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Rationally Additive Semirings[1]

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Abstract: We define rationally additive semirings that are a generalization of (ω)-complete and (ω-)continuous semirings. We prove that every rationally additive semiring is an iteration semiring. Moreover, we characterize the semirings of rational power series with coefficients in N_∞, the semiring of natural numbers equipped with a top element, as the free rationally additive semirings.

Key Words: semiring, complete semiring, iteration semiring, fixed point, power series.

Category: F.4.3

1 Introduction

Rationally additive semirings arise in [Mohri 1998]. Rationally additive semiring possess enough infinite sums to solve any finite system of linear fixed point equations. They are a common generalization of (ω)-complete and (ω-)continuous semirings [see Eilenberg 1974, Kuich 1987, Sakarovitch 1987, Kuich 1997] in which all (countable) sums exist. Two prime examples of rationally additive semirings are the semiring of rational (or regular) sets in A^*, where A is any set, and the semiring $N_\infty^{\mathrm{rat}}\langle\langle A^*\rangle\rangle$ of rational power series over A with coefficients in N_∞, the semiring of natural numbers with a top element ∞.

In our main result, Theorem 10, we prove that every rationally additive semiring is an iteration semiring. This fact extends a result of [Hebisch 1990] by which every complete semiring is a Conway semiring. Iteration semirings appear implicitly in [Conway 1971]. They were explicitly defined in [Bloom, Ésik 1993a, 1993b]. Conway conjectured that a complete axiomatization of the equational theory of rational (regular) languages consists of the Conway semiring equations, defined below, together with the equation $1^* = 1$ and an equation associated with each

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finite group. Conway's conjecture was confirmed in [Krob 1991], see also [Ésik 1999]. In [Bloom, Ésik 1997], the authors conjectured that the valid equations of rational power series with coefficients in N_∞, the semiring of natural numbers equipped with a top element, can be axiomatized by the iteration semiring equations and the equation $1^* = 1^{**}$. This conjecture is still open. In Theorem 15, we characterize the semirings of rational power series with coefficients in N_∞ as the free rationally additive semirings.

2 Conway semirings and iteration semirings

A *-*semiring* is a semiring [see Kuich, Salomaa 1986, Golan 1992]

$$S = (S, +, \cdot, 0, 1)$$

equipped with a star operation $^* : S \to S$. A *Conway semiring* [Bloom, Ésik 1993b] is a *-semiring S which satisfies the *sum-star* and *product-star* equations

$$(x+y)^* = (x^*y)^*x^* \tag{1}$$

$$(xy)^* = 1 + x(yx)^*y. \tag{2}$$

Note that the *fixed point equation*

$$x^* = 1 + xx^* \tag{3}$$

holds in any Conway semiring. (Substitute 1 for y in (2).)

Suppose that S is a *-semiring and $n \geq 0$. We turn the matrix semiring $S^{n\times n}$ into a *-semiring. Let $M \in S^{n\times n}$. When $n = 0$, M^* is the unique 0×0 matrix, and when $M = [a]$, then $M^* = [a^*]$. Suppose now that $n > 1$. Write $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where a is $(n-1)\times(n-1)$ and d is 1×1. We define

$$M^* = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \tag{4}$$

where

$$\alpha = (a + bd^*c)^*$$
$$\beta = \alpha bd^*$$
$$\gamma = \delta ca^*$$
$$\delta = (d + ca^*b)^*.$$

Theorem 1. [Conway 1971, Bloom, Ésik 1993] *If S is a Conway semiring, then so is each matrix semiring $S^{n\times n}$. Moreover, the above matrix formula (4) holds for each way of splitting M into four blocks $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that a and d are square matrices.*

Suppose that G is a finite group of order n with elements $g_1, \ldots, g_n$. For each g_i, let x_i denote a variable associated with g_i. We define $M_G = [\,(M_G)_{ij}\,]$, where $(M_G)_{ij}$ is the variable associated with the group element $g_i^{-1} g_j$, i.e., $(M_G)_{ij} = x_k$ where $g_k = g_i^{-1} g_j$. The matrix M_G^* is defined as in (4) above, so that each entry of M_G^* is a term in the variables $x_1, \ldots, x_n$.

