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Region-based Discrete Geometry

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Abstract: This paper is an essay in axiomatic foundations for discrete geometry intended, in principle, to be suitable for digital image processing and (more speculatively) for spatial reasoning and description as in AI and GIS. Only the geometry of convexity and linearity is treated here. A digital image is considered as a finite collection of regions; regions are primitive entities (they are not sets of points). The main result (Theorem 20) shows that finite spaces are sufficient. The theory draws on both "region-based topology" (also known as mereotopology) and abstract convexity theory.
Key Words: Discrete geometry, regions, mereotopology, convexity.

1 Introduction

Digital topology, as a study of the connectivity properties of digital images and the spaces in which they lie, is by now a fairly well-developed discipline. The approach is either graph-theoretic, as in [14], or else topological in the strict sense, as in [8]. (See also [9] and references given there.)

In our own work in this area, the emphasis has been on an amalgamation of digital topology with ordinary topology, achieved by working with a single category in which all the spaces (including the graphs) exist as objects. In this category (*TopGr*, for topological graphs) the usual "continuous" spaces arise as (inverse) limits of the digital spaces [15, 16, 17].

In the present paper we take up the task of extending this approach to geometry proper. We are not aware of any previous axiomatic theory of digital geometry, although it has occasionally been proposed that such a theory should be developed (for example by Zeeman [18]). Most often what is studied is some version of the "grid" model: in effect, $\mathbf{Z}^\mathbf{n}$ taken with some graph (adjacency) structure, and with n restricted to be 2 or 3.

What we shall undertake here is an axiomatic approach to the geometry of convexity and linearity: no attempt will be made to deal with parallels and congruence (for the time being). Abstract convexity theory, especially as developed by W. Prenowitz [12], is our exemplar; the problem is to adapt this material to "discrete" geometry. (See also Coppel [4] for a recent treatment of abstract convexity theory.) Both Prenowitz and Coppel provide non-trivial finite models of parts of the theory. These, however, like the finite geometries of combinatorial theory [5], cannot be said to resemble the spaces traditionally studied (Euclidean or non-Euclidean), nor to be capable of being viewed as discrete or approximate versions of these.

In order to obtain discrete spaces that are suitable for digital image processing, we shall adopt the *region-based* approach of (what is known as) mereotopology [2, 6, 13, 7]. Pixels (or voxels) and other simple convex regions are not considered as sets of points, but as primitive entities which are subject to a relation of partial order ("part of") and a symmetric binary relation expressing closeness of two regions (the "connection predicate"). Our theory may thus be considered as a sort of fusion of mereotopology with abstract convexity theory.

The primitives we adopt are not too far removed from those which have been adopted in some previous work on spatial reasoning ([3] and references given there). The main differences between what we are attempting here and what was done in those works are as follows. First, we aim for a mathematically adequate account of convexity. A simple criterion for this, often adopted in studies of abstract convexity, is that it permits the derivation of (abstract versions of) three famous theorems in convexity, namely those of Radon, Helly and Caratheodory; see for example [1]. The axioms concerning convexity have to be chosen so as to permit this. A second difference is that we are willing to admit (some) points as regions. We are mainly interested in discrete geometries, and it does not greatly matter whether the finitely many points which can be found in a bounded region of a discrete space are treated as primitive or as (finite) filters of regions. (For the general theory, it would be desirable to resolve this question, however.) A further important technical difference from many region-based theories [13, 10] is that we do not require our structures to be Boolean algebras; in particular, there is no operation corresponding to the intersection of regions. Despite these differences, we think it possible that some development of the theory presented here will prove to be useful for studies in spatial reasoning.

Note: As a reminder, Radon's Theorem for $\Re^n$ says that any set of $n+2$ (or more) points can be partitioned into two disjoint subsets whose convex closures intersect. It is enough to look at the case $n = 2$ to grasp the meaning of the theorem; while the case $n = 1$ is already of interest, as we shall see later. The Helly and Carathéodory theorems are closely related to the Radon theorem, but they will not be considered explicitly here.

2 Axiomatics

Our basic structure is a triple $(Q,\infty,*)$, where Q is a set (of *regions*), ∞ is a symmetric binary relation (*connection*), and $*$ is a commutative associative operation (*product*), satisfying the following Axioms:

A) ∞ is "almost reflexive":

$$\exists X.A \infty X \quad \Rightarrow \quad A \infty A$$

The effect of this will be that a non-null region is connected with itself.

