Journal of Universal Computer Science, vol. 6, no. 12 (2000), 1226-1252
submitted: 3/12/00, accepted: 22/12/00, appeared: 28/12/00

New Tools for Cellular Automata in the Hyperbolic Plane

Maurice Margenstern

(Laboratoire d'Informatique Théorique et Appliquée, EA 3097,
Université de Metz, I.U.T. de Metz,
Département d'Informatique,
Île du Saulcy,
57045 Metz Cedex, France,
margens@lita.univ-metz.fr)

Abstract: In this paper, we introduce a new technique in order to deal with cellular automata in the hyperbolic plane. The subject was introduced in [7] which gave an important application of the new possibility opened by the first part of that paper. At the same time, we recall the results that were already obtained in previous papers.

Here we go further in these techniques that we opened, and we give new ones that should give better tools to develop the matter.

Categories: F.1, F.1.1, G.2

Keywords: hyperbolic plane, tessellations, cellular automata

1 Introduction

Cellular automata have been studied for a long time, see [2, 3, 12], and they are most often used and studied in three spatial contexts: cellular automata displayed along a line, cellular automata in the plane, cellular automata in the three-dimensional space. There are also a few investigations in more general contexts, see for instance [16], where they are studied on graphs, connected with Cayley groups.

About the spatial representations, we should add the precision that in all cases, we are dealing with **euclidean** space. Indeed, that precision is so evident that it seems useless to remind this so *obvious* basis.

Take for instance cellular automata in the plane with von Neumann neighbourhood. If a cell has coordinates (x, y) with x and y integers, its neighbours are $(x, y+1)$, $(x, y-1)$, $(x-1, y)$ and $(x+1, y)$. This description is so simple that we forget the reason of such an elegant systems of coordinates, which extends without problem to the regular grids of the euclidean plane. Indeed, the group of displacements of the euclidean space possesses a normal subgroup, the group of translations and dilatations. That property namely is at the very basis of such elegant and simple coordinates.

The situation is completely different in the hyperbolic case, starting from two dimensions. The problem of finding an easy way to locate cells in that plane is not so trivial as it is in the euclidean case, because in the hyperbolic case, there is no equivalent to the euclidean group of translations and dilatations, because the group of hyperbolic displacements contains no nontrivial normal subgroup, see [14], for instance.

When the hyperbolic plane was considered in tiling problems, see [15], the study of cellular automata in that context was initiated by the technical report [6]. Later, two papers appeared, or will appear by the same authors, [7, 8] and two new technical reports by the present author are published, [4] and [5]. The present paper gives an account of [4] and [5], that are devoted to the representation of the regular rectangular pentagonal grid in the hyperbolic plane, that we call later the *pentagrid*. Here, we give new algorithms to locate cells of a cellular automaton grounded on the pentagrid. These new algorithms are simpler than the algorithm provided in [6]. Moreover, they are linear in the size of the data.

Some basic features of what is needed of hyperbolic geometry are given in [6, 8] and [4]. However, in order to make the paper self-contained, we give here a minimal set of tools coming from elementary (euclidean) geometry, which will allow the reader to prove the properties that are used in the paper. We shall also remind very sketchily the proof of the existence of the pentagrid that is given in the just quoted papers. This reminding is necessary in order to understand the new tools that we indicate here.

2 About the hyperbolic plane

In order to simplify the approach for the reader, we shall present a model of the hyperbolic plane and simply refer to the literature for a more abstract, purely axiomatic exposition.

As it is well known, hyperbolic geometry appeared in the first half of the XIX$^{\text{th}}$ century, in the last attempts to prove the famous parallel axiom of Euclid's *Elements* from the remaining ones. Hyperbolic geometry was yielded as a consequence of the repeated failure of such attempts. Lobachevsky and, independently, Bolyai, discovered a new geometry by assuming that in the plane, from a point out of a given line, there are at least two lines that are parallel to the given line. Later, during the XIX$^{\text{th}}$ century, models were discovered that gave implementations of the new axioms. The constructions of the models, all belonging to euclidean geometry, proved by themselves that the new axioms bring no contradiction to the other ones. Hyperbolic geometry is not less sound than euclidean geometry is. It is also no more sound, in so far as much later, models of the euclidean plane were discovered in the hyperbolic plane.

Among these models, Poincaré's models met with great success because in these models, hyperbolic angles between lines coincide with the euclidean angles of their supports. In this paper, we take Poincaré's disk as a model of the hyperbolic plane.

2.1 Lines of the hyperbolic plane and angles

In Poincaré's disk model, the hyperbolic plane is the set of points lying in the open unit disk of the euclidean plane. The lines of the hyperbolic plane in Poincaré's disk model are either diametral segments (open segments as the points lying on the unit circle do not belong to the hyperbolic plane) or circles, orthogonal to the unit circle. By circle we mean the arc of such circles which are defined by the intersection with the open unit disk. We say that the considered circle *supports* the hyperbolic line, h-line for short, and sometimes simply *line* when there is no ambiguity.

Poincaré's unit disk model of the hyperbolic plane makes an intensive use of some properties of the euclidean geometry of circles.

Consider a fixed circle γ and a fixed point M. A line through M may cut γ in two points, P and Q with, for instance, Q near M. There is a diameter of the circle which passes through M: it cuts γ in A and B, B near M. As the whole circle is seen from angle APQ plus angle ABQ and as the angle from a point of the circle is half the angle from the center that sees the same arc[1], we see that $\widehat{APQ}+\widehat{ABQ} = \pi$. Consequently, $\widehat{APQ} = \widehat{QBM}$. As triangles MPA and MBQ have a common angle in their common vertex M, those triangles are similar. In particular, $\dfrac{MP}{MB} = \dfrac{MA}{MQ}$, which can be rewritten : $MP.MQ = MA.MB$. As the center of γ, say O, is the iddle of AB, $MA.MB = OM^2{-}R^2$, where R is the radius of γ. And so, we obtain $\boxed{MP.MQ = OM^2{-}R^2}$. This shows that the product $MP.MQ$ does not depend on the secant but on the position of M with respect to γ. That product is called the *power* of M with respect to γ and is denoted by $P_\gamma(M)$. Remark that there is an interesting position of MPQ: when the intersection with γ is a double point, *i.e.* when the line is a tangent from M to γ. This is possible if M is not inside the circle. In that case, if we call T the point of contact of the tangent with γ, we have that $P_\gamma(M) = MT^2$. Another way to express the same property is that the circle with center M and radius MT is orthogonal to γ: MT is a radius of the second circle and a tangent for γ, and OM, which is a radius for γ is a tangent to the second circle. By construction, $OM \perp MT$.

If we have now two circles, γ and δ with different centers. We can ask what are the points that have the same power with respect to both γ and δ. The answer is remarkably simple and it is almost easier first to give the proof of it: if $P_\gamma(M) = P_\delta(M)$, we have that $O_\gamma M^2{-}R_\gamma^2 = O_\delta M^2{-}R_\delta^2$, where O_γ and O_δ are the centers of, respectively, γ and δ and R_γ, repsectively R_δ, are the radiuses of the corresponding circles. If H is the orthogonal projection of M on $O_\gamma O_\delta$, the line which joins the centers of the considered cirles, the previous relation can be rewritten : $O_\gamma H^2{-}R_\gamma^2 = O_\delta H^2{-}R_\delta^2$ or, simpler: $\overline{0_\gamma H} = \dfrac{O_\gamma O_\delta^2 + R_\gamma^2 - R_\delta^2}{2.\overline{O_\gamma O_\delta}}$. This shows that H is fixed and that the set of the considered points is a line: the whole line defined by the perpendicular to $O_\gamma O_\delta$ raised from H.