The *group-equation associated with* G [see Conway 1971] is the equation

$$e \cdot M_G^* \cdot u = (x_1 + \ldots + x_n)^*,$$

where e is the $1 \times n$ row matrix whose first entry is 1 and whose other entries are 0, and where u is the $n \times 1$ column matrix whose entries are all 1. (Under the Conway semiring equations (1) and (2), the particular order $g_1, \ldots, g_n$ of the group elements is irrelevant.)

An *iteration semiring* [see Bloom, Ésik 1993, Ésik 1999] is a Conway semiring satisfying all group-equations.

Proposition 2. [Bloom, Ésik 1993b] *Any Conway semiring S satisfying the* functorial implication

$$AC = CB \Rightarrow A^*C = CB^*,$$

for all matrices $A \in S^{n\times n}$, $B \in S^{m\times m}$ and $C \in S^{n\times m}$, is an iteration semiring.

Notation For each nonnegative integer n, we denote the set $\{1, \ldots, n\}$ by $[n]$. Thus, $[0]$ is another name for the empty set.

For any set Σ, we denote by Σ^* the free monoid of all words over Σ including the empty word ϵ. When S is semiring, $S\langle\langle A^*\rangle\rangle$ denotes the semiring of all power series over A with coefficients in S. Moreover, we let $S\langle A\rangle$ denote the collection of all finite sums of terms of the form sa with $s \in S$ and $a \in \Sigma$.

3 Rationally additive semirings

A *weak rationally additive semiring* is a semiring S equipped with a partial summation $\sum_{i\in I} s_i$ defined on countable families $s_i \in S$, $i \in I$ subject to the following conditions:

- **Ax**$_1$. When $s_i \in S$ for $i \in F$ and F is finite, then $\sum_{i\in F} s_i$ is the sum of the s_i as defined in the semiring S.
- **Ax**$_2$. For each $s \in S$, the geometric sum $\sum_{n=0}^{\infty} s^n$ exists.
- **Ax**$_3$. If $\sum_{i\in I} s_i$ exists, then so do $\sum_{i\in I} ss_i$ and $\sum_{i\in I} s_i s$, for each $s \in S$, moreover,

$$s(\sum_{i\in I} s_i) = \sum_{i\in I} ss_i$$
$$(\sum_{i\in I} s_i)s = \sum_{i\in I} s_i s.$$

- **Ax**$_4$. Suppose that the countable set I is the disjoint union of the sets I_j, $j \in J$. Then for any family $s_i \in S$, $i \in I$, if $r_j = \sum_{i\in I_j} s_i$ exists for each $j \in J$, and if $r = \sum_{j\in J} r_j$ exists, then $\sum_{i\in I} s_i$ also exists and equals r.

A *rationally additive semiring* is a weak rationally additive semiring S that satisfies:

- $\mathbf{Ax}_5$. Suppose that the countable set I is the disjoint union of the sets I_j, $j \in J$. Then for any family $s_i \in S$, $i \in I$, if $s = \sum_{i\in I} s_i$ exists and $r_j = \sum_{i \in I_j} s_i$ exist, for all $j \in J$, then $\sum_{j\in J} r_j$ exists and equals s.

Proposition 3. *Suppose that S is a weak rationally additive semiring.*

- *For any countable families s_i, $i \in I$ and r_j, $j \in J$, if $\sum_{i\in I} s_i = s$ and $\sum_{j\in J} r_j = r$ exist, then so does $\sum_{(i,j)\in I\times J} s_i r_j$. Moreover, $\sum_{(i,j)\in I\times J} s_i r_j = sr$.*
- *For any countable families $s_i \in S$ and $s'_j \in S$ with $i \in I$ and $j \in J$, if there is a bijection $\pi : I \to J$ with $s_i = s'_{i\pi}$, for all $i \in I$, then $\sum_{i\in I} s_i$ exists iff $\sum_{j\in J} s'_j$ does, in which case the two sums are equal.*
- *Any countable sum $\sum_{i\in I} s$ exists. Moreover, $\sum_{i\in I} 0 = 0$, i.e., any countable sum of 0 with itself is 0.*
- *For any countable family s_i, $i \in I$, if $\sum_{j\in J} s_j = r$ exists, where J is the set of all $i \in I$ with $s_i \neq 0$, then $\sum_{i\in I} s_i$ exists and equals r.*