B) (Fusion) For any collection $\mathcal{B} \subseteq Q$, there exists a unique region $\bigvee \mathcal{B}$ such that

$$X \infty \bigvee \mathcal{B} \quad \Leftrightarrow \quad \exists B \in \mathcal{B}.X \infty B.$$

C) (Distr.) $*$ distributes over $\bigvee$.

D) (Extension) For any $A, B \in Q$, there exists a region B/A such that

$$X \infty B/A \quad \Leftrightarrow \quad AX \infty B$$

Notice that occurrences of the operator $*$ are generally omitted.

E) $(A \infty C \;\&\; B \infty D) \vee B/A \infty D/C \quad \Rightarrow \quad AD \infty BC$
E*) (A stronger form of Axiom E)

$$A \infty C \;\&\; B \infty D \quad \Rightarrow \quad B/A \infty D/C \quad \Rightarrow \quad AD \infty BC$$

Some immediate consequences of these axioms:

Proposition 1. *The relation $\leq$, defined on Q by*

$$A \leq B \quad \rightleftharpoons \quad \forall X. A \infty X \Rightarrow B \infty X$$

is a partial order, with respect to which $\bigvee$ is the sup operation.

Proof. Trivially, $\leq$ is a pre-order. It is a partial order since, if $\forall X. A \infty X \Leftrightarrow B \infty X$, then, by uniqueness of fusion, $A = B = \bigvee\{A\}$. It is then a trivial verification that fusion is the join for this partial order.

Proposition 2. *Extension distributes over join.*

Proof.

$$\begin{aligned} X \infty \bigvee \mathcal{A} / \bigvee \mathcal{B} &\Leftrightarrow X * \bigvee \mathcal{B} \infty \bigvee \mathcal{A} \\ &\Leftrightarrow \bigvee_{B \in \mathcal{B}} (X * B) \infty \bigvee \mathcal{A} \\ &\Leftrightarrow \exists A \in \mathcal{A}, B \in \mathcal{B}. X * B \infty A \\ &\Leftrightarrow \exists A \in \mathcal{A}, B \in \mathcal{B}. X \infty A/B. \end{aligned}$$

There follow some comments on the axioms:

- Taking Q together with $\bigvee$ and $*$, we have a (non-unital) *quantale*
- Since we have $\bigvee$, we have a complete lattice. However, this is not (required to be) distributive, and $\bigwedge$ has little significance compared with $*$ and ∞.
- It is easy to see that for any $A \in Q$ we have the "complement" $\bigvee\{X | \neg(X \infty A)\}$, which satisfies the usual join and meet conditions for Boolean complement. By the preceding remark, however, we do not thereby obtain a Boolean algebra. Moreover the complement so defined is not in general an involution (thus it is not an orthocomplement).
- An immediate consequence of E* (but not of E) is that, if A, B are non-null regions, then A/B is non-null. A model based on a closed bounded subset of Euclidean space will fail Axiom E* (see the interpretation (3) to be given in a moment), but may nevertheless be of interest for discrete geometry. In the proofs below we have taken care to use E* only when E is not sufficient. (Prenowitz works with point-based axioms corresponding to Axiom E*, whereas Bryant & Webster have, in effect, only the weaker form E.)
- It may sometimes be convenient to drop the uniqueness requirement from Axiom B. The join is then a specified operation satisfying the property given in B, and the order (Prop. 1) can only be claimed to be a pre-order. In the conclusions of one or two of the subsequent Propositions, equality should be replaced by equivalence.

– The main remaining axiom to be considered is an Axiom of Order (see Section 5).

We next consider some of the main intended models of the axiom system.

1. (Euclidean) Q is: the subsets of $\Re^n$. $A \infty B$ means: A meets B. The product is defined in the first instance for points:

$$x * y = \begin{cases} (x, y) & \text{if } x \neq y \\ x & \text{if } x = y \end{cases}$$