It is easy to see that there is no point M if γ and δ have the same center.

[1] This makes use of the euclidean axiom on parallels in the form that the sum of the three angles of a triangle is equal to π.

The just defined line is called the *radical axis* of γ and δ. The radical axis has also an important property:

The set of the centers of the circles that are orthogonal to both γ and δ is the part of the radical axis that is outside to both γ and δ.

Indeed that set is distinct from the radical axis in two cases : γ and δ are tangent in a common point P, or they intersect in two distinct points P and Q. In both cases , the closed segment PQ must be ruled out from the radical axis.

There is an important particular case of those considerations: when the second circle is reduced to a point, say A. The radical axis is the set of points M whose power to γ is the square of their distance to A. As far as $O_\delta = A$ and as $R_\delta = 0$, we may rewrite O_γ and the formula that gives $O_\gamma H$ is now: $\overline{0H} = \dfrac{OA^2 + R_\gamma^2}{2.\overline{OA}}$. In particular, if $A \notin \gamma$, $OH > R$, which means that the radical axis is completely outside γ. In particular, the set of the centers of the circles that are orthogonal to γ and that pass through A is the whole radical axis.

From these considerations, it is easy to prove that from two distinct points of the unit disk there is a single h-line which passes through the points.

Indeed, let A and B be the considered points and let γ be the unit circle. From these considerations, it is easy to prove that from two distinct points of the unit disk there is a single h-line which passes through the points.

Indeed, let A and B be the considered points and let γ be the unit circle. If A and B are on a diameter of γ we have our h-line. If not, the radical axis of A and γ, say Δ_A, and the radical axis of B and γ, say Δ_B meet on a point P: both axes are outside γ and as they are perpendicular to, respectively, OA and OB that make an angle in O which is different of π, Δ_A and Δ_B cannot be parallel, and so they meet in a point P which is outside γ. The circle centered in P that passes through A and B is our h-line. There is no other one, as P is unique. This gives also the unicity of the solution when A and B are on a diameter: in that case Δ_A and Δ_B are parallel; they meet at infinity and the circle drawn from an infinite center is the line AB.

Consider the points of the unit circle as *points at infinity* for the hyperbolic plane: it is easy to see that an h-line defines two points at infinity by the intersection of its euclidean support with the unit circle. They are called points at infinity of the h-line. The following easily proved properties will often be used: any h-line has exactly two points at infinity; two points at infinity define a unique h-line passing through them; a point at infinity and a point in the hyperbolic plane uniquely define an h-line.

The previous considerations that proved the existence of a unique h-line through two distinct points of the hyperbolic plane extend to the points at infinity: the radical axis of a point on γ with γ is the tangent to γ on the considered point. This explains why the previous constructions extend to the above cases.

The angles between h-lines are defined as the euclidean angle between the tangents to the arcs which are taken as the support of the corresponding h-lines. This is one reason for choosing that model: hyperbolic angles between h-lines are, in a natural way, the euclidean angle between the corresponding supports. In particular, orthogonal circles support perpendicular h-lines.

In the hyperbolic plane, given a line, say ℓ, and a point A not lying on ℓ, there are infinitely many lines passing through A which do not intersect ℓ. In the euclidean plane, two lines are parallel if and only if they do not intersect. If the points at infinity are added to the euclidean plane, parallel lines are characterized as the lines passing through the same point at infinity. Hence, as for lines, to have a common point at infinity and not to intersect is the same property in the euclidean plane. This is not the case in the hyperbolic plane, where two lines may not intersect and have no common point at infinity. We shall distinguish those two cases by calling *parallel*, h-lines that share a common point at infinity, and *non secant*, h-lines which have no common point at all neither in the hyperbolic plane nor at infinity. So, considering the situation illustrated by Figure 1 below, there are exactly two h-lines parallel to a given h-line that pass through a point not lying on the latter line and infinitely many ones that pass through the point but are non-secant with the given h-line. This is easily checked in Poincaré's disk model, see Figure 1. Some authors call *hyperparallel* or *ultraparallel* lines that we call *non-secant*.

Another aspect of the parallel axiom lies in the sum of interior angles at the vertices of a polygon. In the euclidean plane, the sum of angles of any triangle is exactly π. In the hyperbolic plane, this is no more true: the sum of the angles of a triangle is *always less* than π. The difference from π is, by definition, the *area* of the triangle in the hyperbolic plane. Indeed, one can see that the difference of the sum of the angles of a triangle from π has the additive property of a measure on the set of all triangles. As a consequence, there is no rectangle in the hyperbolic plane. Consequently two non-secant lines, say ℓ and m, have, at most, one common perpendicular. It can be proved that this is the case: two non-secant lines of the hyperbolic plane have exactly one common perpendicular.

As that result plays an important part in the proof of the existence of the grid which we shall later define in the hyperbolic plane, we give a sketchy proof of that fact here.

The proof is given in the frame of Poincaré's model, although there are proofs purely belonging to hyperbolic geometry, see for instance [11]. We shall use again the notion of radical axis that we already introduced. Two non-secant h-lines, say ℓ and m, define two radical axes with the unit circle that intersect outside the unit disk. They intersect in a point P that is at infinity if the radical axes are parallel (in the euclidean sense). If P is at finite distance, the tangents from P to the unit circle touch it in R and S. The intersection with the unit open disk of the circle centered in P that passes through R and S gives us the needed common perpendicular. If P is at infinity, the circles are symmetric to the diameter of the unit circle which passes through P, say Δ, and the just given construction gives the diameter, which is perpendicular to Δ, as the common perpendicular of ℓ and m.

It can be added that parallel h-lines have no common perpendicular.

Consider two parallel h-lines, and let P be their common point at infinity. The above construction would give P as the center of the circle supporting the common perpendicular. But no circle with center P is orthogonal to the unit circle, except the circle with zero as a radius.

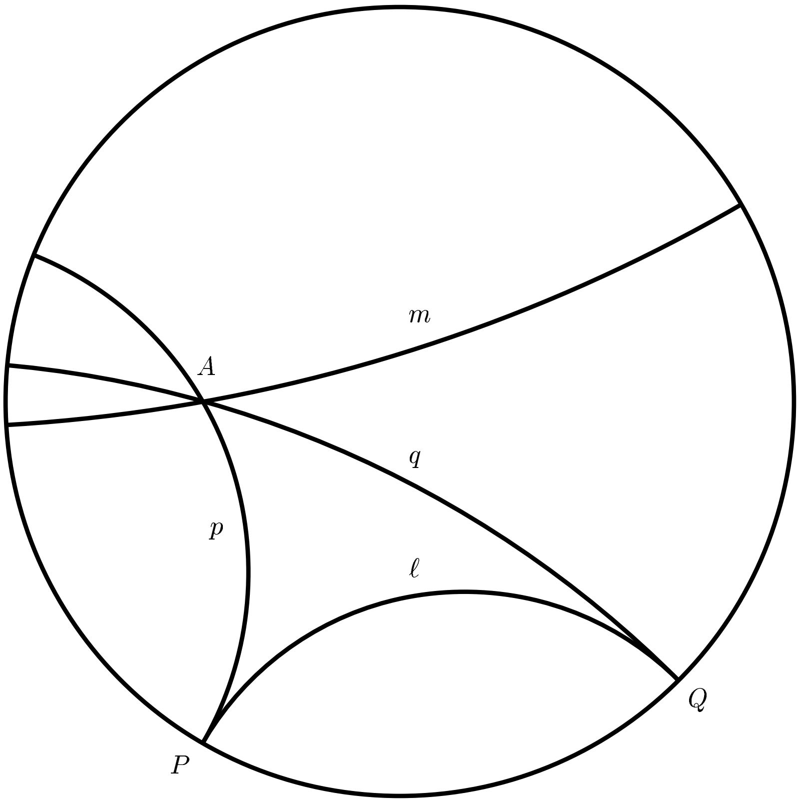


Figure 1: Lines p and q are parallel to line ℓ, with points at infinity P and Q; h-line m is non-secant with ℓ.