Proof. The first claim follows from $\mathbf{Ax}_3$ and $\mathbf{Ax}_4$. For the second, suppose that $\sum_{i\in I} s_i = s$ exists. Let $J_i = \{i\pi\}$, for each $i \in I$. Thus the sets J_i determine a partition of J. Each sum $\sum_{k\in J_i} s'_k = s'_{i\pi} = s_i$ exists, moreover, $\sum_{i\in I} s_i$ exists. Thus, by $\mathbf{Ax}_4$, we have that $\sum_{j\in J} s'_j$ exists and equals $\sum_{i\in I} s_i$. In the same way, it follows that if $\sum_{j\in J} s'_j$ exists then $\sum_{i\in I} s_i$ also exists. For the third claim, assume first that $s = 1$. If I is finite with n elements, then $\sum_{i\in I} s = \sum_{i\in I} 1$ exists by $\mathbf{Ax}_1$, and equals the usual n-fold sum of 1 with itself. Assume now that I is infinite. Then $\sum_{i\in I} s = \sum_{i\in I} 1 = \sum_{i=0}^{\infty} 1^i$ exists by $\mathbf{Ax}_2$. Thus for any s, we have that $\sum_{i\in I} s = \sum_{i\in I}(s \cdot 1) = s(\sum_{i\in I} 1)$ exists. When s is 0, this sum is also 0. The last claim now follows from $\mathbf{Ax}_4$. $\square$

Remark. When S is rationally additive, the converse of the last fact also holds, so that using the same notation, $\sum_{j\in J} s_j$ exists iff $\sum_{i\in I} s_i$ exists.

Suppose that S and S' are (weak) rationally additive semirings. A homomorphism $h : S \to S'$ is a semiring homomorphism that preserves all existing countable sums. Thus, if $\sum_{i\in I} s_i$ exists, where $s_i \in S$ for each $i \in I$, then so does $\sum_{i\in I} s_i h$ and $(\sum_{i\in I} s_i)h = \sum_{i\in I} s_i h$.

Example 1. A countably additive (or ω-complete) semiring is a rationally additive semiring S such that $\sum_{i\in I} s_i$ exists for all countable families $s_i \in S$, $i \in I$. For example, the power set semiring of a semiring is countably additive, where summation is defined by set union. An example of a rationally additive semiring which is not countably additive is the semiring of regular sets in A^*, where A is any set. In this semiring only those sums (unions) exist that are regular languages. An ω-continuous (or just continuous) semiring is a countably additive semiring which is naturally ordered and such that $\sum_{i\in I} s_i$ is the supremum of the finite sums $\sum_{i\in F} s_i$, for all finite subsets $F \subseteq I$. Since any countably additive semiring is rationally additive, so is any ω-continuous semiring. For more on complete and continuous semirings, the reader is referred to [Eilenberg 1974, Kuich 1987, Sakarovitch 1987, Kuich 1997].

Example 2. A prime example of a countably additive semiring is the semiring $N_\infty = \{0, 1, \dots, \infty\}$ obtained by adding a top element to the natural numbers N equipped with the following summation. For all $n_i \in N_\infty$, $i \in I$, where I is countable, define $\sum_{i \in I} n_i = \infty$ if $n_i = \infty$ for some i, or if $n_i > 0$ for infinitely many numbers i. Otherwise let $\sum_{i \in I} n_i$ be the ordinary sum. Note that all countable sums exist in N_∞. Moreover, we have $x \cdot \infty = \infty \cdot x = \infty$ for all $x \neq 0$. We call the above countably additive structure on N_∞ the *standard countably additive structure.*