   By distributivity, the definition extends to arbitrary regions. Taking $x * y$ as the set of points strictly between x and y (the open interval), in case $x \neq y$, is Prenowitz' preferred interpretation. Notice that if, in apparent conformity with this, we were to try to take $x * x$ as empty, we should lose associativity (consider $x * x * y$). A variant (corresponding to what is usually done in defining "interval convexities") is to define $x * y$ as the closed interval $[x, y]$ in all cases.
2. More in the spirit of mereotopology would be to take Q as the set of regular open subsets of $\Re^n$, and to take $A \infty B$ as meaning that the closures of A,B meet, with the product defined exactly as before (it clearly makes no difference here whether intervals are taken as open or closed). Of course with this interpretation we would obtain a (complete) Boolean algebra.
3. The first interpretation may be generalized by starting with an arbitrary convex subset C of the Euclidean n-space (while defining ∞ and $*$ exactly as before). It is easily seen that, if C is open, all the axioms are satisfied. If C is not open, then we do not (in general) have E*, but only the weak form Axiom E. This is because, if x is an extreme point of C, x/y may be empty (assuming Prenowitz' interpretation of $*$). In any case, let us denote by $G(C)$ the geometry defined in this way. Suppose next that S is a finite subset of C. Then we may consider the geometry "generated" by S: the collection of all the regions that may be obtained from S by repeatedly taking products and extensions (and union), with ∞ defined as in $G(C)$. From the point of view of discrete geometry, the question that is now of interest is, whether the geometry so obtained is itself finite (that is, whether only finitely many distinct regions arise in the construction). We shall return to this later (for the case $C = \Re^n$).

We may remark that, implicit in the third interpretation just given, is the notion of a subgeometry. Given two geometries $(Q, \infty, *)$, $(Q', \infty', *')$, the first is said to be a *subgeometry* of the second if $Q \subseteq Q'$ and the connection, product, extension and join (fusion) of Q coincide with the restrictions of those of Q' to Q. Thus, if C is a proper (convex) subset of $\Re^n$, then $G(C)$ is not a subgeometry of $G(\Re^n)$, since the extension operation differs. On the other hand, the question just raised is a question as to whether a certain finitely generated subgeometry of $G(C)$ is finite.

Note that the subgeometry construction is one which can lead to geometries lacking uniqueness of fusion ("pre-ordered geometries"), since there may be too few elements remaining to make (via ∞ and Axiom B) all the distinctions between regions which hold in the larger geometry. See comments following Theorem 20.

3 Development of geometry I

In developing geometry on the basis of an axiomatization of the above type, a key role is played by a series of basic lemmas, first developed by Prenowitz [12]. (A similar development is given by Coppel [4].) With their aid, much geometric reasoning is reduced to simple algebraic manipulations (though in a completely different fashion from the way in which this happens in ordinary analytic geometry). The lemmas (as will be seen) resemble the formulas found in the elementary calculus of fractions, though with equality usually replaced by inequality ($\leq$). The development of these lemmas given by the authors just cited is, of course, entirely point-based (regions are assumed to be sets of points). We have to ensure that, by adopting a region-based approach, we have not lost anything essential.

Lemma 3. $(A/B)/C \quad = \quad A/BC$

Proof. For any region X

$$\begin{aligned} X \infty (A/B)/C \quad &\Leftrightarrow \quad XC \infty A/B \\ &\Leftrightarrow \quad XCB \infty A \\ &\Leftrightarrow \quad X \infty A/BC \end{aligned}$$

(all by the Extension axiom). Due to the Fusion axiom, this proves the result.

Lemma 4. *Assume* $A \neq 0$. *Then*
a) $B \leq A(B/A)$
b) $B \leq AB/A$
c) $B \leq A/(A/B)$

Proof. Suppose that $X \infty B$. Then $X/A \infty B/A$, $AX \infty AB$, and $A/X \infty A/B$ (all by Axiom E*). Invoking Extension, we immediately obtain a),b),c) respectively.

Lemma 5. $A(B/C) \leq AB/C$

Proof. Suppose $X \infty A(B/C)$. Then
$X/A \quad \infty \quad B/C$ (Extension)
$XC \quad \infty \quad AB$ (Axiom E)
$X \quad \infty \quad AB/C$

Lemma 6. $(A/B)(C/D) \leq AC/BD$

Proof. $(A/B)(C/D) \leq (A/B)C/D$ (Lemma 5)
$\leq (CA/B)/D$ (Lemma 5, monotonicity)
$\leq AC/BD$ (Lemma 3).

The monotonicity of $*$ and $/$, which we have not troubled to spell out, is of course a consequence of the distributivity of $\bigvee$ over these operators.
The following lemma is proved similarly to the preceding one:

Lemma 7. $(A/B)/(C/D) \leq AD/BC$

The pattern of these proofs should be clear. Where, in point-based work, an inclusion $A \subseteq B$ is proved via the assumption "$x \in A$", the region-based argument will yield $A \leq B$, via the assumption "$X \infty A$".