Consider the following problem of euclidean geometry:

(P) Let α, β, γ be positive real numbers such that $\alpha+\beta+\gamma = \pi$ and let be given two lines ℓ, m intersecting in A with angle α. How many triangles ABC can be constructed with $B \in \ell$, $C \in m$ and BC making angle β in B with ℓ?

The answer is clearly: infinitely many. That property of the euclidean plane defines the notion of *similarity*.

Another consequence of the non-validity of Euclid's axiom on parallels in the hyperbolic plane is that there is no notion of similarity in that plane: *if α, β, γ are positive real numbers such that $\alpha+\beta+\gamma < \pi$, ℓ and m are h-lines intersecting* *in A with angle α, there are exactly two triangles ABC such that $B \in \ell$, $C \in m$ and BC makes angle β in B with ℓ and angle γ in C with m.* Each of those triangles is determined by the side of ℓ with respect to A in which B is placed.

2.2 Reflections with respect to a h-line

Any h-line, say ℓ, defines a *reflection* through that line denoted ρ_ℓ. Let Ω be the center of the euclidean support of ℓ, R its radius. Two points M and M' are *symmetric* with respect to ℓ if and only if Ω, M and M' belong to the same euclidean line and if $\Omega M.\Omega M' = R^2$. Moreover, M and M' do not lie in the same connected component of the complement of ℓ in the unit disk. We also say that M' is obtained from M by the reflection through ℓ. It is clear that M is obtained from M' by the same reflection.

All the transformations of the hyperbolic plane that we shall later consider are reflections or constructed by reflections.

By definition, an *isometry* of the hyperbolic plane is a finite product of reflections. Two segments AB and CD are called *equal* if and only if there is an isometry transforming AB into CD.

It is proved that finite products of reflections can be characterized as either a single reflection or the product of two reflections or the product of three reflections. In our sequel, we will mainly be interested by single reflections or products of two reflections (the reader interested in the properties of a product of three reflections is referred to the literature).

At this point, we can compare *reflections* through a line in the hyperbolic plane with symmetries with respect to a line in the euclidean plane. Indeed, these respective transformations share many properties on the objects on which they respectively operate. However, there is a very deep difference between the isometries of the euclidean plane and those of the hyperbolic plane: while in the first case, the group of isometries possesses non trivial normal subgroups, in the second case, it is simple.

The product of two reflections with respect to lines ℓ and m is a way to focus on that difference. In the euclidean case, according to whether ℓ and m do intersect or are parallel, the product of the two corresponding symmetries is a rotation around the point of intersection of ℓ and m, or a shift in the direction perpendicular to both ℓ and m. In the hyperbolic case, if h-lines ℓ and m intersect in a point A, the product of the corresponding reflections is again called a rotation around A as far as the obtained transformation can be considered as what is intuitively called a rotation. But, if ℓ and m do not intersect, there are two cases: either ℓ and m intersect at infinity, or they do not intersect at all. This gives rise to different cases of shifts. The first one, called *shift*, is a kind of degenerated rotation, as in the euclidean case, and the second one is called *hyperbolic shift* or *displacement* along the common perpendicular to ℓ and m.

Such a displacement can be characterized by the image P' of any point P on the common perpendicular, say n. We shall speak of the displacement along n transforming P into P' and sometimes simply of *hyperbolic shift* or *displacement* when the explicit indication of the line along which the displacement is performed is not needed.

It can be proved that for any couple of two h-lines ℓ and m, there is an h-line n such that ℓ and m are exchanged in the reflection through n. In the case when ℓ and m are non-secant, n is the perpendicular bisector of the segment that joins the intersections of ℓ and m with their common perpendicular. The construction of h-line n is the same in all cases: let L_1, L_2 be the points at infinity of ℓ and M_1, M_2 those of m. We can always label the points in such a way that the euclidean lines L_1M_1 and L_2M_2 intersect in Ω, outside the closed unit disk, possibly at infinity. In the case that $L_1 = M_1$, the euclidean line L_1M_1 is the tangent at L_1 to the unit circle. From Ω, the tangents to the unit circle define two points at infinity through which a single h-line passes which is namely h-line n. Indeed, it is not difficult to check that L_1 and L_2 are the images of, respectively, M_1 and M_2 under the reflection defined by n constructed as just indicated. It immediately ensues that ℓ is the image of m under the reflection through n.

3 Representations of the pentagrid

3.1 Tessellations in the hyperbolic plane

Tessellations in the plane − the definition is independant of the geometry that we consider − consist in the following operations. First, take a convex polygon P. Let $\mathcal{S}(P)$ be the set of the lines that support its sides. If $\mathcal{E}$ is a set of polygons, one extends $\mathcal{S}$ to $\mathcal{E}$ by setting that $\mathcal{S}(\mathcal{E}) = \bigcup\limits_{P\in\mathcal{E}} \mathcal{S}(P)$. Given $\mathcal{K}$ a set of lines and $\mathcal{E}$ a set of polygons, we define that $\rho_{\mathcal{K}}(\mathcal{E}) = \bigcup\limits_{k\in\mathcal{K},Q\in\mathcal{E}} \rho_k(Q)$. Setting $\mathcal{T}_0 = \{P\}$, we inductively define $\mathcal{T}_{k+1}$ by $\mathcal{T}_{k+1} = \rho_{\mathcal{S}(\mathcal{T}_k)}(\mathcal{T}_k)$. Finally, we define $\mathcal{T}^* = \bigcup\limits_{k=0}^{\infty} \mathcal{T}_k$ to be the *tessellation* generated by P. We say that the tessellation is a *tiling* one if and only if the following conditions hold:

- any point of the plane belongs to at least one polygon in $\mathcal{T}^*$;
- the interiors of the elements of $\mathcal{T}^*$ are pairwise disjoint.

In that definition, lines are defined according to the considered geometry. It may have a consequence on the existence of a tessellation, depending on which polygon is taken in the first step of the construction. As an example, starting from a regular figure, there are basically three possible tessellations giving rise to a tiling of the euclidean plane: the tessellation based on the equilateral triangle, or on the square, or on the regular hexagon.

The situation is completely different in the hyperbolic plane where there are infinitely many tilings generated by tessellation. It is a consequence of the following result:

Poincaré's Theorem, ([13]) – *Any triangle with angles* $\pi/\ell, \pi/m, \pi/n$ *such that*

$$\frac{1}{\ell} + \frac{1}{m} + \frac{1}{n} < 1$$

generates a unique tiling tessellation.