Remark. The same semiring S may sometimes be turned into a weak rationally additive semiring in several different ways. Suppose that we have a weak rationally additive structure on S with summation denoted $\sum$. Then there is a smallest weak rationally additive structure on S contained in $\sum$. If we denote the summation operation of this structure by $\sum'$, we have that $\sum'_{i \in I} s_i$ exists iff I is finite, or there is an element $s \in S$ such that for some linear order $i_0, i_1, \dots$ of the set I, we have that $s_{i_n} = s^n$, for all $n \geq 0$, or there is a family s'_i, $i \in I$ and an element $s' \in S$ such that either $s_i = s'_i s'$ for all i or $s_i = s' s'_i$ for all i, or there exist disjoint sets I_j, $j \in J$ with $I = \cup_{j \in J} I_j$ such that $r_j = \sum'_{i \in I_j} s_j$ exists for each $j \in J$ and $\sum'_{j \in J} r_j$ exists. In either case, $\sum'_{i \in I} s_i$, when exists, is defined to be $\sum_{i \in I} s_i$. In the same way, each rationally additive structure on S contains a least rationally additive structure.

Remark. There exists a weak rationally additive semiring which is not rationally additive. For one example, take the (standard) countably additive semiring N_∞ defined above. It will be shown below in Corollary 14 that N_∞ has no other rationally additive structure properly included in the standard structure. On the other hand, consider the least weak rationally additive structure contained in it. Let $\sum'$ denote the corresponding summation. Then there is only a countable number of sets $K \subseteq N$ such that $\sum'_{k \in K} k$ exists. Hence this weak additive semiring structure is not the standard countably additive structure.

In any (weak) rationally additive semiring, we define

$$\begin{aligned} {}^* : S &\to S \\ s &\mapsto \sum_{n=0}^{\infty} s^n. \end{aligned}$$

It is clear that morphisms of (weak) rationally additive semirings preserve the *-operation.

Proposition 4. *Any weak rationally additive semiring S is a Conway semiring.*

Proof. Suppose that $a, b \in S$ and let $\overline{a}$ and $\overline{b}$ denote distinct letters corresponding to a and b, respectively. Below we will use regular languages in $(\overline{a} + \overline{b})^*$ as index sets. For any word $\overline{w} \in (\overline{a} + \overline{b})^*$, let w denote the corresponding element in S. Since $(a + b)^* = \sum_{n=0}^{\infty} (a + b)^n$ exists, it follows by $\mathbf{Ax}_4$ that $\sum_{\overline{w} \in (\overline{a} + \overline{b})^*} w$ also exists, and $(a + b)^* = \sum_{\overline{w} \in (\overline{a} + \overline{b})^*} w$. Let us partition $(\overline{a} + \overline{b})^*$ into the disjoint union of the sets $L_k = (\overline{a}^* \overline{b})^k \overline{a}^*$, $k \geq 0$. It follows by $\mathbf{Ax}_2$ and $\mathbf{Ax}_3$ that each sum $\sum_{\overline{w}\in L_k} w$ exists, and $\sum_{\overline{w}\in L_k} w = (a^*b)^k a^*$. Thus, again by $\mathbf{Ax}_2$ and $\mathbf{Ax}_3$, $\sum_{k=0}^{\infty}(a^*b)^k a^* = (a^*b)^* a^*$ exists. Since for each k we have $\sum_{\overline{w}\in L_k} w = (a^*b)^k a^*$, it follows from $\mathbf{Ax}_4$ that $\sum_{\overline{w}\in L} w = (a^*b)^* a^*$. Hence $(a+b)^* = \sum_{\overline{w}\in L} w = (a^*b)^* a^*$.

Also, $\sum_{k=0}^{\infty} a(ba)^k b = a(\sum_{k=0}^{\infty}(ba)^k)b = a(ba)^* b$ exists, hence by $\mathbf{Ax}_4$, $(ab)^* = \sum_{k=0}^{\infty}(ab)^k = 1 + \sum_{k=0}^{\infty} a(ba)^k b = 1 + a(ba)^* b$. □

Corollary 5. *The* fixed point equation *(3) holds in any weak rationally additive semiring.*

Proposition 6. *Any weak rationally additive semiring S satisfies* $1^* = 1^{**}$*,* $1^*1^* = 1^*$ *and* $1^* + 1^* = 1^*$.

