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On Quasi-Products of Tree Automata[1]

Ferenc Gécseg
(University of Szeged, Hungary [2]
Email: gecseg@inf.u-szeged.hu)

Abstract: In this paper we introduce the concept of the quasi-product of tree automata. In a quasi-product the inputs of the component tree automata are operational symbols in which permutation and unification of variables are allowed. It is shown that in sets of tree automata which are homomorphically complete with respect to the quasi-product the essentially unary operations play the basic role among all operations with nonzero ranks. Furthermore, we give a characterization of homomorphically complete sets which is similar to the classical one.
Key Words: tree automata, products, complete sets
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1 Introduction

The concept of products of automata has been introduced by V. M. Gluškov as an abstract model of electronic circuits (see, [Gluškov 1961]). In the general form of the product all the component automata are fed back to each other. To decrease the feed-back complexity, several special forms of the product have been introduced. One of them is the loop-free product given by J. Hartmanis in [Hartmanis 1962]. In a loop free product we have only steering, there is no feed-back. A celebrated structure theory for loop-free products has been developed by K. B. Krohn and J. L. Rhodes. A nice presentation of the Krohn-Rhodes theory can be found in [Ginzburg 1968]. It is shown in [Gécseg 1965] that there is no finite set of finite automata which is complete in the sense that every automaton can be represented homomorphically by a loop-free product of automata from this set. To find products simpler than the general product under which there are finite homomorphically complete sets, in [Gécseg 1974] we introduced a hierarchy of products called α_i-products, where i runs over the set of non-negative integers. In an α_i-product the set of component automata is linearly ordered and to the input of the ith component at most the next $i-1$ components are fed-back. The first member of the hierarchy is equivalent to the loop-free product. Thus, there is no finite homomorphically complete set for the α_0-product. It is easy to show that this is true also for the α_1-product. It was Z. Ésik who showed the existence of finite homomorphically complete sets for the α_2-product (see, [Ésik 1985]). Based on this result, in [Ésik and Horváth 1983] Z. Ésik and Gy. Horváth proved that the α_2-product is equivalent to the general product from the point of view of homomorphic representation. The system of the μ_i-products introduced by P.

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Dömösi and B. Imreh in [Dömösi and Imreh 1983] is another comprehensively studied hierarchy of products. In a μ_i-product, where i is a non-negative integer, to the input of a component automaton at most i component automata may be fed back. In [Gécseg and Jürgensen 1991] the power of α_0-products, μ_1-products, and $\alpha_0 - \nu_1$-products are compared. Moreover, in [Gécseg and Jürgensen 1990] the computational power of soliton automata is studied by products of automata.

In [Steinby 1977] Magnus Steinby introduced the concept of products of tree automata as a generalization of the Gluškov-type product of finite state automata. In the same paper he shows that the characterization of isomorphically complete sets of ordinary automata can be carried over to tree automata in a natural way. This is not so in the case of homomorphic completeness. For example, in the classical case every homomorphically complete set of automata contains a single automaton which itself is homomorphically complete. On the other hand, depending on the rank type of the considered tree automata, for every natural number k, there exists a k-element set of tree automata which is homomorphically complete and minimal (see [Gécseg 1994]). One reason for this behavior is that by the definition of the product of tree automata no operation symbol can be replaced by another one if they have different ranks even in the case when both of them depend on the same variables. To remedy it, in this paper we introduce the quasi-product which is slightly more general than the product given in [Steinby 1977].

Finally, we note that the tree automata to be considered here are the so-called frontier-to-root tree automata processing a tree from the frontier towards the root. We shall not deal with systems working in the opposite direction which are called root-to-frontier tree automata. Results concerning completeness of root-to-frontier tree automata can be found in [Virágh 1983].

2 Notions and notations

Sets of operational symbols will be denoted by Σ with or without superscripts. If Σ is finite, then it is called a *ranked alphabet*. For the subset of Σ consisting of all m-ary operational symbols from Σ we shall use the notation Σ_m $(m \geq 0)$. By a *Σ-algebra* we mean a pair $\mathcal{A} = (A, \{\sigma^{\mathcal{A}} | \sigma \in \Sigma\})$, where $\sigma^{\mathcal{A}}$ is an m-ary operation on A if $\sigma \in \Sigma_m$. If there will be no danger of confusion then we omit the superscript $\mathcal{A}$ in $\sigma^{\mathcal{A}}$ and simply write $\mathcal{A} = (A, \Sigma)$. Finally, all algebras considered in this paper will be finite, i.e., A is finite and Σ is a ranked alphabet.

A *rank type* is a non void set R of nonnegative integers. A ranked alphabet Σ is of *rank type* R if $\{m \mid \Sigma_m \neq \emptyset\} \subseteq R$. An algebra $\mathcal{A} = (A, \Sigma)$ is of *rank type* R if Σ is of rank type R. Let us note that in [Steinby 1977] the concept of the rank type is used in a stronger sense. Namely, there an algebra $\mathcal{A} = (A, \Sigma)$ is of rank type R if $\{m | \Sigma_m \neq \emptyset\} = R$. Under the definition of the quasi-product such a strong concept of rank type will not be needed.