The theorem immediately shows that a tiling tessellation is generated by the triangle with the following angles : $\frac{\pi}{5}, \frac{\pi}{4}, \frac{\pi}{2}$. It is easy to see that ten of those triangles share the same vertex corresponding to the angle $\frac{\pi}{5}$ and that such a grouping defines a regular pentagon with right angles. This tiling defines what we call from now on the *pentagrid*, a representation of which in the south-west quarter of the hyperbolic plane is shown below in Figure 2.

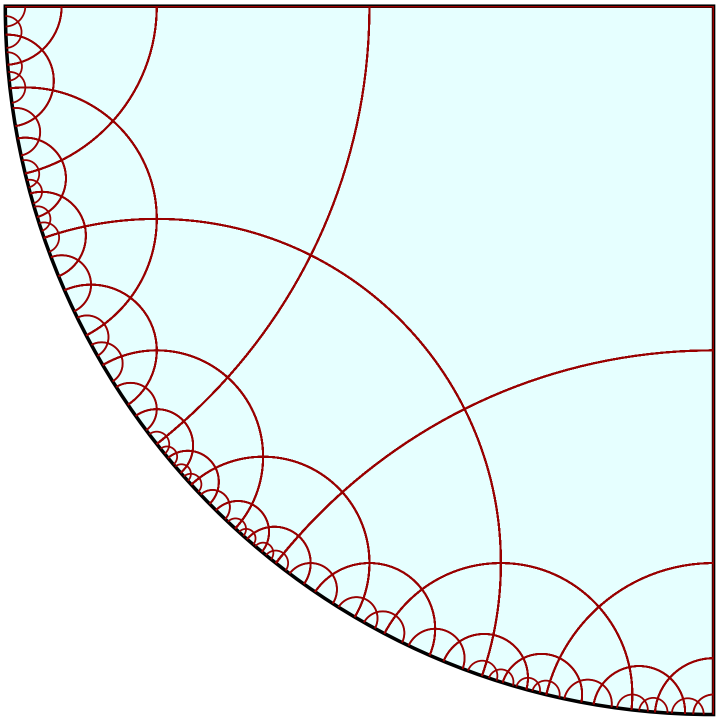


Figure 2: The pentagrid in the south-western quarter

It should be noticed that the pentagrid is the simplest regular grid of the hyperbolic plane. The triangular equilateral grid and the square grid of the euclidean plane cannot be constructed here as they violate the law about the sum of angles in a triangle which is always less than π in the hyperbolic plane.

Poincaré's theorem was first proved by Henri Poincaré, [13], and other proofs were given later, for example in [1] and [10]. In [6], another proof is provided for the existence of the pentagrid which gives rise to a feasible algorithm in order to locate cells. In this paper, we improved such an algorithm by constructing a new one, based on another principle. This gives us a family of algorithms, and we show that among them, there is a simplest one from the point of view of computer science.

3.2 Construction of the Fibonacci tree

The independent proof of the existence of the pentagrid is established in [6] by means of a bijection which is constructed between the tiling of the south-western quarter of the hyperbolic plane, say $\mathcal{Q}$, with a special infinite tree: the *Fibonacci tree*. Notice that $\mathcal{Q}$ is isometric to any quarter of the hyperbolic plane.

Here, we remind sketchily the construction of that bijection.

Let P_0 be the regular rectangular pentagon contained in $\mathcal{Q}$ that has one vertex on the center of the unit disk and two sides supported by the sides of $\mathcal{Q}$. Say that P_0 is the *leading* pentagon of $\mathcal{Q}$.

Number the sides of P_0 clockwise by **1**, **2**, **3**, **4** and **5** as indicated above, on Figure 3. As **1** is perpendicular to **2** and **5** and as **4** is perpendicular to **3** and **5**, **2** and **3** do not intersect **5**. The complement of P_0 in $\mathcal{Q}$ can be split into three regions as follows. Line **2** splits $\mathcal{Q}$ into two components, say R_1 and R_1' with R_1' containing P_0. Line **3** splits R_1' into R_2 and R_2' with R_2' containing P_0. Line **4** splits R_2' into P_0 and R_3. This defines the initial part of a tree: P_0 is associated to the root of the tree, and let us consider that the root has three sons, ordered from left to right and respectively associated to R_3, R_2 and R_1. We can denote it as indicated by Figure 3. We shall say that the root is a 3-node because it has three sons.

R_1 and R_2 are isometric images of $\mathcal{Q}$ by simple displacements: R_1 is obtained from $\mathcal{Q}$ by the displacement along **1** that transforms **5** into **2**. Similarly for R_2 with the displacement along **4** that transforms **5** into **3**. The same splitting into four parts can be repeated for these regions. Their leading pentagons are also 3-nodes.

Now, let us see the status of region R_3. It is plain that R_3 *is not* isometric to $\mathcal{Q}$. Let P_1 be the reflection of P_0 through **4** with sides which are now numbered anti-clockwise, so that the same number is given to the edges supported by the same h-line. In order to avoid possible confusion, we put the name of the considered pentagon as an index, if needed. Say that P_1 is the leading pentagon of R_3. Notice that $R_3 \cup P_0$ is transformed into a region $\mathcal{S}$ by the displacement along **5** that transforms $\mathbf{1}_{P_0}$ into $\mathbf{4}_{P_0}$, say Δ, see Figure 3. Define S_1 and S_2 as the respective images of R_2 and R_3 by Δ. Then notice that $\mathcal{S} = S_2 \cup P_1$. Say that S_1 and S_2 are the sons of R_3 and associate also these nodes to their leading pentagon. We say that the node associated to R_3 is a 2-node.

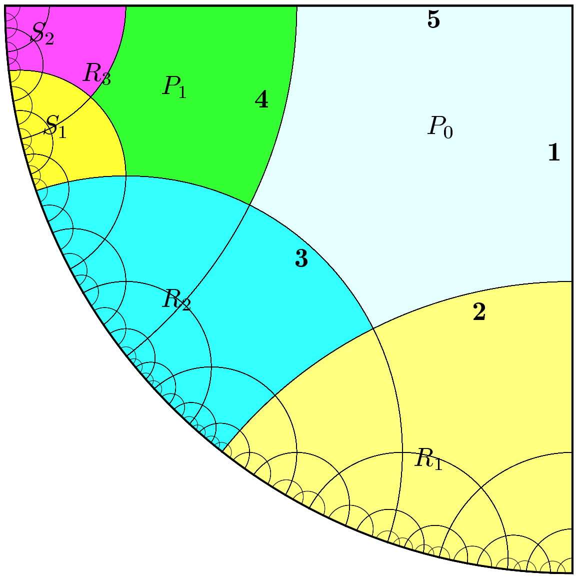


Figure 3: Splitting the quarter into four parts:
First step: regions P_0, R_1, R_2 and R_3, where region R_3 is constituted of regions P_1, S_1 and S_2;
Second step: regions R_1 and R_2 are split as the quarter (not represented) while region R_3 is split into three parts: P_1, S_1 and S_2 as indicated in the figure.

One can clearly see how we may proceed now. Define the following two rules:

- a 3-node has three sons: to left, a 2-node and, in the middle and to right, in both cases, 3-nodes;
- a 2-node has 2 sons: to left a 2-node, to right a 3-node.

Those two rules, combined with the axiom which tells that the root is a 3-node, uniquely define a tree which we call *Fibonacci* tree, see Figure 4, above.

The properties of the Fibonacci tree are indicated in [6], [7] and [8], and they are thoroughly proved in [4]. We shall not recall all of them here, where our attention is focused on the *location* of the elements of the pentagrid. These properties are used in order to establish the following important result proved in [6], [7] and [8] that uses a cellular automaton based on the pentagrid.

