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On the Simplification of HD0L Power Series

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Abstract: Nielsen, Rozenberg, Salomaa and Skyum have shown that HD0L languages are CPDF0L languages. We will generalize this result for formal power series. We will also give a new proof of the result of Nielsen, Rozenberg, Salomaa and Skyum.

Key Words: Formal power series, Lindenmayer systems, D0L power series

Category: F.4.3

1 Introduction

Nielsen, Rozenberg, Salomaa and Skyum have shown that HD0L languages are CPDF0L languages (see [9]). In other words, the effect of an arbitrary morphism on an arbitrary D0L language can be reduced to a coding of a propagating D0L language with multiple axioms. This result is discussed in detail also in [7], where it is described as one of the most sophisticated results about the elimination of ε-rules in the theory of Lindenmayer systems.

In this paper we will generalize the result of [9] for power series. It has been shown in [2] that infinite D0L power series having coefficients in an arbitrary commutative semiring are CPDF0L power series. On the other hand, CPDF0L power series over the semiring of nonnegative integers are properly included in HD0L power series. Below we will show, however, that HD0L power series are CPDF0L power series if certain natural restrictions are posed on HD0L power series. As a byproduct we will see that there is a close connection between the result of [9] and the usual simplification of D0L systems by elementary morphisms (see [10]).

Results connecting HD0L power series with CPDF0L power series might turn out to be very useful in studying the equivalence problem of HD0L power series. It has recently been shown that we can cope with multiple axioms (see [5]). At present there are no methods to deal simultaneously with codings and multiple axioms.

It is assumed that the reader is familiar with the basics concerning formal power series and L systems (see [1, 8, 10, 11, 12]). Notions and notations that are not explained are taken from these references. For further background and motivation we refer to [2, 3, 4] and their references.

2 Definitions and results

In what follows A will always be a commutative semiring. We assume also that either A is a subsemiring of a field or A is the Boolean semiring. Let X be an alphabet. The set of *formal power series with noncommuting variables* in X and coefficients in A is denoted by $A\langle\langle X^*\rangle\rangle$. The subset of $A\langle\langle X^*\rangle\rangle$ consisting of *polynomials* is denoted by $A\langle X^*\rangle$. If $a \in A$ is nonzero and $w \in X^*$, the *length* of aw equals by definition the length of w. In symbols,

$$|aw| = |w|.$$

Let X and Y be finite alphabets. A semialgebra morphism $h : A\langle X^*\rangle \longrightarrow A\langle Y^*\rangle$ is called a *monomial morphism* if for each $x \in X$ there exist a nonzero $a \in A$ and $w \in Y^*$ such that $h(x) = aw$. A monomial morphism $h : A\langle X^*\rangle \longrightarrow A\langle Y^*\rangle$ is called *nonerasing* (or *propagating*) if $h(x)$ is quasiregular for all $x \in X$. (Recall that a series r is called *quasiregular* if $(r, \varepsilon) = 0$.) A monomial morphism $h : A\langle X^*\rangle \longrightarrow A\langle Y^*\rangle$ is called a *coding* if

$$|h(x)| = 1$$

for all $x \in X$.

A series $r \in A\langle\langle Y^*\rangle\rangle$ is called an *HD0L power series* if there exist monomial morphisms $g : A\langle X^*\rangle \longrightarrow A\langle X^*\rangle$, $h : A\langle X^*\rangle \longrightarrow A\langle Y^*\rangle$, a nonzero $a \in A$ and a word $w \in X^*$ such that

$$r = \sum_{n=0}^{\infty} hg^n(aw) \tag{1}$$

and, furthermore, the family

$$\{hg^n(aw) \mid n \geq 0\} \tag{2}$$

is locally finite. (Recall that, by definition, (2) is *locally finite* if for any $v \in Y^*$ there exist finitely many values of n such that $(hg^n(aw), v) \neq 0$.) If $X = Y$ and h is the identity, r is called a *D0L power series*. Finally, a series $r \in A\langle\langle Y^*\rangle\rangle$ is called a *CPDF0L power series* if there exist a nonerasing monomial morphism $g : A\langle X^*\rangle \longrightarrow A\langle X^*\rangle$, a coding $c : A\langle X^*\rangle \longrightarrow A\langle Y^*\rangle$ and a polynomial P such that

$$r = \sum_{n=0}^{\infty} cg^n(P) \tag{3}$$

and, furthermore, the family

$$\{cg^n(P) \mid n \geq 0\} \tag{4}$$

is locally finite.