Let Ξ be a set of variables. The set $T_\Sigma(\Xi)$ of *$\Sigma\Xi$-trees* is defined as follows:

(i) $\Xi \subseteq T_\Sigma(\Xi)$,
(ii) $\sigma(p_1, \ldots, p_m) \in T_\Sigma(\Xi)$ whenever $m \geq 0$, $\sigma \in \Sigma_m$ and $p_1, \ldots, p_m \in T_\Sigma(\Xi)$, and
(iii) every $\Sigma\Xi$-tree can be obtained by applying the rules (i) and (ii) a finite number of times.

In the sequel Ξ will stand for the countable set $\{\xi_1, \xi_2, \ldots\}$ and for every $m \geq 0$, Ξ_m will denote the subset $\{\xi_1, \ldots, \xi_m\}$ of Ξ.

If $p \in T_\Sigma(\Xi_l)$ and $p_1, \ldots, p_l \in T_\Sigma(\Xi_m)$ are trees, then $p(p_1, \ldots, p_l) \in T_\Sigma(\Xi_m)$ is the tree obtained by replacing each occurrence of ξ_i $(i = 1, \ldots, l)$ in p by p_i.

A tree $p \in T_\Sigma(\Xi_1)$ is *completely balanced* if

1) all paths leading from the root of p to leaves of p are of the same length, and
2) if p_1 and p_2 are subtrees of p with the same height, then $p_1 = p_2$.

The set of all completely balanced trees from $T_\Sigma(\Xi_1)$ will be denoted by $\hat{T}_\Sigma(\Xi_1)$.

The basic part of a tree recognizer or (deterministic) tree transducer is an algebra: both types of systems are built on algebras. From the point of view of the structure theory of tree automata, important classes of algebras are those which are complete by one of the following definitions.

A class $\mathcal{K}$ of algebras of rank type R is *homomorphically complete* if for every algebra $\mathcal{A}$ of rank type R there is an algebra $\mathcal{B}$ in $\mathcal{K}$ such that $\mathcal{A}$ is a homomorphic image of a subalgebra of $\mathcal{B}$.

A class $\mathcal{K}$ of algebras of rank type R is *forest complete* if for every forest T recognizable by a tree recognizer built on an algebra of rank type R there is an algebra $\mathcal{A}$ in $\mathcal{K}$ such that T can be recognized by a tree recognizer built on $\mathcal{A}$.

A class $\mathcal{K}$ of algebras of rank type R is *transformation complete* if for every tree transformation τ induced by a deterministic frontier-to-root tree transducer built on an algebra of rank type R there is an algebra $\mathcal{A}$ in $\mathcal{K}$ such that τ can be induced by a tree transducer built on $\mathcal{A}$.

We say that a class $\mathcal{K}$ of algebras is *closed under deletion* of operations, if for all $\mathcal{A} = (A, \Sigma) \in \mathcal{K}$ with $|\Sigma| \geq 2$ and $\sigma \in \Sigma$, the algebra $\mathcal{A}' = (A, \Sigma')$, where $\Sigma'_m = \Sigma_m \setminus \{\sigma\}$ for all $m \geq 0$ and $\bar{\sigma}^{\mathcal{A}'}(a_1, \ldots, a_l) = \bar{\sigma}^{\mathcal{A}}(a_1, \ldots, a_l)$ $(\bar{\sigma} \in \Sigma'_l,\ l \geq 0,\ a_1, \ldots, a_l \in A)$, is also in $\mathcal{K}$.

It can be seen from the proof of Theorem 7 in [Gécseg 1994] that for every class of algebras which is closed under the deletion of operations the three conditions of completeness coincide. Since classes of algebras obtained from given sets of algebras by means of quasi-product are closed under the deletion of operations, in this paper we shall restrict ourselves to homomorphic completeness.

Let R be a rank type and take the algebras $\mathcal{A}_i = (A_i, \Sigma_i)$ $(i = 1, \ldots, k > 0)$ with rank type R and let

$$\varphi = \{\varphi_m : (A_1 \times \ldots \times A_k)^m \times \Sigma_m \to T_{\Sigma_1}(\Xi_m) \times \ldots \times T_{\Sigma_k}(\Xi_m) \mid \sigma \in \Sigma_m,\ m \in R\}$$

be a family of mappings, where Σ is an arbitrary ranked alphabet of rank type R. Then by the *generalized product* of $\mathcal{A}_1, \ldots, \mathcal{A}_k$ with respect to Σ and φ we mean the algebra $\mathcal{A} = (A, \Sigma)$ with $A = A_1 \times \ldots \times A_k$ such that for arbitrary $\sigma \in \Sigma_m$ $(m \in R)$ and $(a_{11}, \ldots, a_{1k}), \ldots, (a_{m1}, \ldots, a_{mk}) \in A$,