Theorem 1 (Margenstern-Morita) − NP-*problems can be solved in polynomial time in the space of cellular automata in the hyperbolic plane.*

3.3 A new tool, using the Fibonacci tree

In principle, the technique that is used in [6, 7] and [8] allows us to find the location of a cell and its neighbours in the pentagrid. This is the reason why the quoted papers assume that the Fibonacci tree is implemented in the hardware of the cellular automaton. A way to implement the tree consists in assuming that the path from the root to each node is known: it may be stored as a sequence of the sides (numbered from 1 to 5) through which reflections are performed starting from the pentagon of the root until the right node is reached. In order to locate the neighbours of the cells in that setting, it can be assumed that for each node, the path to the next node on the same level is also given. Otherwise, it would be possible to compute it, but at the price of a complete induction.

In the technical report [4], a new and more efficient way is defined to locate the cells which lie in the quarter, by numbering the nodes of the tree with the help of the positive numbers. We attach 1 to the root and then, the following numbers to its sons, going on on each level from left to right and one level after another one, see Figure 4, below.

That numbering is fixed once and for all in the paper. We fix also a representation of the numbers by means of the Fibonacci sequence, $\{F_i\}_{i\in I\!N}$.

It is known that every positive number n is a sum of distinct Fibonacci numbers: $n = \sum\limits_{i=1}^{k} \alpha_i.F_i$ with $\alpha_i \in \{0,1\}$. Such a representation defines a word $\alpha_k \ldots \alpha_1$ which is called a *Fibonacci representation* of n.

It is known that such a representation is not unique, but it can be made unique by adding a condition. Namely, we can assume that in the representation, there is no occurrence of the pattern 11: if $\alpha_i = 1$ in the above word, then $i = k$ or $\alpha_{i+1} = 0$. Following [4], we shall say that this new representation is the *standard* one. In [4], we give a proof of these well-known features.

From the standard representation, which can easily be computed from the number itself, it is possible to find the information that we need to locate the considered node in the tree: we can find its *status*, *i.e.* whether it is a 2-node or a 3-node; the number of its father; the path in the tree that leads from the root to that node; the numbers attached to its neighbours. This is done in great detail in [4] for the considered tree. Although the property is not stated in [4], it is not difficult to see that the algorithms that are there indicated work in linear time as we shall prove for the algorithms that we shall indicate in section 5.

As we shall use another kind of Fibonacci tree, we shall not give more details about those tools that the interested reader may find in [4].

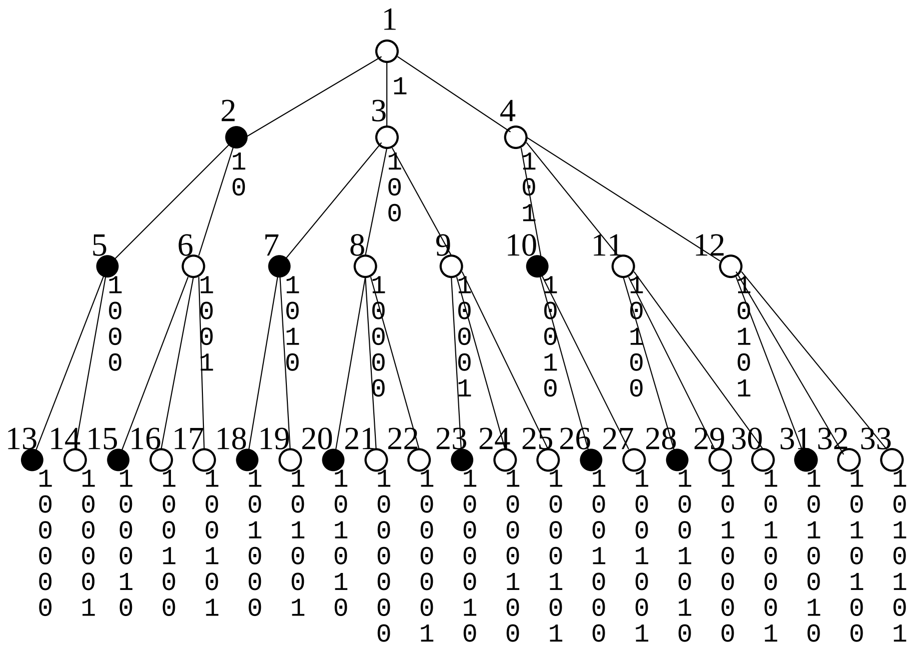


Figure 4: The standard Fibonacci tree:
above a node: its number; below: its standard representation.

4 Constructing a continuum of Fibonacci trees

Now we show that there are indeed infinitely many ways to attach Fibonacci trees to the restriction of the pentagrid in a quarter of the hyperbolic plane.

4.1 A new Fibonacci tree

In order to see that, consider again Figure 3. Indeed, that figure contains all the information that is needed in order to state the rules that lead to the tree represented in Figure 4.

Indeed, we can split the quarter in another way, as shown by Figure 5, below.

This defines a new splitting which differs from the one defined in [6, 7, 8] and [4], only on the way with which the regions that are isometric to a quarter are chosen.

At this point, we can notice that we can apply the arguments given in [6, 8, 4] in order to prove the bijection between the new tree and the tiling of the quarter. Indeed, when we consider the diameter of a region that tends to zero as the index of the step of splitting tends to infinity, the estimates that we then established are still in force here.

Let us now focus on the trees that are obtained. The Fibonacci tree defined in the first papers can be rewritten as indicated in Figure 4 where the numbers of the nodes are also displayed with their standard representation.

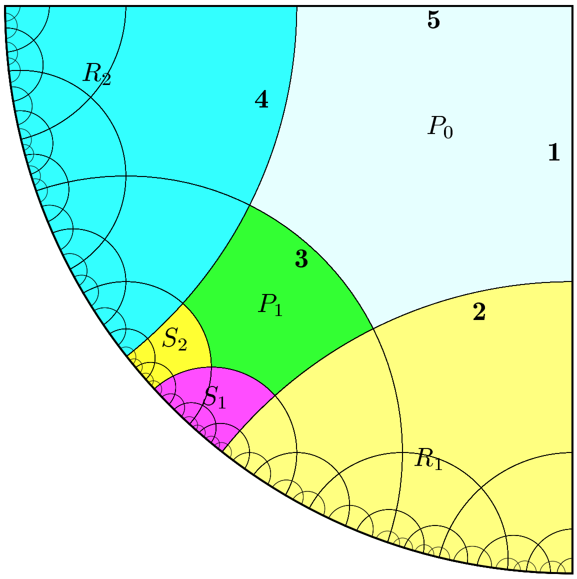


Figure 5: Splitting the quarter into four parts in another way:
Region R_3 consists of P_1, S_1 and S_2.

The new splitting that we define with the help of Figure 5 gives rise to a new kind of Fibonacci tree, where the rules for the nodes are different for the 3-nodes. In the case of the Fibonacci tree, the rules can also be expressed as follows: $2 \rightarrow 2\,3$ and $3 \rightarrow 2\,3\,3$. In the case of this new tree, let us call it *central* Fibonacci tree, the rules are: $2 \rightarrow 2\,3$ and $3 \rightarrow 3\,2\,3$. The central Fibonacci tree is illustrated by Figure 6, below.