4 Convex and Linear Regions

A region A should be considered convex if, whenever U,$V \leq A$, $U * V \leq A$. This simplifies to:

Definition 8. Region A is *convex* if $AA \leq A$.

Theorem 9. *The collection of all convex regions is closed under arbitrary meets and directed joins.*

Proof. Let X be the meet of an arbitrary collection $\mathcal{A}$ of convex regions. Then $XX \leq AA$, for every $A \in \mathcal{A}$. Hence $XX \leq \bigwedge_{A \in \mathcal{A}} AA \leq X$. Next, let Y be the join of a directed collection $\mathcal{B}$ of convex regions. By (Distr.), $YY = \bigvee_{B,C \in \mathcal{B}} BC$. Since $\mathcal{B}$ is directed, $YY \leq \bigvee_{D \in \mathcal{B}} DD \leq Y$.

This means that convex closure (or hull) can be defined, with the usual general properties, including (Scott-)continuity.

As for the other operators, it is immediate that the product of (finitely many) convex regions is convex. Only a little less trivial is:

Proposition 10. *A,B convex $\Rightarrow A/B$ convex.*

Proof. $AA \leq A$ and $BB \leq B$. Hence by Lemma 6,

$$(A/B)(A/B) \leq AA/BB \leq A/B.$$

A linear region should contain, with any parts U,V, not only what lies between them, but also what lies in the extension V/U. On simplifying, we get:

Definition 11. Region A is *linear(ly closed)* provided that A is convex and $A/A \leq A$.

In deference to analytic geometry, it might be preferable to dub these regions *affine* rather than linear. (Prenowitz, and also Bryant and Webster, have "linear", Coppel has "affine".) We have the same general properties for linearity, including the existence of the linear hull of any region, as indicated above for convexity. For the special operators, however, matters are a little different. Evidently, the product of two linear regions need not be linear. For the extension, we have a positive result, corresponding to Theorem 6.5 of Prenowitz [12]. Prenowitz' proof (including the proof of his Lemma 6.5) cannot be adapted to our setting, as it depends essentially on the use of points, and specifically on the assumption that, if two regions are connected, they have a point in common. So we provide the region-based proof here in detail:

Lemma 12. *Let A,L be regions such that $A \infty L$. If L is convex, then $L \leq L/A$. If L is linear, then also $L \leq A/L$.*

Proof. Suppose that L is convex, and $X \infty L$. Then (Axiom E) $XA \infty LL \leq L$. So (Extension) $X \infty L/A$. This shows that $L \leq L/A$. Next, assume L linear, and suppose that $X \infty L$. Then (Axiom E*) $A/X \infty L/L \leq L$. Hence $A \infty XL$, and so $X \infty A/L$.

Theorem 13. *Suppose that A,B are linear, and $A \infty B$. Then A/B is linear.*

Proof.

$$\begin{aligned}(A/B)/(A/B) &\leq (A/B)/(B/A)/(A/B) \text{(Lemma 12)}\\ &\leq (A/B)/(BB/AA) \quad \text{(Lemma 7)}\\ &\leq (A/B)/(B/A)\\ &\leq A/B \qquad\qquad \text{(Lemma 7)}\end{aligned}$$

This shows that A/B is linear.

An easy application of Lemma 7 also yields:

Proposition 14. *If A is convex, then A/A is linear.*

Proposition 15. *The linear hull of a finite set $\{A_1,...,A_n\}$ of convex regions is given by R/R, where $R = A_1 * ... * A_n$.*

Proof. The region R/R is linear by the preceding Proposition, and is evidently contained in any linear region which contains all the A_i. It remains only to show that $A_i \leq R/R$ for each i. Wlg, let us show this for $i = 1$. Suppose $X \infty A_1$. Then for any P (in particular, for $P = A_2 * ... * A_n$), $XA_1P \infty A_1A_1P$. Since $A_1A_1 \leq A_1$, $XA_1P \leq A_1P$. Hence $X \infty A_1P/A_1P = R/R$.

A simple result which will be useful in Section 6 is:

Proposition 16. *If A is a convex region, and B, C are any regions, then $(A/B)* (A/C) \leq A/BC$ and $(B/A) * (C/A) \leq BC/A$.*

Proof. By two uses of Lemma 5 and one use of Lemma 3:

$$\frac{A}{B} * \frac{A}{C} \quad \leq \quad \frac{(A/B) * A}{C} \quad \leq \quad \frac{AA/B}{C} \quad \leq \quad \frac{A}{BC}$$

The other part is similar.