Proof. Equation $1^* + 1^* = 1^*$ follows from $\mathbf{Ax}_4$. By Proposition 3,

$$1^*1^* = (\sum_{i=0}^{\infty} 1)(\sum_{i=0}^{\infty} 1) = \sum_{i,j=0}^{\infty} 1 = \sum_{k=0}^{\infty} 1 = 1^*,$$

and by $\mathbf{Ax}_3$,

$$1^*1^* = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} 1.$$

Hence, $(1^*)^n = 1^*$, for all $n \geq 1$. Moreover,

$$1^{**} = 1 + \sum_{n=1}^{\infty}(1^*)^n = 1 + \sum_{n=0}^{\infty} 1^* = 1 + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} 1 = 1 + 1^* = 1^*,$$

where the last step follows from the fixed point equation. □

Remark. In fact, the equations $1^*1^* = 1^*$ and $1^* + 1^* = 1^*$ hold in any Conway semiring satisfying $1^{**} = 1^*$.

Suppose that S is a weak rationally additive semiring. Then, as shown above, S is a Conway semiring. Thus, by Theorem 9, the semirings $S^{n\times n}$, $n \geq 0$ are also Conway semirings. Moreover, for each decomposition of a matrix $A \in S^{n\times n}$ in the form $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where a and d are square matrices, we have

$$A^* = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \tag{5}$$

where

$$\begin{aligned} \alpha &= (a + bd^*c)^* \\ \beta &= \alpha b d^* \\ \gamma &= \delta c a^* \\ \delta &= (d + ca^*b)^*. \end{aligned}$$

Suppose now that S is rationally additive. We turn $S^{n\times n}$ into a rationally additive semiring. Suppose that $A_i \in S^{n\times n}$, $i \in I$ where I is countable. We say that $\sum_{i\in I} A_i$ exists if $\sum_{i\in I}(A_i)_{jk}$ exists for all $j,k \in [n]$. Moreover, we define $(\sum_{i\in I} A_i)_{jk} = \sum_{i\in I}(A_i)_{jk}$, for each $j,k \in [n]$. We define the summation on countable families of matrices in $S^{n\times m}$, for $n,m \geq 0$ in the same way.

Proposition 7. *Suppose that S is rationally additive. If $A_i \in S^{n\times m}$, $i \in I$ such that $\sum_{i\in I} A_i$ exists, then for any $B \in S^{m\times p}$, $\sum_{i\in I} A_iB$ exists and equals $(\sum_{i\in I} A_i)B$.*

Proof. It suffices to prove the proposition for $p = 1$. We argue by induction on m. The case that $m = 0$ is trivial. When $m = 1$, the proposition holds by $\mathbf{Ax}_3$. Suppose now that $m > 1$. Then let $m = m_1 + m_2$, where $m_1, m_2 < m$, and let us write $A_i = [\, a_i\ b_i\,]$, $i \in I$, and $B = \begin{bmatrix} x \\ y \end{bmatrix}$, where a_i is $n \times m_1$, etc. Let $a = \sum_{i\in I} a_i$ and $b = \sum_{i\in I} b_i$, so that $A = \sum_{i\in I} A_i = [\, a\ b\,]$. By the induction assumption, both $\sum_{i\in I} a_ix$ and $\sum_{i\in I} b_iy$ exist, moreover, $\sum_{i\in I} a_ix = ax$ and $\sum_{i\in I} b_iy = by$. Since $\mathbf{Ax}_5$ holds in S, it follows that $\sum_{i\in I}(a_ix + b_iy)$ exists and equals $ax + by$. Thus, $\sum_{i\in I} A_iB = (\sum_{i\in I} A_i)B$ exists. Note that only a weak form of $\mathbf{Ax}_5$ was used: the case when each set I_j is finite. □

In the same way, we have:

Proposition 8. *Suppose that S is rationally additive. If $A_i \in S^{n\times m}$, $i \in I$ such that $\sum_{i\in I} A_i$ exists, then for any $B \in S^{p\times n}$, $\sum_{i\in I} BA_i$ exists and equals $B(\sum_{i\in I} A_i)$.*