$$\sigma^{\mathcal{A}}((a_{11}, \ldots, a_{1k}), \ldots, (a_{m1}, \ldots, a_{mk})) =$$

$$(p_1^{\mathcal{A}_1}(a_{11}, \ldots, a_{m1}), \ldots, p_k^{\mathcal{A}_k}(a_{1k}, \ldots, a_{mk})),$$

where $(p_1, \ldots, p_k) = \varphi_m((a_{11}, \ldots, a_{1k}), \ldots, (a_{m1}, \ldots, a_{mk}), \sigma)$. For this generalized product we use the notation $\prod_{i=1}^{k} \mathcal{A}_i[\Sigma, \varphi]$.

A similar generalization of the cascade product can be found in [Ésik 1996].

Let us extend φ_m to a mapping from $(A_1 \times \ldots \times A_k)^m \times T_\Sigma(\Xi_m)$ to $T_{\Sigma_1}(\Xi_m) \times \ldots \times T_{\Sigma_k}(\Xi_m)$ in the following way: for arbitrary $\mathbf{a} \in A^m$ and $p \in T_\Sigma(\Xi_m)$

(i) if $p = x_j$ $(1 \leq j \leq m)$, the $\varphi_m(\mathbf{a}, x_j) = (x_j, \ldots, x_j)$,
(ii) if $p = \sigma(p_1, \ldots, p_l)$ $(\sigma \in \Sigma_l,\ l \geq 0)$, then

$$\varphi_m(\mathbf{a}, p) =$$

$$(q_1(\pi_1(\varphi_m(\mathbf{a}, p_1)), \ldots, \pi_1(\varphi_m(\mathbf{a}, p_l))), \ldots,$$
$$q_k(\pi_k(\varphi_m(\mathbf{a}, p_1)), \ldots, \pi_k(\varphi_m(\mathbf{a}, p_l))))$$

where $(q_1, \ldots, q_k) = \varphi_l(p_1(\mathbf{a}), \ldots, p_l(\mathbf{a}), \sigma)$ and π_j is the jth projection.

3 Quasi-products

A generalized product $\mathcal{A} = (A, \Sigma) = \prod_{i=1}^k \mathcal{A}_i[\Sigma, \varphi]$, where $\mathcal{A}_i = (A_i, \Sigma^{(i)})$ $(i = 1, \ldots, k)$, is called a *quasi-product* if for $\varphi_m(\mathbf{a}_1, \ldots, \mathbf{a}_m, \sigma) = (p_1, \ldots, p_k)$ $(\mathbf{a}_1, \ldots, \mathbf{a}_m \in A_1 \times \ldots \times A_k,\ \sigma \in \Sigma_m,\ m \geq 0)$ we have $p_i = \sigma_i(\xi_{i_1}, \ldots, \xi_{i_{m_i}})$ $(1 \leq i_1, \ldots, i_m \leq m)$ and $\sigma_i \in \Sigma^{(i)}_{m_i}$ $(i = 1, \ldots, k)$. If in this quasi-product $\mathcal{A}_1 = \ldots = \mathcal{A}_k = \mathcal{B}$ then we speak of a *quasi-power* of $\mathcal{B}$, and write $\mathcal{B}^k[\Sigma, \varphi]$. Let us note that the above quasi-product is called a *(general) product* if $m_i = m$ and $\xi_{i_j} = \xi_j$ for all $i = 1, \ldots, k$ and $j = 1, \ldots, m$.

The cascade product introduced by G. Ricci in [Ricci 1973] is a special quasi-product.

The next lemma gives necessary conditions for a class of algebras to be homomorphically complete.

Lemma 1. *Let R be a rank type with $0 \notin R$. If a class $\mathcal{K}$ of algebras of rank type R is homomorphically complete, then there exist an algebra $\mathcal{A} = (A, \Sigma) \in \mathcal{K}$, an element $a_0 \in A$, a natural number $l \in R$, two operational symbols $\sigma_1, \sigma_2 \in \Sigma_l$, and two completely balanced trees $p, q \in \hat{T}_\Sigma(\Xi_1)$ such that $\sigma_1(a_0, \ldots, a_0) \neq \sigma_2(a_0, \ldots, a_0)$ and $p(\sigma_1(a_0, \ldots, a_0)) = q(\sigma_2(a_0, \ldots, a_0)) = a_0$.*