As already indicated, the numbering of the nodes in the tree is fixed and so, the standard representation fixes the chosen Fibonacci representation. However, the algorithms which gives the status of a node, the number of the father, the path from the root to the considered node and the numbers of its neighbours will be different, see [5] for more details.

4.2 Infinitely many Fibonacci trees

It is now clear, that there are still other possible kinds of Fibonacci trees.

For a systematic study, one could proceed as follows. Using the previous notations, all possible rules for 2-nodes are $2 \rightarrow 2\,3$ and $2 \rightarrow 3\,2$ whereas all possible rules for 3-nodes are $3 \to 2\,3\,3$, $3 \to 3\,2\,3$ and $3 \to 3\,3\,2$. Denote these rules by, respectively, $2L$, $2R$, $3L$, $3M$ and $3R$. Also recall that the *status* of a 2-node is 2 and the status of a 3-node is 3.

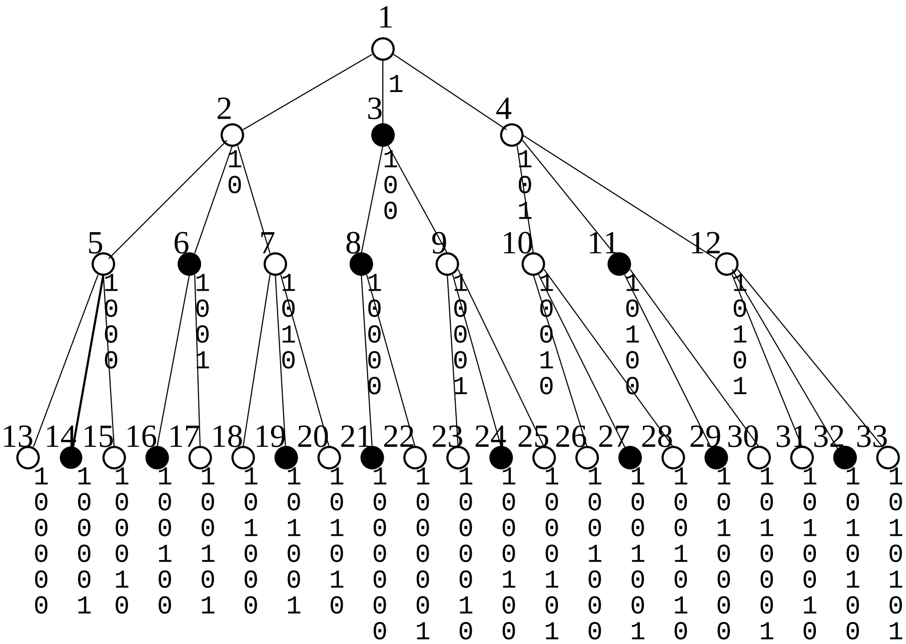


Figure 6: The central Fibonacci tree:
above a node: its number; below: its standard representation

Definition 1 − *Call general Fibonacci tree, an infinite tree whose nodes have either two sons or three sons and such that there is a mapping τ of the nodes into the set $\{2L, 2R, 3L, 3M, 3R\}$ satisfying the following property:*

for all node ν, $\tau(\nu) \in \{2L, 2R\}$ if and only if ν is a 2-node.

Say then that τ matches the status of each node and that τ is a Fibonacci assignment, in short an assignment, over the tree.

Due to the application to the tiling of a quarter of the hyperbolic plane by regular pentagons with right angles, we shall always assume that the root of the tree is a 3-node.

There are infinitely many general Fibonacci trees. They can be all constructed by a random algorithm using a dice[2] as follows:

- construct the root as a 3 node, which is at level 0;
- iteratively construct levels one after another:

[2] We use a *cubic*, hence euclidean, dice in a three-dimensional euclidean space.

for each node of the current level:
  throw the dice and let r be the result
  for a 2-node apply rule $2L$ iff $r < 4$, otherwise $2R$
  for a 3-node apply rule $3L$ iff $r < 3$, else $3M$ if $r < 5$, otherwise $3R$

As we have only permutations in the position of the 2-node among the sons of a node, this does not change the number of nodes which occur and, by induction on the level of the considered tree, it is easy to see that the number of nodes in a considered level is always the same for any general Fibonacci tree. Consequently, the numbering is always the same and, hence the standard representation attached to the numbers of the nodes only depends on the depth of the node in the tree, and on its rank on its level.

We can now state the following result:

Theorem 2 $-$ *There is a continuum of general Fibonacci trees and the trees are determined by their assignment.*

Among all possible assignments, we shall be also interested by the *fixed* ones: assign always the *same* 2-rule to 2-nodes and, similarly, always the same 3-rule to 3-nodes. There are six of them, in particular the Fibonacci tree, which corresponds to the assignment defined by rules $2L$ and $3L$, and the central Fibonacci tree, which is associated to the assignment defined by rules $2L$ and $3M$. From now on, call *standard* the assignment attached to the Fibonacci tree.

4.3 The preferred son property

In order to simplify the writing, we identify a node with its number and also with the standard representation of its number. We shall say that the representation of node ν ends in β, in short that ν ends in β, where $\beta \in \{0,1\}^*$, if β is a suffix of the standard representation of the number attached to node ν.

In [4], we noticed and proved the following property:

Proposition 1 $-$*Let ν be any node in the standard Fibonacci tree. Among the sons of ν there is exactly one of them, say ω, that ends in 00; moreover, the standard representation of ω is obtained by appending two 0's to the standard representation of ν.*

and we called *preferred son* the son with such a representation.

That property plays an important rôle in the algorithms that are given in [4]. It has also an important part in the algorithms that we introduce here, see [5].

The property of the preferred son is also true for the central Fibonacci tree. However, it is not true for any generalised Fibonacci tree. For instance, the tree built according to the rules $2R$ and $3R$ does not possess that property. Also the tree built according to the rules $2L$ and $3R$ does not satisfy that property. We refer the reader to [5] for more details.

Accordingly, following [5], we introduce the present definition:

Definition 2 – *The continuator of the node numbered by μ is the node whose number ν is such that its standard representation is obtained by appending two 0's to the standard representation of μ.*

When the continuator of a node happens to occur among its sons, the considered son will be called the *preferred son* of the node. If this happens for all the nodes of the tree, we shall say that the tree possesses the *preferred son* property.

In order to characterise the preferred son property, we first need to study the relations between any assignment with the standard one.

4.3.1 Relations between an assignment and the standard one

Let ν be a node of level $k+1$ in the tree. We shall also identify ν with its *rank* among the nodes of the same level. Let α be a Fibonacci assignment. For each node different from the root, we associate a function f_α such that $f_\alpha(\nu)$ is the father, under that assignment, of the node with ν as a number. We also identify $f_\alpha(\nu)$ with its *rank* on its own level.

We shall denote the standard assignment by σ, and we shall see that in some sense, it possesses some maximal property.

Now, we define the following function on the nodes of a tree. Denote by $\omega_\alpha(\nu)$ the number of the rightmost son of ν under assignment α. The same node may have different sons under different assignments. To which extent can they be different? Not much as it is proved by the following relations:

Proposition 2 – *For all k and ν we have:*

$$(1) \qquad \omega_\alpha(\nu)-1 \leq \omega_\sigma(\nu) \leq \omega_\alpha(\nu)$$

$$(2) \qquad f_\sigma(\nu) - 1 \leq f_\alpha(\nu) \leq f_\sigma(\nu).$$

The full proof of that proposition is given in [5]. Here we give its main points.