Theorem 9. *When S is rationally additive, so is $S^{n\times n}$, for any $n \geq 0$. Moreover, for the star operation defined in (5), we have*

$$A^* = \sum_{k=0}^{\infty} A^k.$$

Proof. Our claims are clear for $n = 0, 1$. We proceed by induction on n. Assume that $n > 1$. It is clear that $\mathbf{Ax}_1$, $\mathbf{Ax}_4$ and $\mathbf{Ax}_5$ hold in $S^{n\times n}$. The fact that $\mathbf{Ax}_3$ holds was shown above.

Suppose now that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in S^{n\times n}$, where a, b, c, d are submatrices of A such that a and d are square matrices of size smaller than n. We want to show that $\sum_{k=0}^{\infty} A^k = A^*$, i.e., that $\sum_{k=0}^{\infty} A^k$ exists and

$$\sum_{k=0}^{\infty} A^k = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

where α, β, γ and δ were given as above. We will only show that the submatrix of $\sum_{k=0}^{\infty} A^k$ at the left upper corner exists and equals α.

Consider the regular language $L = (\overline{a} + \overline{b}\overline{d}^*\overline{c})^*$. Then L is the union of the disjoint sets L_1^k, $k \geq 0$, where $L_1 = \overline{a} + \overline{b}\overline{d}^*\overline{c}$. By the induction assumption,

$$a + bd^*c = a + b(\sum_{j=0}^{\infty} d^j)c = a + \sum_{j=0}^{\infty} bd^jc = a + \sum_{\overline{w}\in\overline{b}\overline{d}^*\overline{c}} w = \sum_{\overline{w}\in L_1} w.$$

Hence, by Proposition 7 and Proposition 8,

$$(a+bd^*c)^2 = (\sum_{\overline{w}\in L_1} w)(\sum_{\overline{w}\in L_1} w) = \sum_{\overline{u},\overline{v}\in L_1} uv = \sum_{\overline{w}\in L_1^2} w,$$

since each word in L_1^2 has a unique factorization as a product of two words in L_1. In the same way, it follows that

$$(a+bd^*c)^k = \sum_{\overline{w}\in L_1^k} w,$$

for all $k \geq 0$. Thus, by the induction assumption,

$$(a+bd^*c)^* = \sum_{k=0}^{\infty}(a+bd^*c)^k = \sum_{k=0}^{\infty}\sum_{\overline{w}\in L_k} w = \sum_{\overline{w}\in L} w.$$

In particular, $\sum_{\overline{w}\in L} w$ exists. Now let us write $A^k = \begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix}$, $k \geq 0$. To complete the proof, we need show that $\sum_{k=0}^{\infty} a_k$ exists and equals $\sum_{\overline{w}\in L} w$. But for each k, $a_k = \sum_{\overline{w}\in L,\ |\overline{w}|=k} w$. Thus, since $\mathbf{Ax}_5$ holds by the induction assumption, $\sum_{k=0}^{\infty} a_k$ exists and equals $\sum_{\overline{w}\in L} w$. (Again note that only the weak form of $\mathbf{Ax}_5$ when the sets I_j are finite has been used.) □

Theorem 10. *Any rationally additive semiring S is an iteration semiring satisfying $1^* = 1^{**}$.*

Proof. We have already proved that any rationally additive semiring S is a Conway semiring and satisfies $1^* = 1^{**}$. The fact that the group-equations hold follows from the functorial implication, see Proposition 2, which can be established as follows. Suppose that $A \in S^{n\times n}$, $B \in S^{m\times m}$ and $C \in S^{n\times m}$ with $AC = CB$. Then $A^kC = CB^k$, for all $k \geq 0$. Thus, by Propositions 7 and 8,

$$A^*C = (\sum_{k=0}^{\infty} A^k)C = \sum_{k=0}^{\infty} A^kC = \sum_{k=0}^{\infty} CB^k = C(\sum_{k=0}^{\infty} B^k) = CB^*.$$

□

Assume that S is a rationally additive semiring and A is a set. We turn the power series semiring $S\langle\langle A^*\rangle\rangle$ into a rationally additive semiring. For any countable family $r_i \in S\langle\langle A^*\rangle\rangle$, $i \in I$, we say that $\sum_{i\in I} r_i$ is defined if the sum $\sum_{i\in I}(r_i, u)$ is defined for all $u \in A^*$. Moreover, in this case, we let $(\sum_{i\in I} r_i, u) = \sum_{i\in I}(r_i, u)$.