As already mentioned, we do not at present see how to eliminate points altogether from the development. In particular, we seem to need them for formulating the Axiom of Order (to be considered shortly), and thus for establishing the key theorems which depend on it. It is also true that, in describing some of the geometries in which we are mainly interested in this paper, namely grid models for digital image processng, we find it convenient to admit, besides the pixels, also the edges and vertices of pixels as primitive regions. We shall regard points as linear regions which are as "small as possible" (without being null). There are (at least) two notions of smallness which suggest themselves here. The first is complete primality: if a point is covered by a collection of regions, it must be contained in one (or more) of those regions. The second is that, if a point is connected with a region, it must be contained in that region. Fortunately, these are equivalent:

Proposition 17. *A region A is a complete prime if and only if it satisfies:*

$$\forall X. A \infty X \quad \Rightarrow \quad A \leq X \qquad (1)$$

Proof. ONLY IF: Suppose $A \leq \bigvee \mathcal{B} \Rightarrow \exists B \in \mathcal{B}.\, A \leq B$ and $A \infty X$. Take $\mathcal{B}$ as

$$\{X\} \cup \{Y | Y \infty A \quad \& \quad \neg(Y \infty X)\}$$

Clearly, $A \leq \bigvee \mathcal{B}$. But X is the only member of $\mathcal{B}$ which can contain A .
IF: Assume (1). Then

$$\begin{aligned} A \leq \bigvee \mathcal{B} \quad &\Rightarrow \quad \exists B \in \mathcal{B}. A \infty B \\ &\Rightarrow \quad \exists B \in \mathcal{B}. A \leq B \end{aligned}$$

Definition 18. A point is a non-null completely prime linear region.

Proposition 19. *If P is a point, then $PP = P/P = P$.*

5 Development of geometry II

This section on further development of geometry could of course be rather extensive, but will on the contrary be very brief. This is because we shall have recourse to points, after which we need do little more than show how the development in [12] or [4] may be imitated.

The Axiom of Order, as usually formulated, concerns the order of points on a line ("Given any three distinct collinear points, exactly one of them lies between the other two", say). This is problematic for us in a number of ways. Ignore, for the moment, scruples about the emphasis on points. A *line* is presumably to be defined as the linear hull of two distinct points. Now, it may very well happen, in a non-trivial model of our theory, that lines - even, all lines - have only two points. In that case,the Axiom of Order as formulated above would be vacuous. We can try to reformulate the Axiom so that it refers to a "linear arrangement" of regions that need not be points; but this seems hard to do, in a way that preserves the power of the axiom in point-based work.

A partial solution may be found by switching attention from lines to rays. A ray is a half-line. Mention of lines, however, is inessential in treating rays. If p,q are any two points,the ray from p in the direction of q may be expressed as $p * q/p$. (Notice that the case $p = q$ is not excluded. If $p = q$, we get the "degenerate" ray $p * p/p = p$.) In the expression for a ray, the point q may with advantage be replaced by an arbitrary region A. The point p may be replaced, for reasons which we cannot go into here, by an arbitrary *linear* region, yielding the notion of a *generalized ray* (or *cone*) as a region of the form MA/M, where M is linear. The beautiful theory of rays and spherical geometries of Prenowitz [11, 12] can now, if desired, be adapted to the region-based style. But here we have introduced rays only in order to address the Axiom of Order.

One of several principles shown by Prenowitz to be equivalent (in point-based theory) to the Axiom of Order is the following: for any point P and region (for Prenowitz: set of points) R,

$$PR/P = PR \vee R \vee R/P$$

(In words: the ray PR/P consists of what lies between P and R, R itself, and the extension of R away from P.) Notice that the point P cannot be replaced even by a general linear region. (Taking P as a line, the resulting statement would be false already in Euclidean geometry.) This "ray principle" is taken as the basic form of the Order Axiom by Bryant & Webster [1], and this is what we shall do too. But whereas for these authors the ray version is simply a variant of the line version, for us it is essentially stronger: in a region geometry, the ray principle may very well be non-vacuous even when every line contains only two points.