Proof. Let $\mathcal{B} = (B, \Sigma)$ be an algebra with $\Sigma = \Sigma_l = \{\sigma_1, \sigma_2\}$ for an $l \in R$, where $B = \{0, 1\}$, $\sigma_1^{\mathcal{B}}(0, \ldots, 0) = 1$, $\sigma_1^{\mathcal{B}}(1, \ldots, 1) = 0$, $\sigma_2^{\mathcal{B}}(0, \ldots, 0) = 0$ and $\sigma_2^{\mathcal{B}}(1, \ldots, 1) = 1$, and in all other cases $\sigma_i^{\mathcal{B}}(b_1, \ldots, b_l)$ $(i = 1, 2,\ b_1, \ldots, b_l \in B)$ is given arbitrarily. Let $\mathcal{A} = (A, \Sigma) \in \mathcal{K}$ be an algebra such that a subalgebra $\mathcal{A}' = (A', \Sigma)$ of $\mathcal{A}$ can be mapped homomorphically onto $\mathcal{B}$ under a homomorphism τ. Let a'_0 be a counter image of 0 under τ. Now consider the following subsets A_i $(i = 1, 2, \ldots)$. Set $A_1 = \{p(a'_0) \mid p \in \hat{T}_\Sigma(\Xi_1),\ h(p) > 0\}$. Assume that A_{i-1} $(i > 1)$ has been defined. Then let $A_i = \{p(a) \mid p \in \hat{T}_\Sigma(\Xi_1),\ h(p) > 0\}$, for an arbitrarily chosen $a \in A_{i-1}$ if $\{p(a) \mid p \in \hat{T}_\Sigma(\Xi_1),\ h(p) > 0\}$ is a proper subset of A_{i-1}. If there is no such $a_i \in A_{i-1}$, then $A_i = A_{i-1}$. Since $A_{i+1} \subseteq A_i$ $(i = 1, 2, \ldots)$, there is a j such that $A_j = A_{j+1}$. This j has the property that $A_j = \{p(a) \mid p \in \hat{T}_\Sigma(\Xi_1),\ h(p) > 0\}$ for each $a \in A_j$. Furthermore, by the choice of $\mathcal{B}$, $\tau(A_j) = B$. Let $a_0 \in \tau^{-1}(0) \cap A_j$ be arbitrary. Since $\sigma_1^{\mathcal{B}}(0, \ldots, 0) \neq \sigma_2^{\mathcal{B}}(0, \ldots, 0)$, thus $\sigma_1^{\mathcal{A}}(a_0, \ldots, a_0) \neq \sigma_2^{\mathcal{A}}(a_0, \ldots, a_0)$, and both $\sigma_1^{\mathcal{A}}(a_0, \ldots, a_0)$ and $\sigma_2^{\mathcal{A}}(a_0, \ldots, a_0)$ are in A_j. Therefore, by the above property of A_j, there are

$p, q \in \hat{T}_\Sigma(\Xi_1)$ with $p^{\mathcal{A}}(\sigma_1^{\mathcal{A}}(a_0, \ldots, a_0)) = q^{\mathcal{A}}(\sigma_2^{\mathcal{A}}(a_0, \ldots, a_0)) = a_0$, which ends the proof of the lemma.

A class $\mathcal{K}$ of algebras of rank type R is called *homomorphically complete for R with respect to the quasi-product* if the class of all quasi-products of algebras from $\mathcal{K}$ is homomorphically complete.

The following result is from [Letičevskiĭ1961] (see, also [Gécseg 1986]).

Theorem 2. *Let $R = \{1\}$. A set $\mathcal{K}$ of algebras of rank type R is homomorphically complete for R with respect to the quasi-product if and only if there are an $\mathcal{A} = (A, \Sigma) \in \mathcal{K}$, an element $a_0 \in A$, two operational symbols $\sigma_1, \sigma_2 \in \Sigma (= \Sigma_1)$, and two trees $p_1, p_2 \in F_\Sigma(\Xi_1)$ such that $\sigma_1(a_0) \neq \sigma_2(a_0)$ and $p_1(\sigma_1(a_0)) = p_2(\sigma_2(a_0)) = a_0$.*

¿From this theorem we directly obtain

Lemma 3. *Let R be a rank type with $R \neq \{0\}$. Moreover, let $\mathcal{K}$ be a set of algebras of rank type R such that there are an algebra $\mathcal{A} = (A, \Sigma) \in \mathcal{K}$, an element $a_0 \in A$, two natural numbers $l_1, l_2 \in R$, two operational symbols $\sigma_1 \in \Sigma_{l_1}$, $\sigma_1' \in \Sigma_{l_2}$, and two completely balanced trees $p, q \in \hat{T}_\Sigma(\Xi_1)$ satisfying $\sigma_1(a_0, \ldots, a_0) \neq \sigma_1'(a_0, \ldots, a_0)$ and $p(\sigma_1(a_0, \ldots, a_0)) = q(\sigma_1'(a_0, \ldots, a_0)) = a_0$. Take the algebra $\mathcal{B} = (\{0, 1\}, \Sigma)$ with $\Sigma' = \Sigma_1' = \{\sigma, \sigma'\}$ defined by*

$$\sigma(a) = \begin{cases} 0, \text{ if } a = 0, \\ 1, \text{ if } a = 1 \end{cases}$$

and

$$\sigma'(a) = \begin{cases} 1, \text{ if } a = 0, \\ 0, \text{ if } a = 1 \end{cases}$$

for all $a \in \{0, 1\}$. Then $\mathcal{B}$ is a homomorphic image of a subalgebra of a quasi-power of $\mathcal{A}$.