Proposition 11. *Suppose that S is a rationally additive semiring and A is a set. Then $S\langle\langle A^*\rangle\rangle$ is also a rationally additive semiring.*

Proof. We only show that $\mathbf{Ax}_2$ and $\mathbf{Ax}_3$ hold in $S\langle\langle A^*\rangle\rangle$. So suppose that $r \in S\langle\langle A^*\rangle\rangle$. We clearly have that

$$\sum_{n=0}^{\infty}(r^n, \epsilon) = \sum_{n=0}^{\infty}(r, \epsilon)^n = (r, \epsilon)^*.$$

Suppose now that $u \neq \epsilon$. Then $(r^n, u) = \sum_{u_1 \dots u_n = u}(r, u_1)\dots(r, u_n)$. Thus, by $\mathbf{Ax}_5$, $\sum_{n=0}^{\infty}(r^n, u)$ exists if the sum $\sum_{u_1\dots u_n=u,\ n\geq 0}(r, u_1)\dots(r, u_n)$ does. But this latter sum indeed exists. This can be seen as follows. For each fixed $u_1, \dots, u_k \neq \epsilon$ with $u_1 \dots u_k = u$,

$$\sum_{m_0,\dots,m_k\geq 0}(r, \epsilon)^{m_0}(r, u_1)\dots(r, u_k)(r, \epsilon)^{m_k} = (r, \epsilon)^*(r, u_1)\dots(r, u_k)(r, \epsilon)^*$$

exists. Since $\sum_{u_1\dots u_n=u}(r, u_1)\dots(r, u_n)$ is just a finite sum of sums of this form, it follows by $\mathbf{Ax}_4$ that this sum also exists. Again, only a weak form of $\mathbf{Ax}_5$ has been used. □

The following fact is clear.

Proposition 12. *Suppose that S is a (weak) rationally additive semiring and S' is a subsemiring of S which is closed under *. Say that $\sum_{i\in I} s_i$ exists in S', where $s_i \in S'$ for all $i \in I$, if $\sum_{i\in I} s_i$ exists in S and belongs to S', and in that case, let the sum in S' be the same as in S. Then S' is also a (weak) rationally additive semiring.*

When S is a *-semiring and $B \subseteq S$, the *B-rational elements* of S are those contained in the *-semiring generated by B. Thus the B-rational elements form a *-semiring denoted $Rat_S(B)$, or just $Rat(B)$. Let S be rationally additive and let A be a set. Then, as shown above, $S\langle\langle A^*\rangle\rangle$ is also rationally additive and each $a \in A$ and $s \in S$ can be conveniently identified with a series in $S\langle\langle A^*\rangle\rangle$. We denote $S^{\mathrm{rat}}\langle\langle A^*\rangle\rangle = Rat_{S\langle\langle A^*\rangle\rangle}(A \cup S)$.

The countably additive semiring N_∞ was defined above.

Proposition 13. *Suppose that S is rationally additive. Then there is a unique morphism $N_\infty \to S$.*

Proof. Clearly, any morphism $h : N_\infty \to S$ is forced to map each integer n to the n-fold sum of 1 with itself and ∞ to 1^*. The fact that this function is in turn a morphism will follow by Remark 3 once we prove that for any countably infinite family $n_i,\ i \in I$ of nonzero elements of N_∞, the sum $\sum_{i\in I} n_i h$ exists in S and equals 1^*. But this follows by Proposition 3. □

Corollary 14. *There exits no rationally additive semiring structure on N_∞ properly included in the standard structure.*

Theorem 15. *For each set A, $N_\infty^{\mathrm{rat}}\langle\langle A^*\rangle\rangle$ is freely generated by A in the class of rationally additive semirings.*