We shall proceed by giving an indication (little more than a hint) of how the Order Axiom, in its ray version, underlies powerful theorems on convexity. Notice that in the equation embodying the ray principle, the "quotient" on the left has the factor P occurring in both numerator and denominator, whereas no such repeated factor occurs on the right hand side. Indeed, in the geometric calculus, the principle functions mainly as an aid in eliminating repetitions of factors of this kind. An important application is in connection with the formula for linear hull (Proposition 15). In case the A_i are points, repeated use of the principle enables an expansion of the linear hull formulas to be given, in which no repeated factors occur. For the details of this, we refer to [12, 1]. Here we just consider the case n=2. It is an easy exercise to show, using the ray principle, that, if A_1,A_2 are points, we have :

$$A_1A_2/A_1A_2 = A_1/A_2 \vee A_1 \vee A_1A_2 \vee A_2 \vee A_2/A_1$$

This clearly implies the following : if A_3 is any third point on the line through A_1,A_2, then one of the three points lies between the other two. This shows the connection with the line version of the Order Axiom. More interesting in the present context, however, is that it shows that the ray principle, applied to the linear hull formula for two points, yields what is essentially the 1-dimensional Radon theorem. In a similar way, the expansion for n points enables the full Radon theorem to be obtained. With a little more work, general versions of the Helly and Caratheodory theorems may be obtained as well: see [12], or the extremely concise [1], for details.

6 Finite Models

It might be asked: why do we not admit intersection (of regions) as a primitive operation, enabling the whole development to be simplified? The answer is that we would thereby lose the (possibilty of) finite models of the kind in which we are interested.

Suppose that we construct a geometry on a finite grid (1 x 3 suffices), with unit squares. Let A,B,C_1,D_1 be the points (0,0), (3,0), (0,1), (3,1) respectively. The segments AD_1, BC_1 meet the ordinates through (0,1), (0,2) at, say, U,V. The line through U,V meets AC_1, BD_1 at points which we shall call C_2,D_2. Taking C_2,D_2 in place of C_1, D_1, and repeating the construction, we obtain sequences of distinct points (C_i),(D_i), which we are forced to admit into the geometry if intersections of regions are accepted as regions.

Now, it must be observed that, in certain circumstances, the extension operator is capable of generating infinite sequences in a similar way. To see this, let us start with the 2 x 3 grid, and let the points A,B,C_1,D,U,V be the points (0,1), (3,1),(3,1.5), (0,0), (1,1),(2,1) respectively. The point (region) C_1 does not belong to the model, but the region R_1=VBC_1 does, as it is V/ADV. It is easily seen that the region R_2=$V/U/R_1$ is VBC_2, where C_2 lies strictly between B and C_1. Continuing in this manner, we obtain a strictly decreasing sequence of regions R_i.

In terms of the discussion in Section 2 (third interpretation), what this argument has shown is that, if K is the rectangle with vertices (0,0),(3,0),(3,2),(0,2), then the subgeometry of $G(K)$ generated by the (unit) grid points of K is infinite.

To approach the construction of finite models, we begin by considering a property enjoyed by some, but not all, geometries:

$$(F) \quad A/BC \leq (A/B) * (A/C) \quad \& \quad BC/A \leq (B/A) * (C/A)$$

It is easy to see that (F) is satisfied by the geometry $G(\Re^n)$ (it suffices to verify the property in the case that A,B,C are points). By Proposition 16, we know that, in case A is convex, the inequalities in property (F) may be replaced by equalities. The significance of this is that any term built up, using * and /, from constants A_1,...,A_n representing convex regions can be reduced to a product of terms in A_1,...,A_n built using / alone.

The preceding reduction drives occurrences of / inwards. A further reduction will drive them to the left. Indeed, we have

$$(A/B)/C \quad = \quad A/BC \qquad (Lemma 3) \qquad = \quad (A/B) * (A/C) \quad (F)$$

this being a product of terms of the form A/X, each having fewer occurrences of / than $(A/B)/C$. Let an expression $A_1/A_2/.../A_n$ be understood as $A_1/(A_2/(.../A_n)...)$ (association to right). Then by an easy ordinal induction based on the above reductions, we have:

Theorem 20 (Normal form). *Let T be any term built, using * and /, from constants representing convex regions. Then, assuming that Property (F) holds, T can be expressed as a product of terms of the form $A_1/A_2/.../A_n$ (A_1,...,A_n constants)*

We can now obtain our main result, namely the existence of finite geometries which may be considered to approximate $\Re^n$.