Lemma 4. *Let $\mathcal{B}$ be the algebra given in Lemma 3. Then for every algebra $\mathcal{A} = (A, \Sigma)$ of rank type R with $0 \notin R$ there is a quasi-power $\mathcal{C} = (C, \Sigma)$ of $\mathcal{B}$ such that $\mathcal{A}$ is isomorphic to a subalgebra of $\mathcal{C}$.*

Proof. Let $k \geq \log_2 |A|$ be a natural number and take an arbitrarily fixed one-to-one mapping τ of A into the Cartesian power $\{0, 1\}^k$. Form the quasi-power $\mathcal{C} = (C, \Sigma) = \mathcal{B}^k[\Sigma, \varphi]$, where φ is given in the following way. Take a $\bar{\sigma} \in \Sigma_m$ ($m \in R$) and $a_1, \ldots, a_m \in A$. Let $\tau(a_i) = (c_{i1}, \ldots, c_{ik})$ ($i = 1, \ldots, m$) and $\tau(\bar{\sigma}(a_1, \ldots, a_m)) = (c_1, \ldots, c_k)$. Then

$$\pi_i(\varphi_m(\tau(a_1), \ldots, \tau(a_m), \bar{\sigma})) = \begin{cases} \sigma(\xi_1) \text{ if } c_{1i} = c_i = 0 \text{ or} \\ \qquad c_{1i} = c_i = 1, \\ \sigma'(\xi_1) \text{ if } c_{1i} = 0 \& c_i = 1 \text{ or} \\ \qquad c_{1i} = 1 \& c_i = 0 \end{cases}$$

for all $i (= 1, \ldots, k)$. In all other cases φ is given arbitrarily. It is clear that $\tau(A)$ forms a subalgebra of $\mathcal{C}$ which is isomorphic to $\mathcal{A}$ under τ.

The following example will be needed later.

Example 1. Let $R = \{0, 1\}$ be a rank type. Take the algebra

$$\mathcal{B} = (\{0, 1\}, \Sigma')$$

with $\Sigma' = \Sigma'_0 \cup \Sigma'_1$, $\Sigma'_0 = \{\sigma_0, \sigma'_0\}$, $\Sigma'_1 = \{\sigma, \sigma'\}$ defined by $\sigma_0^{\mathcal{B}} = 0$, ${\sigma'_0}^{\mathcal{B}} = 1$,

$$\sigma(a) = \begin{cases} 0, \text{ if } a = 0, \\ 1, \text{ if } a = 1 \end{cases}$$

and

$$\sigma'(a) = \begin{cases} 1, \text{ if } a = 0, \\ 0, \text{ if } a = 1. \end{cases}$$

Then $\mathcal{B}$ is homomorphically complete for R with respect to the quasi-product. For this it is enough to show that for every algebra $\mathcal{A} = (A, \Sigma)$ of rank type R there is a quasi- power $\mathcal{C} = (C, \Sigma)$ of $\mathcal{B}$ such that $\mathcal{A}$ is isomorphic to a subalgebra of $\mathcal{C}$. This can be shown in the same way as in the proof of Lemma 4 with the additional condition that if $\bar{\sigma} \in \Sigma_0$ and $\tau(\bar{\sigma}^{\mathcal{A}}) = (c_1, \ldots, c_k)$, then

$$\pi_i(\varphi_0(\bar{\sigma})) = \begin{cases} \sigma_0 \text{ if } c_i = 0, \\ \sigma'_0 \text{ if } c_i = 1. \end{cases}$$

Next we give necessary and sufficient conditions for a set of algebras of rank type R to be homomorphically complete for R with respect to the quasi-product, under the condition that $0 \notin R$.

Theorem 5. *Let R be a rank type with $0 \notin R$. A set $\mathcal{K}$ of algebras of rank type R is homomorphically complete for R with respect to the quasi-product if and only if there exist an algebra $\mathcal{B} = (B, \Sigma') \in \mathcal{K}$, an element $b_0 \in B$, two operational symbols $\sigma_1 \in \Sigma'_{l_1}$, $\sigma_2 \in \Sigma'_{l_2}$ and two completely balanced trees $p', q' \in \hat{T}_{\Sigma'}(\Xi_1)$ such that $\sigma'_1(b_0, \ldots, b_0) \neq \sigma'_2(b_0, \ldots, b_0)$ and $p'(\sigma'_1(b_0, \ldots, b_0)) = q'(\sigma'_2(b_0, \ldots, b_0)) = b_0$.*