The following results have been established in [2].

Theorem 1. *If r is a D0L power series such that the support of r is infinite, then r is a CPDF0L power series.*

Theorem 2. *Suppose that the basic semiring A is the semiring of nonnegative integers. Then CPDF0L power series are properly included in HD0L power series.*

The following theorem will be proved in the next section.

Theorem 3. *Let*

$$r = \sum_{n=0}^{\infty} hg^n(w)$$

be an HD0L power series where $g : A\langle X^*\rangle \longrightarrow A\langle X^*\rangle$ and $h : A\langle X^*\rangle \longrightarrow A\langle Y^*\rangle$ are monomial morphisms and $w \in X^*$. Suppose that

$$(hg^n(x), \varepsilon) \in \{0, 1\} \text{ for all } x \in X \text{ and } n \geq 1.$$

Then there exist a positive integer k, an alphabet Σ, a nonerasing monomial morphism $f : A\langle \Sigma^*\rangle \longrightarrow A\langle \Sigma^*\rangle$, a coding $c : A\langle \Sigma^*\rangle \longrightarrow A\langle Y^*\rangle$ and monomials $w_i \in A\langle \Sigma^*\rangle$, $0 \leq i < k$, such that

$$hg^{k(n+1)+i}(w) = cf^n(w_i) \tag{5}$$

for all $0 \leq i < k$ and for almost all $n \geq 0$.

Theorem 3 implies a generalization of the result of Nielsen, Rozenberg, Salomaa and Skyum, [9].

Corollary 4. *Let*

$$r = \sum_{n=0}^{\infty} hg^n(w)$$

be an HD0L power series where $g : A\langle X^*\rangle \longrightarrow A\langle X^*\rangle$ and $h : A\langle X^*\rangle \longrightarrow A\langle Y^*\rangle$ are monomial morphisms and $w \in X^*$. Suppose that

$$(hg^n(x), \varepsilon) \in \{0, 1\} \text{ for all } x \in X \text{ and } n \geq 1.$$

Then there exist a polynomial $s_1 \in A\langle Y^*\rangle$ and a CPDF0L power series $s_2 \in A\langle\langle Y^*\rangle\rangle$ such that

$$r = s_1 + s_2.$$

3 Proofs

To prove Theorem 3 we will use elementary morphisms in a power series framework. By definition, a monomial morphism $h : A\langle X^*\rangle \longrightarrow A\langle Y^*\rangle$ is *simplifiable* if there exist an alphabet Z and monomial morphisms $h_1 : A\langle X^*\rangle \longrightarrow A\langle Z^*\rangle$ and $h_2 : A\langle Z^*\rangle \longrightarrow A\langle Y^*\rangle$ such that $h = h_2h_1$ and $\mathrm{card}(Z) < \mathrm{card}(X)$. If h is not simplifiable it is called *elementary.* (If $\mathrm{card}(X) = 1$ then h is regarded as elementary if and only if h is nonerasing.) Elementary morphisms are closely related to cyclic morphisms. Here, we call a monomial morphism $h : A\langle X^*\rangle \longrightarrow A\langle X^*\rangle$ *cyclic* if for all $x \in X$, the letter x occurs at least once in the support of $h(x)$. For the proof of the following lemma see [3, 6].

Lemma 5. *Let $h : A\langle X^*\rangle \longrightarrow A\langle X^*\rangle$ be an elementary morphism. Then there exists a positive integer t such that h^t is cyclic.*

First we show how a nonerasing morphism can be replaced by a coding.

Lemma 6. *Let*

$$r = \sum_{n=0}^{\infty} hg^n(w)$$

be an HD0L power series where $g : A\langle X^*\rangle \longrightarrow A\langle X^*\rangle$ and $h : A\langle X^*\rangle \longrightarrow A\langle Y^*\rangle$ are monomial morphisms and $w \in X^*$. Assume that g is cyclic and h is nonerasing. Then there is an alphabet Δ, a nonerasing monomial morphism $g_1 : A\langle \Delta^*\rangle \longrightarrow A\langle \Delta^*\rangle$, a coding $c : A\langle \Delta^*\rangle \longrightarrow A\langle Y^*\rangle$ and a word $w_1 \in \Delta^*$ such that

$$hg^n(w) = cg_1^n(w_1)$$

for all $n \geq 0$.