Theorem 21. *Let S be a finite set of points in $\Re^n$. Then the subgeometry of $G(\Re^n)$ generated by S is finite.*

Proof. In view of the Normal Form Theorem, we have only to show that there are no more than finitely many distinct regions $A_1/A_2/.../A_n$, where $A_1,...,A_n \in S$. We adopt the following notation: $A_{i(jk)}$ is the ray with end-point A_i in the direction $\overleftrightarrow{A_j A_k}$ (in case $A_j = A_k$, $A_{i(jk)}$ is the degenerate ray A_i). In particular, $A_{k(jk)}$ is A_k/A_j. We claim that $A_1/A_2/.../A_n$ is equal to a product of rays $A_{i(jk)}$, where $i,j,k \in 1,...,n$. This is trivial in case $n = 1$. For an induction step, notice that $A_l/A_{i(jk)} = A_{l(il)}{}^*A_{l(kj)}$, so that if Z is a product of rays $A_i(jk)$, then so is A_l/Z. Since there are only finitely many such rays, it follows that the (sub)geometry generated by S is finite.

In particular, if S is a finite grid in $\Re^n$, then it generates a finite geometry which has the (convex) polytopes with vertices in S among its regions. Strictly speaking, we have to take into account the possibility that the subgeometry is only pre-ordered (Section 2). Actually, this is the case with the set-up as we have it as present. Consider, for example, the cell with vertices $a = (0,0), b = (0,1), c = (1,1), d = (1,0)$, with an integer planar grid of points. Then the regions $abcd$ (open square) and $abc \vee abd$ have to be considered as equivalent, as they cannot be distinguished via the connection predicate. Although not necessarily disastrous, this is somewhat anomalous. In future work we shall consider how best to remove the anomaly.

If it is desired to have a (finite) model which does not include points among the regions, this is now easily obtained:

Corollary 22. *Let S be a finite grid in $\Re^n$, and S' the set of (open) voxels of the grid. Then S' generates a finite subgeometry of $G(\Re^n)$.*

Proof. Each voxel is a product of points of S. Hence the voxels generate a geometry which is contained in that generated by S.

7 Conclusion

The work reported here represents only the first steps towards a system of discrete geometry that could be adequate for image processing. It is possible that it is a little closer to adequacy in relation to “qualitative spatial reasoning”, as in AI and GIS (Geographic Information Systems).

Our focus has been on showing that it is possible to have geometric models which support notions of convexity and linearity, which “resemble” Euclidean space, and yet which have only a finite ontology. It is interesting to compare this finding with the recent results of Pratt & Lemon [10] on ontologies for the theory of polygons in mereotopology. Their main result is, roughly speaking, that any model for the theory of polygons in the plane (based on a quite minimal set of mereotopological primitives) must contain the rational polygons. We did not here explicitly develop the theory of polygons, but there would be no difficulty in doing so (as indeed the theory of polytopes generally), following Prenowitz [12]. Yet we have finite models for our region-based theory. The explanation is that, although the Pratt & Lemon polygonal theory is certainly parsimonious in relation to what is usually assumed in such studies, it does include the assumption that the polygonal regions form a Boolean algebra. As we have emphasized, this assumption is lacking in our framework; moreover the meet of regions does not, in the intended discrete models, represent the intersection of geometric loci.

Our theory is designed to capture (features of) the topology and geometry of a space at a given level of resolution. The question of multiresolution analysis, and of the approximation of "ideal" spaces by spaces of finite resolution, has received much attention in our previous topological work (mentioned in the Introduction), but has yet to be addressed in the present context. Any such analysis would involve the use of suitable approximation mappings; mappings (morphisms) of geometries are, however, conspicuously lacking in the outline given above.

Our eventual introduction of points as regions is something of a stopgap. We intend, in due course, to take up the construction of points out of non-pointlike regions, as is customary in point-free versions of topology. What is also involved here -which makes this task rather demanding - is the construction of segments, lines, and other entities of lower dimension than the ambient space, and which are indispensable for geometric work.

A surprising amount can be done with a primitive for convexity as the sole geometric primitive, as Prenowitz and other writers on abstract convexity theory have shown. We have tried to indicate how such work can be accommodated in a region-based approach suitable for discrete geometric modelling. In subsequent work we shall endeavour to accommodate notions that take us beyond convexity theory (parallels, congruence, metrics).

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