Proof. Assume that $\mathcal{K}$ is homomorphically complete for R with respect to the quasi-product. Then, by Lemma 1, there exist a quasi-product

$$\mathcal{A} = (A, \Sigma) = \prod_{i=1}^{k} \mathcal{B}_i[\Sigma, \varphi]$$

with $\mathcal{B}_i = (B_i, \Sigma^{(i)}) \in \mathcal{K}$ $(i = 1, \ldots, k)$, an element $\mathbf{a}_0 \in A$, an $l \in R$, two operational symbols $\sigma_1, \sigma_2 \in \Sigma_l$ and two trees $p, q \in \hat{T}_{\Sigma}(\Xi_1)$ such that $\sigma_1(\mathbf{a}_0, \ldots, \mathbf{a}_0) \neq \sigma_2(\mathbf{a}_0, \ldots, \mathbf{a}_0)$ and $p(\sigma_1(\mathbf{a}_0, \ldots, \mathbf{a}_0)) = q(\sigma_2(\mathbf{a}_0, \ldots, \mathbf{a}_0)) = \mathbf{a}_0$. Let $\varphi_l(\mathbf{a}_0, \ldots, \mathbf{a}_0, \sigma_j) = (\sigma_{1j}(\xi_{1j1}, \ldots, \xi_{1jl_{1j}}), \ldots, \sigma_{kj}(\xi_{kj1}, \ldots, \xi_{kjl_{kj}}))$ $(j = 1, 2)$, $\mathbf{a}_0 = (b_{01}, \ldots, b_{0k})$, and $\sigma_j(\mathbf{a}_0, \ldots, \mathbf{a}_0) = \mathbf{a}_j = (b_{j1}, \ldots, b_{jk})$ $(j = 1, 2)$. Moreover, let $\varphi_1(\mathbf{a}_1, p) = (p_1, \ldots, p_k)$ and $\varphi_1(\mathbf{a}_2, q) = (q_1, \ldots, q_k)$. Since $\mathbf{a}_1 \neq \mathbf{a}_2$, there is an i $(1 \leq i \leq k)$ such that $b_{1i} \neq b_{2i}$. It can be easily seen that $\mathcal{B}_i$ with b_{0i}, σ_{i1}, σ_{i2}, $p_i(\xi_1, \ldots, \xi_1)$ and $q_i(\xi_1, \ldots, \xi_1)$ satisfies the conditions of our theorem.

Conversely, assume that $\mathcal{K}$ satisfies the conditions of the theorem. Since the formation of the quasi-product is transitive, by Lemmas 3 and 4, $\mathcal{K}$ is homomorphically complete for R with respect to the quasi-product.

¿From Theorem 5 we directly obtain

Corollary 6. *Let R be a rank type with $0 \notin R$. If a set $\mathcal{K}$ of algebras of rank type R is homomorphically complete for R with respect to the quasi-product, then $\mathcal{K}$ contains an algebra $\mathcal{A}$ such that $\mathcal{A}$ itself is homomorphically complete for R with respect to the quasi product.*

Furthermore, by the proof of Theorem 5, we also have

Corollary 7. *Let R and R' be rank types such that $R \subseteq R'$ and $\emptyset \notin R'$. If a set $\mathcal{K}$ of algebras of rank type R is homomorphically complete for R with respect to the quasi-product, then $\mathcal{K}$ is homomorphically complete for R' with respect to the quasi-product.*

The next example shows that Corollary 6 is not valid if R contains 0.

Example 2. Let $R = \{0, 1\}$. Consider the algebras $\mathcal{A}_1 = (\{0,1\}, \Sigma^{(1)})$ and $\mathcal{A}_2 = (\{0,1,2\}, \Sigma^{(2)})$ with $\Sigma^{(1)} = \Sigma_0^{(1)} \cup \Sigma_1^{(1)}$, $\Sigma_0^{(1)} = \{\sigma_0, \sigma_0'\}$, $\Sigma_1^{(1)} = \{\sigma\}$, and $\Sigma^{(2)} = \Sigma_0^{(2)} \cup \Sigma_1^{(2)}$, $\Sigma_0^{(2)} = \{\sigma_0, \sigma_0'\}$, $\Sigma_1^{(2)} = \{\sigma, \sigma'\}$. Moreover, $\sigma_0^{\mathcal{A}_1} = 0$, $\sigma_0'^{\mathcal{A}_1} = 1$, $\sigma^{\mathcal{A}_1}(0) = 0$, $\sigma^{\mathcal{A}_1}(1) = 1$ and $\sigma_0^{\mathcal{A}_2} = \sigma_0'^{\mathcal{A}_2} = 2$, $\sigma'^{\mathcal{A}_2}(2) = 0$, $\sigma^{\mathcal{A}_2}(2) = 1$, $\sigma'^{\mathcal{A}_2}(0) = 1$, $\sigma'^{\mathcal{A}_2}(1) = 0$, $\sigma^{\mathcal{A}_2}(0) = 0$, $\sigma^{\mathcal{A}_2}(1) = 1$.