We first define the following function on the nodes of a tree. Denote by $\beta_\alpha(\nu)$ the number of nodes of the same level of ν which are not on the right of ν and that are 2-nodes under the assignment α. It will be useful to place emphasis on the level and we shall write $\beta_{\alpha,k}(\nu)$ when ν is on the level k and we may also consider that ν is replaced by its rank.

Location lemma – *For any node of level $k+1$ with rank w that is also the rightmost son of its α-father, we have that $w = 3.f_\alpha(w) - \beta_{\alpha,k}(f_\alpha(w))$ and we have also that $f_\alpha(w) = \beta_{\alpha,k+1}(w)$.*

By induction on the levels and, on the current level, on the rank of the considered node, we prove the following technical lemma:

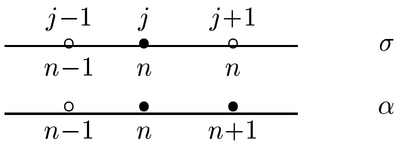
Lemma – *For all k and ν we have:*

$$(3) \qquad \beta_{\sigma,k}(\nu)-1 \leq \beta_{\alpha,k}(\nu) \leq \beta_{\sigma,k}(\nu)$$

Assuming that the property is proved until node j, the proof consists in examining the different possible statuses of nodes j and $j+1$. We consider, as an example of the used argument the following piece of the proof:

Assume that j is a 2-node under σ.

It is easy to see that that $j-1$ is a 3-node under σ, and that the values of $\beta_{\sigma,k+1}$ on $j+1$, j and $j-1$ are, respectively, n, n and $n-1$. Notice that function β_α cannot decrease and that it increases at most by 1 when going from one node to its right-hand neighbour. That property can be expressed by $(*)\beta_{\alpha,k}(m+1) \in \{\beta_{\alpha,k}(m), \beta_{\alpha,k}(m)+1\}$. From $(*)$ and from the assumption, we obtain that the corresponding values of $\beta_{\alpha,k+1}$ are $n+1$, n and $n-1$. As $\beta_{\alpha,k+1}$ increases two times by one, this means that j and $j+1$ are 2-nodes under α. This is represented by the following figure:



As j is a 2-node under σ, we know that $j-1$ is the rightmost son of its σ-father which is $n-1$. This provides us with the following relation:

$$(a) \qquad j-1 = 3(n-1) - \beta_{\sigma,k}(n-1)$$

On another hand, as $j+1$ and j are 2-nodes under α, we entail that they cannot have the same α-father and so, j is the rightmost son of its α-father, which is n. Accordingly, we obtain:

$$(b) \qquad j = 3n - \beta_{\alpha,k}(n)$$

Substracting (b) from (a), we get (c) $\beta_{\alpha,k}(n) = \beta_{\sigma,k}(n-1) + 2$. But, on another hand, by the induction assumption and by $(*)$, we have that $\beta_{\alpha,k}(n) \leq \beta_{\sigma,k}(n) \leq \beta_{\sigma,k}(n-1) + 1$, which is a contradiction with (c).

We refer the reader to [5] for the consideration of the other cases.

4.3.2 Characterisation of the preferred son property

Call an assignment 00-*assignment* if and only if it possesses the following property: *under the assignment, each node has among its sons exactly one son whose number ends in* 00.

We have the following result:

Theorem 3 $-$ *A generalised Fibonacci tree possesses the preferred son property if and only if it is associated to a* 00*-assignment.*

This property is proved in [5], using the previous lemmas, starting from the fact that the standard Fibonacci tree possesses the property and also using the maximal property of β_σ.

In [5], other characterisations of the preferred son property are proved. We indicate the following one:

Theorem 4 $-$ *An assignment* α *is a* 00*-assignment if and only if any node that ends in* 10 *is a son of a* 3*-node.*

As we know that there is a continuum of generalised Fibonacci trees and that not all assignments possess the preferred son property, it is interesting to know whether that property is exceptional or not.

The following result, see [5], gives a certain answer to that question, namely:

Theorem 5 $-$ *There is a continuum of generalised Fibonacci trees with the preferred son property.*

The idea of the proof is the following: we start with the standard Fibonacci tree, and we see whether it is possible to introduce a perturbation from some node by applying another rule than the expected one and whether it is possible to "control" the just introduced perturbation and, in case it is possible, in which manner. The answer is yes: we can introduce such a perturbation in the root, by deciding to apply rule $3M$ instead of rule $3L$. It can be seen that on the next level, only two nodes differ from the standard configuration, call them *exceptional nodes*. They are consecutive nodes. The leftmost one is a 2-node in the standard tree and a 3-node in the new tree. The statuses are the reverse for the right hand exceptional node. If we apply the standard rules to all the nodes of the level in the new tree, again there are two exceptional nodes on the next level by comparison with the configuration in the standard tree. The new exceptional nodes are the standard leftmost and middle sons of the previous rightmost exceptional node. By induction on the level, it can be seen that this situation is repeated endlessly for the middle son of the rightmost exceptional node. Figure 7, below, illustrates the situation.

This shows that the perturbation is "linear" and that it does not propagate elsewhere in the tree. It is also not difficult to see that the new tree does possess the property of the preferred son: it is enough to check that this is the case with the exceptional nodes. Accordingly, the same perturbation can be introduced on a lower level, say on its leftmost node. It is enough to do that say, two levels below: we are sure, by the above argumentation, that both perturbation will go on without crossing one another. And so, we may define a countable number of nodes in the tree for which we can decide or not to perform that perturbation.

This provides us with a way to obtain a continuum of generalised Fibonacci trees that possess the preferred son property.

Omitted here details can be found in [5].

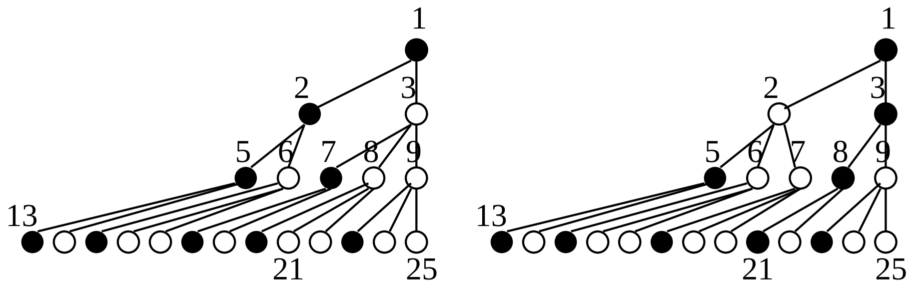


Figure 7: Introducing a point disturbation in the standard assignment:
on left: the standard assignment, on right: the effect of the change
the change concerns nodes 2 and 3, then 7 and 8, then 20 and 21
notice that the continuator of 3 is 8 whose continuator is 21

4.3.3 Fixed assignments

Among the assignments, some of them are good candidates for a convenient representation of the hyperbolic plane. In particular, *fixed* assignments are *a priori* to be first investigated.

However, as shown by the studies of [4, 5], fixed assignments do not have better properties than the standard one. Worse: as already noticed, two of them are not good assignments, see [5].

In connection with the 00-assignments, an ideal assignment would be a fixed 00-assignment such that the 2-son of a node is exactly its continuator. Unfortunately, there is no such fixed assignment, as it can be checked by a straightforward computation. However, there is a 00-assignment such that the continuator of the nodes are exactly the 2-nodes. The assignment is *almost* a fixed one: the rule applied to the node depends on the ending of the node: 00, 01 or 10. See [5] for an exact construction with its proof.