Proof. Let

$$\Delta = \{(x, i) \mid x \in X, 1 \leq i \leq |h(x)|\}$$

be a new alphabet. Define the morphism $\alpha : X^* \longrightarrow \Delta^*$ by

$$\alpha(x) = (x, 1) \dots (x, |h(x)|)$$

for $x \in X$. Denote $w_1 = \alpha(w)$. Let $g_1 : A\langle \Delta^*\rangle \longrightarrow A\langle \Delta^*\rangle$ be a nonerasing monomial morphism such that

$$g_1\alpha(x) = \alpha g(x)$$

for all $x \in X$. The existence of g_1 follows because g is cyclic and, hence, the word $(x, 1) \dots (x, |h(x)|)$ is a factor of $\alpha g(x)$. Finally, let $c : A\langle \Delta^*\rangle \longrightarrow A\langle Y^*\rangle$ be a coding such that

$$c\alpha(x) = h(x)$$

for all $x \in X$. The existence of c follows because for all $x \in X$ the length of $\alpha(x)$ equals the length of $h(x)$. Then we have

$$hg^n(w) = c\alpha g^n(w) = cg_1^n(\alpha(w)) = cg_1^n(w_1)$$

for all $n \geq 0$. □

If $a \in A$ is nonzero and $w \in X^*$, then $\mathrm{alph}(aw)$ is the set of all letters of X which have at least one occurrence in w.

Lemma 7. *Let*

$$r = \sum_{n=0}^{\infty} hg^n(w)$$

be an HD0L power series where $g : A\langle X^*\rangle \longrightarrow A\langle X^*\rangle$ *and* $h : A\langle X^*\rangle \longrightarrow A\langle Y^*\rangle$ *are monomial morphisms and* $w \in X^*$. *Assume that*

$$(hg^n(x), \varepsilon) \in \{0, 1\} \text{ and } x \in alph(g(x)) = alph(g^2(x))$$

for all $x \in X$ *and* $n \geq 1$. *Then there exist an alphabet* X_1, *a cyclic monomial morphism* $g_1 : A\langle X_1^*\rangle \longrightarrow A\langle X_1^*\rangle$, *a nonerasing monomial morphism* $h_1 : A\langle X_1^*\rangle \longrightarrow A\langle Y^*\rangle$ *and a word* $w_1 \in X_1^*$ *such that*

$$hg^{n+1}(w) = h_1 g_1^n(w_1)$$

for all $n \geq 0$.

Proof. Denote

$$X_1 = \{x \in X \mid hg(x) \neq \varepsilon\}.$$

Equivalently, $x \in X_1$ if and only if $hg(x)$ is quasiregular. Let $\beta : A\langle X^*\rangle \longrightarrow A\langle X_1^*\rangle$ be the monomial morphism defined by

$$\beta(x) = \begin{cases} x \text{ if } x \in X_1 \\ \varepsilon \text{ otherwise} \end{cases}$$

and let $g_1 : A\langle X_1^*\rangle \longrightarrow A\langle X_1^*\rangle$ be the monomial morphism defined by

$$g_1(x) = \beta g(x)$$

for $x \in X_1$. Then we have $g_1\beta(x) = \beta g(x)$ for all $x \in X$. This is clear if $x \in X_1$. Otherwise, $hg(x) = \varepsilon$ and $hg^2(x) = \varepsilon$, hence $\beta g(x) = \varepsilon$.

Finally, let $h_1 : A\langle X_1^*\rangle \longrightarrow A\langle Y^*\rangle$ be the monomial morphism defined by

$$h_1(x) = hg(x)$$

for all $x \in X_1$ and denote $w_1 = \beta(w)$. Then

$$hg^{n+1}(w) = hg\beta g^n(w) = hgg_1^n\beta(w) = h_1 g_1^n(w_1)$$

for all $n \geq 0$. $\square$

As a penultimate step we prove Theorem 3 if the monomial morphism g is elementary.