Take the product $\mathcal{A} = (A, \Sigma) = \mathcal{A}_1 \times \mathcal{A}_2[\Sigma, \varphi]$, where $\Sigma = \Sigma_0 \cup \Sigma_1$, $\Sigma_0 = \{\sigma_0, \sigma_0'\}$, $\Sigma_1 = \{\sigma, \sigma'\}$ and φ is given in the following way: $\varphi_0(\sigma_0) = (\sigma_0, \sigma_0)$, $\varphi_0(\sigma_0') = (\sigma_0', \sigma_0')$, $\varphi_1(0,2,\sigma) = (\sigma, \sigma')$, $\varphi_1(0,0,\sigma) = \varphi_1(1,0,\sigma) = \varphi_1(1,2,\sigma) = \varphi_1(1,1,\sigma) = \varphi_1(0,1,\sigma) = \varphi_1(0,2,\sigma') = (\sigma, \sigma)$, $\varphi_1(0,0,\sigma') = \varphi_1(1,0,\sigma') = \varphi_1(1,2,\sigma') = \varphi_1(1,1,\sigma') = \varphi_1(0,1,\sigma') = (\sigma, \sigma')$.

Finally, consider the mapping $\tau : A \to \{0,1\}$ given by $\tau((0,2)) = \tau((0,0)) = \tau((1,0)) = 0$ and $\tau((1,2)) = \tau((1,1)) = \tau((0,1)) = 1$. A straightforward computation shows that the algebra $\mathcal{B}$ of Example 1 is a homomorphic image of $\mathcal{A}$ under τ. Therefore, $\{\mathcal{A}_1, \mathcal{A}_2\}$ is homomorphically complete for R with respect to the quasi-product.

The algebra $\mathcal{A}_1$ itself does not form a homomorphically complete set for R with respect to the quasi-product since each unary operation in every quasi-power of $\mathcal{A}_1$ is idempotent. The algebra $\mathcal{A}_2$ is not homomorphically complete for R with respect to the quasi-product either, since different 0-ary operational symbols have the same realization in each quasi-power of $\mathcal{A}_2$. Therefore, $\{\mathcal{A}_1, \mathcal{A}_2\}$ is minimal.

Let us call a quasi-product $\mathcal{A} = (A, \Sigma) = \prod_{i=1}^k \mathcal{A}_i[\Sigma, \varphi]$ a *unification product* if for $\varphi_m(\mathbf{a}_1, \ldots, \mathbf{a}_m, \sigma) = (p_1, \ldots, p_k)$ ($\mathbf{a}_1, \ldots, \mathbf{a}_m \in A_1 \times \ldots \times A_k$, $\sigma \in \Sigma_m$, $m \geq 0$) we have $p_i = \sigma_i(\xi_{i'}, \ldots, \xi_{i'})$ ($1 \leq i' \leq m_i$) and $\sigma_i \in \Sigma_{m_i}^{(i)}$ ($i = 1, \ldots, k$), i.e. the rank of σ_i is arbitrary and all variables coincide in p_i ($i = 1, \ldots, k$). Moreover, let us say that a quasi-product $\mathcal{A} = (A, \Sigma) = \prod_{i=1}^k \mathcal{A}_i[\Sigma, \varphi]$ is a *permutation* product if for $\varphi_m(\mathbf{a}_1, \ldots, \mathbf{a}_m, \sigma) = (p_1, \ldots, p_k)$ ($\mathbf{a}_1, \ldots, \mathbf{a}_m \in A_1 \times \ldots \times A_k$, $\sigma \in \Sigma_m$, $m \geq 0$) we have $p_i = \sigma_i(\xi_{i_1}, \ldots, \xi_{i_m})$ ($1 \leq i_1, \ldots, i_m \leq m$), $\sigma_i \in \Sigma_m^{(i)}$ ($i = 1, \ldots, k$), and $i_1, \ldots, i_m$ is a permutation of $1, \ldots, m$. The concept of the *permutation power* is defined in a natural way. Moreover, a rank type R is *homogeneous* if $R = \{m\}$ for some $m > 0$. Let Q_1-product and Q_2-product mean any of the quasi-product, unification product, permutation product or general product. We say that for a rank type R the Q_1-product is *homomorphically more general* than the Q_2-product with respect to the homomorphic completeness, if the following two conditions are satisfied:

(1) If a set $\mathcal{K}$ of algebras of rank type R is homomorphically complete for R with respect to the Q_2-product then $\mathcal{K}$ is homomorphically complete for R with respect to the Q_1-product.
(2) There is a set $\mathcal{K}$ of algebras of rank type R which is homomorphically complete for R with respect to the Q_1-product and $\mathcal{K}$ is not homomorphically complete for R with respect to the Q_2-product.

The Q_1-product is *homomorphically equivalent* to the Q_2-product for a rank type R, if for any set $\mathcal{K}$ of algebras of rank type R, $\mathcal{K}$ is homomorphically complete for R with respect to the Q_1-product if and only if $\mathcal{K}$ is homomorphically complete for R with respect to the Q_2-product.