Proof. We need to show that if S is a rationally additive semiring and h is a function $A \to S$, then h has a unique extension to a morphism $h^\sharp : N_\infty^{\mathrm{rat}}\langle\langle A^*\rangle\rangle \to S$ of rationally additive semirings. Suppose that $r \in N_\infty^{\mathrm{rat}}\langle\langle A^*\rangle\rangle$. We are forced to define

$$rh^\sharp = \sum_{(r,u)\neq 0} (r,u)uh, \tag{6}$$

where for any word $u = a_1 \dots a_n \in A^*$ of length n, we define $uh = (a_1h) \cdot \ldots \cdot (a_nh)$. Note that the coefficient (r,u) of uh in (6) is taken in S. This is meaningful, since to each integer n there corresponds in S the n-fold sum of 1 with itself, and to ∞ the element 1^*. See Proposition 13.

In a natural way, we may extend h to a function $N_\infty\langle A\rangle \to S$, and then to a function $(N_\infty\langle A\rangle)^{n\times n} \to S^{n\times n}$, for each $n \geq 0$. For each $n \in N_\infty$ and $a \in A$ we define $(na)h = n(ah)$. For a finite sum $\sum_{i\in F} n_i a_i$, we define $(\sum_{i\in F} n_i a_i)h = \sum_{i\in F} n_i(a_ih)$.

We must show that the sum on the right-hand side of (6) exists. Since r is rational, by (a generalization of) Schützenberger's theorem [see Bloom, Ésik 1993b], there exists $\alpha \in N_\infty^{1\times n}$, $M \in N_\infty\langle A\rangle^{n\times n}$ and $\beta \in N_\infty^{n\times 1}$ with $r = \alpha M^*\beta$. Now, by Theorem 9 and Propositions 7 and 8, we have that $\alpha(Mh)^*\beta = \sum_{k=0}^\infty \alpha(Mh)^k\beta$ exists. But for each k,

$$\alpha(Mh)^k\beta = \sum_{|u|=k,\ (r,u)\neq 0} (r,u)uh.$$

Thus, by $\mathbf{Ax}_4$, the right-hand side of (6) exists and equals $\alpha(Mh)^*\beta$.

Note that for any finite set $B \subseteq A$ such that $u \in B^*$ holds for all words $u \in A^*$ with $(r,u) \neq 0$, i.e., such that $supp(r) \subseteq B^*$, we have that $rh^\sharp = \sum_{u\in B^*}(r,u)uh$. We use this fact in our proof that $h^\sharp$ preserves all existing sums. Suppose that $r_i \in N_\infty^{\mathrm{rat}}\langle\langle A^*\rangle\rangle$, $i \in I$ such that $\sum_{i\in I} r_i$ exists in $N_\infty^{\mathrm{rat}}\langle\langle A^*\rangle\rangle$, so that $\sum_{i\in I} r_i$ is rational. Since $r = \sum_{i\in I} r_i$ is rational, there is a finite set $B \subseteq A$ with $supp(r) \subseteq B^*$. Clearly then, $supp(r_i) \subseteq B^*$ for all $i \in I$. By $\mathbf{Ax}_4$ and $\mathbf{Ax}_5$, the fact that $\sum_{i\in I} r_ih^\sharp$ exists and equals $rh^\sharp$ will follow if we can show that the sum $\sum_{u\in B^*,\ i\in I}(r_i,u)uh$ exists and equals $rh^\sharp$. This in turn will hold if for each fixed $u \in B^*$,

$$\sum_{i\in I}(r_i,u)uh = (\sum_{i\in I}(r_i,u))uh$$

exists and is equal to $(r,u)uh$. But by Proposition 13, the sum $\sum_{i\in I}(r_i,u)$ exists in S, and equals (r,u). □

Corollary 16. *There is no rationally additive structure on $N_\infty^{\mathrm{rat}}\langle\langle A^*\rangle\rangle$ properly contained in the rationally additive structure inherited from the countably additive structure on $N_\infty\langle\langle A^*\rangle\rangle$.*

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