by Condition (4), $w(\lfloor i\frac{m}{n}\rfloor)$ runs through each digit in Σ_a once. Hence $g^{-1}(w(\lfloor i\frac{m}{n}\rfloor))$ runs through each string in Σ_c^n once. As $h^{-1}((hg^{-1}(w))(i))$ is a substring of length m of $g^{-1}(w(\lfloor i\frac{m}{n}\rfloor))$ it runs through all the strings in Σ_c^m exactly $c^n/c^m = \frac{a}{b}$ times. Hence, $(hg^{-1}(w))(i)$ runs through all digits in Σ_b exactly $\frac{a}{b}$ times. □

Claim 6. *For any $i \in I$ and $w \in M$, $(hg^{-1}(w))(i) \neq 0$.*

That means: any digit in $hg^{-1}(w)$ in the second class is never equal to zero, for any string $w \in M$.

Proof. Fix a string $w \in M$ and a $j \in \{1,\dots,m-1\}$, and consider $i := \lfloor j\frac{n}{m}\rfloor \in I$. Then the string $h^{-1}((hg^{-1}(w))(i))$ in Σ_c^m corresponding to the digit $(hg^{-1}(w))(i)$ contains the rightmost digit of $g^{-1}(w(j-1))$ and the leftmost digit of $g^{-1}(w(j))$. Since at least one of these two digits in Σ_c is not equal to zero according to Condition (5), also the digit $(hg^{-1}(w))(i)$ in Σ_b cannot be equal to zero. □

By the previous two claims we obtain

$$\sum_{w\in M} \#_0(hg^{-1}(w)) = |\{0,\dots,n-1\} \setminus I| \cdot \frac{a}{b} = (n-m+1)\cdot\frac{a}{b} < n\cdot\frac{a}{b},$$

hence we obtain (2). Note that here we needed the assumption $m \geq 2$, i.e., that a is not a power of b: it implies that the set I, respectively the second class of digits of $hg^{-1}(w)$ for $w \in \Sigma_a^n$, is nonempty. This ends the proof of Lemma 4. □

Finally we have to show that Lemma 4 implies the assertion of Theorem 2 in Case II. For a set $M = \{w^{(0)},\dots,w^{(a-1)}\}$ satisfying Conditions (1) and (2) we define

$$V(M) := \{w^{(\pi(0))}\dots w^{(\pi(a-1))} \mid \pi \text{ is a permutation of } \Sigma_a\} \subseteq \Sigma_a^{m\cdot a}\,.$$

For any $v \in V(M)$, Condition (1) implies $\#_d(v)/|v| = \frac{1}{a}$ for any $d \in \Sigma_a$. Hence any $p \in V(M)^\omega$ is simply normal, i.e. the number $\nu_a^{-1}(p)$ is simply normal to base a. But we shall see that it is not simply normal to base b. If $p = v_0v_1v_2\dots$ with $v_j \in V(M)$ then $\nu_b\nu_a^{-1}(p) = hg^{-1}(v_0)hg^{-1}(v_1)hg^{-1}(v_2)\dots$. Since for any $v \in V(M)$ and $d \in \Sigma_b$ one has

$$\frac{\#_d(hg^{-1}(v))}{|hg^{-1}(v)|} = \frac{\sum_{w\in M}\#_d(hg^{-1}(w))}{a\cdot n}$$

the asymptotic frequency of 0 in $\nu_b\nu_a^{-1}(p)$ is

$$A(0,\nu_b\nu_a^{-1}(p)) = \frac{\sum_{w\in M}\#_0(hg^{-1}(w))}{a\cdot n}.$$

By Condition (2) this is not equal to $\frac{1}{b}$. Hence $\nu_a^{-1}(p)$ is simply normal to base a but not simply normal to base b for any $p \in V(M)^\omega$. Since the set $V(M)^\omega$ has the cardinality of the continuum the assertion follows. This ends the proof of Theorem 2.

We conclude this section with the following corollary of the proof.

Corollary 7. *If $a, b \geq 2$ are equivalent bases, but a is not a power of b, then there are rational numbers which are simply normal to base a but not to base b.*

Proof. We have seen above that any sequence $p \in V(M)^\omega$ is simply normal, i.e., the number $\nu_a^{-1}(p)$ is simply normal to base a. But this number is not simply normal to base b. Thus, for any string $v \in V(M)$, the number $\nu_a^{-1}(v^\omega)$ is simply normal to base a but not to base b. This number is rational since its expansion to base a is (ultimately) periodic; compare e.g. [Bundschuh 1992]. □

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