Lemma 8. *Let*

$$r = \sum_{n=0}^{\infty} hg^n(w)$$

be an HD0L power series where $g : A\langle X^*\rangle \longrightarrow A\langle X^*\rangle$ *and* $h : A\langle X^*\rangle \longrightarrow A\langle Y^*\rangle$ *are monomial morphisms and* $w \in X^*$. *Assume that* g *is elementary and that*

$$(hg^n(x), \varepsilon) \in \{0, 1\} \textit{ for all } x \in X \textit{ and } n \geq 1.$$

Then there exist a positive integer k, *an alphabet* Σ, *a nonerasing monomial morphism* $f : A\langle \Sigma^*\rangle \longrightarrow A\langle \Sigma^*\rangle$, *a coding* $c : A\langle \Sigma^*\rangle \longrightarrow A\langle Y^*\rangle$ *and monomials* $w_i \in A\langle \Sigma^*\rangle$, $0 \leq i < k$, *such that (5) holds for all* $0 \leq i < k$ *and for all* $n \geq 0$.

Proof. By Lemma 5 there is a positive integer t such that g^t is cyclic. Hence there is a multiple k of t such that

$$x \in \mathrm{alph}(g^k(x)) = \mathrm{alph}(g^{2k}(x))$$

for all $x \in X$. By Lemmas 6 and 7 there exist alphabets Σ_i, nonerasing monomial morphisms $f_i : A\langle \Sigma_i^*\rangle \longrightarrow A\langle \Sigma_i^*\rangle$, codings $c_i : A\langle \Sigma_i^*\rangle \longrightarrow A\langle Y^*\rangle$ and monomials $w_i \in A\langle \Sigma_i^*\rangle$ for $0 \leq i < k$ such that

$$hg^{k(n+1)+i}(w) = c_i f_i^n(w_i)$$

for all $0 \leq i < k$ and $n \geq 0$. This implies the claim. Indeed, by renaming we may assume that the alphabets Σ_i, $0 \leq i < k$, are pairwise disjoint. Therefore, if $\Sigma = \Sigma_0 \cup \ldots \cup \Sigma_{k-1}$, there exist a nonerasing monomial morphism $f : A\langle \Sigma^*\rangle \longrightarrow A\langle \Sigma^*\rangle$ and a coding $c : A\langle \Sigma^*\rangle \longrightarrow A\langle Y^*\rangle$ such that

$$f(\sigma) = f_i(\sigma) \text{ and } c(\sigma) = c_i(\sigma)$$

if $\sigma \in \Sigma_i$, $0 \leq i < k$. Then (5) holds for all $0 \leq i < k$ and $n \geq 0$. $\square$

Now we are in a position to conclude the proof of Theorem 3. We use induction on the cardinality of X. If $\mathrm{card}(X) = 1$, the claim follows by Lemma 8 because g is elementary. Consider then an alphabet X and suppose that the claim holds for smaller alphabets. If g is elementary, Theorem 3 again follows by Lemma 8. So, assume there are an alphabet Z and monomial morphisms $g_1 : A\langle X^*\rangle \longrightarrow A\langle Z^*\rangle$ and $g_2 : A\langle Z^*\rangle \longrightarrow A\langle X^*\rangle$ such that $g = g_2 g_1$ and $\mathrm{card}(Z) < \mathrm{card}(X)$. Without restriction we assume that $g_2(z) \in X^*$ for all $z \in Z$. Then

$$r = h(w) + \sum_{n=0}^{\infty} hg^{n+1}(w) = h(w) + \sum_{n=0}^{\infty} hg_2(g_1 g_2)^n g_1(w)$$

where $g_1 g_2 : A\langle Z^* \rangle \longrightarrow A\langle Z^* \rangle$ is a monomial morphism. Because $(hg^n(x), \varepsilon) \in \{0, 1\}$ for all $x \in X$ and $n \geq 1$, also $(hg^n(u), \varepsilon) \in \{0, 1\}$ for all $u \in X^*$ and $n \geq 1$. Hence

$$(hg_2(g_1 g_2)^n(z), \varepsilon) = (h(g_2 g_1)^n g_2(z), \varepsilon) \in \{0, 1\}$$

for all $z \in Z$ and $n \geq 1$. Now the claim follows by induction. □

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