By the proof of Theorem 5 we have

Theorem 8. *For arbitrary rank type R with $0 \notin R$ the quasi-product is homomorphically equivalent to the unification product.*

Finally, we show that even for homogeneous rank types the unification product is homomorphically more general than the permutation product and the latter one is homomorphically more general than the product.

Theorem 9. *For all rank types $R = \{m\}$ with $m > 1$ the quasi-product is homomorphically more general than the permutation product.*

Proof. Take the algebra $\mathcal{A} = (A, \Sigma')$, where $A = \{0,1\}$, $\Sigma' = \Sigma'_m = \{\sigma'_1, \sigma'_2\}$, $\sigma'_1(0,\dots,0) = \sigma'_1(1,\dots,1) = \sigma'_2(1,\dots,1) = 0$, $\sigma'_2(0,\dots,0) = 1$ and for all $a_1,\dots,a_m \in A$, $\sigma'_i(a_1,\dots,a_m) = 1$ $(i = 1,2)$ if there are indices $1 \le k < l \le m$ such that $a_k \ne a_l$. By Theorem 5, $\{\mathcal{A}\}$ is homomorphically complete for R with respect to the quasi-product. Moreover, let $\mathcal{I} = (I, \Sigma)$ be the algebra, where $I = \{0,1\}$, and $\sigma_1(a_1,\dots,a_m) = 1$ and $\sigma_2(a_1,\dots,a_m) = 0$ for all $a_1,\dots,a_m \in I$. Assume that a subalgebra $\mathcal{C} = (C, \Sigma)$ of a permutation-power $\mathcal{B} = \mathcal{A}^n[\Sigma, \varphi]$ can be mapped homomorphically onto $\mathcal{I}$. Let $\mathbf{a} \in C$ be an arbitrary element with maximal, say k, numbers of occurrences of 1. It is clear that $k > 0$. By the choice of $\mathcal{I}$, $\sigma_1(\sigma_1(\mathbf{a},\dots,\mathbf{a}),\mathbf{a},\dots,\mathbf{a}) \ne \sigma_2(\sigma_1(\mathbf{a},\dots,\mathbf{a}),\mathbf{a},\dots,\mathbf{a})$. Therefore, since $\sigma'_1(1,\dots,1) = \sigma'_2(1,\dots,1) = 0$ and $\sigma'_1(0,1,\dots,1) = \sigma'_2(0,1,\dots,1) = 1$, one of the vectors $\sigma_1(\sigma_1(\mathbf{a},\dots,\mathbf{a}),\mathbf{a},\dots,\mathbf{a})$ or $\sigma_2(\sigma_1(\mathbf{a},\dots,\mathbf{a}),\mathbf{a},\dots,\mathbf{a})$ must have at least $k+1$ occurrences of 1, which is a contradiction. Consequently, $\mathcal{A}$ is not homomorphically complete for R with respect to the permutation product.

Theorem 10. *For all rank types $R = \{m\}$ with $m > 1$ the permutation product is homomorphically more general than the product.*

Proof. Take the algebra $\mathcal{A} = (A, \Sigma')$, where $A = \{0,1\}$, $\Sigma' = \Sigma'_m = \{\sigma'_1, \sigma'_2\}$, $\sigma'_1(0,\dots,0) = \sigma'_2(1,\dots,1) = 0$, $\sigma'_1(1,\dots,1) = \sigma'_2(0,\dots,0) = 1$, and finally, $\sigma'_1(a_1,\dots,a_m) = \sigma'_2(a_1,\dots,a_m) = a_1$ if there are indices $1 \le k < l \le m$ with $a_k \ne a_l$. It can be easily shown that every algebra of rank type R is isomorphic to a subalgebra of a permutation power of $\mathcal{A}$. Therefore, $\{\mathcal{A}\}$ is homomorphically complete for R with respect to the permutation product. Assume that a subalgebra $\mathcal{C} = (C, \Sigma)$ of a power $\mathcal{B} = \mathcal{A}^n[\Sigma, \varphi]$ can be mapped homomorphically onto the algebra $\mathcal{I}$ given in the proof of the previous theorem. Let $\mathbf{a}, \mathbf{b} \in C$ be a pair such that they are different in maximal, say k, numbers of their components. By the construction of $\mathcal{I}$, $\sigma_1(\mathbf{a},\mathbf{b},\dots,\mathbf{b})$ and $\sigma_2(\mathbf{a},\mathbf{b},\dots,\mathbf{b})$ are different.

Moreover, they and **a** are equal in those components in which **a** and **b** differ. Therefore, **b** and $\sigma_1(\mathbf{a}, \mathbf{b}, \ldots, \mathbf{b})$ or **b** and $\sigma_2(\mathbf{a}, \mathbf{b}, \ldots, \mathbf{b})$ are different at least in $k+1$ components, which is a contradiction. Therefore, $\mathcal{A}$ is not homomorphically complete for R with respect to the product.

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