However, there is an assignment which is very near to what would be an ideal one. The next section is devoted to its definition and to the description of the algorithms that can be devised from its construction.

5 Tools for the pentagrid

5.1 The best assignment

We have seen that there is no fixed 00-assignment such that the 2-nodes would be exactly the continuators. What would happen if we would replace continuators by nodes that end in 01?

It can be seen that the rules $2R$ and $3M$ almost give the answer. Indeed, under that assignment, 2-nodes end in 01 except the nodes that are on the rightmost branch or that are direct sons of nodes on that branch, see [5]. But now, a slight modification gives the answer:

Theorem 6 – *There is a single* 00-*assignment such that* 2-*nodes are exactly the nodes ending in* 01. *The assignment consists in applying rule* $3R$ *to the root and then for all the other nodes, to apply rule* $2R$ *on the* 2-*nodes and rule* $3M$ *on the* 3-*nodes.*

From now on call 01-*assignment* the assignment constructed in theorem 6, which is illustrated on Figure 8.

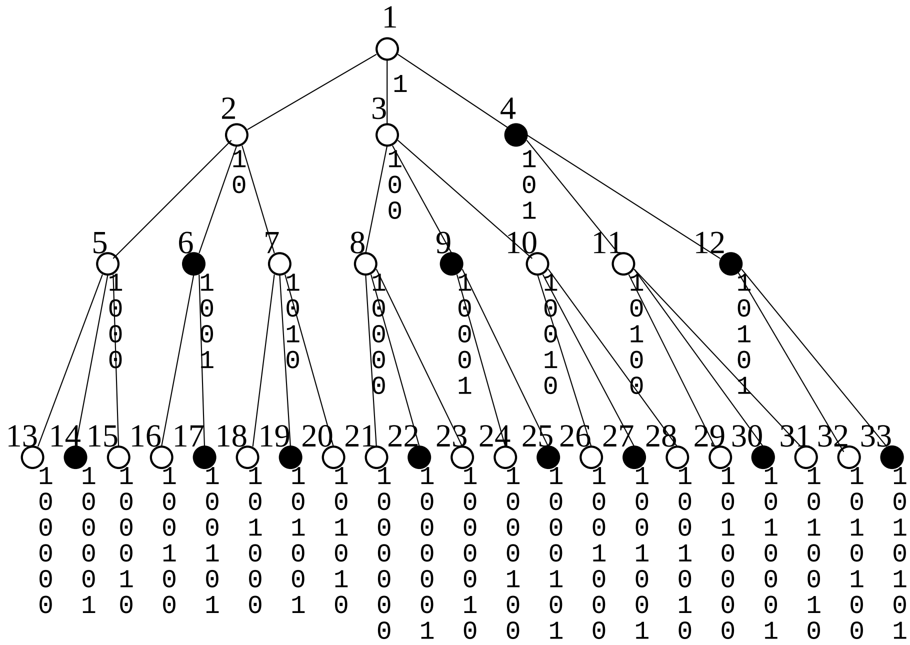


Figure 8: The Fibonacci tree associated to the 01-assignment

As it is proved in [5], the following properties hold:

- the sons of a 3-node end respectively in 00 for the leftmost one, in 01 for the middle one and in 10 for the rightmost one;
- the sons of a 2-node end respectively in 00 and 01;
- the continuator of a node is always its leftmost son.

5.2 The algorithms

It is now clear that the 01-assignment is better fitted to our goals that any other one that we constructed before. Indeed, the rules to determine the status of a node are extremely simple and this simplifies also the rules for finding the path from the root to the node when we know the number attached to a node.

However, there is a small price to pay: the rules that give the reflections performed along a path are not exactly the same since the paths themselves are different from the paths of the standard Fibonacci tree. However, the new rules are not much more complex than the rules used in the standard case. We give them here, for positive orientation, that are illustrated by Figure 9:

- if the node is a 2-node with i as the reflection leading to its father, then reflection i+2 leads to its left son which always ends in 00, and reflection i+3 leads to its 2-son;
- if the node is a 3-node that ends in 00 with reflection i leading to its father, i+2 leads to its leftmost son, its continuator, i+3 leads to its middle son, the 2-son, and i+4 leads to the rightmost son, which ends in 10;
- if the node is a 3-node that ends in 10 with reflection i leading to its father, then i+1 leads to its leftmost son, its continuator, i+2 leads to its middle son, the 2-son, and i+3 leads to its rightmost son, which ends in 10.

In all cases, replace + by − if the orientation of the node is negative.

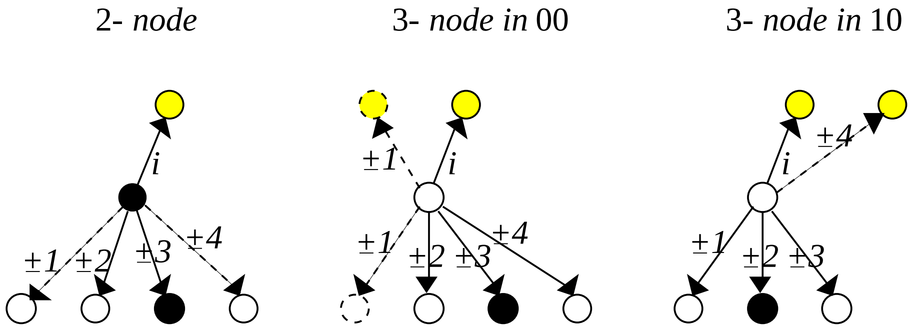


Figure 9: The new rules of the numbering in the Fibonacci tree associated to the 01-assignment:
Signs are the same: + for nodes with positive orientation, otherwise, −.
Notice, for the nodes in 00, that there are two cases, depending on the previous connections: if the left brother of the father is a 2-node, then take the upper arrow, else the lower one

The proof is analogous to the one given in [4] for the standard situation. It can be noticed that a purely combinatorial argument is used: the rule applied to the leftmost node of the level is fixed by the connection with the neighbouring quarter; for further nodes on the right hand, the rules are fixed by the connections established at the previous level, by the number of connections from one node to its neighbours, including the missing connections, and by the *quadrangle* *constraint*. That latter condition simply says that as four pentagons share the same vertex, in terms of nodes of the tree, this means that two consecutive nodes of the same level are connected by a reflection to a node of the previous level and to a node of the next level; the connection is direct (from a father to a son) or it is established by a reflection corresponding to a missing connection of the tree.

For the completeness of this paper, now we indicate the algorithms needed for constructions using the pentagrid. Their correctness is straightforward from the proofs of theorem 6 and proceed by induction on the levels of the tree and on the rank for the level that follows the current one.

First of all, we start with the status of a node, the status being here with three values, detecting whether the node ends in 00, 01 or 10, giving respectively values 0, 1 and 2: take a to be the last digit of the standard representation of the node, b to be the penultimate digit, the status is then $a+2b$.

Finding the father is also easy: erase the last two digits of the standard representation in order to have a representation of the father.

5.2.1 Algorithm to find the path

This algorithm needs an auxiliary one, which, to the path, associates the reflection through which the node is transformed into its father and conversely. This defines function *Index_Father* which is given by the first set of instructions that we give below in Figure 10. Notice that in order to compute this function, the initialisation step proceeds outside the loop since an exceptional rule is applied to the root.

With the help of functions *Index_Father* and *Father* as well as *Continuator* which appends two 0's to the standard representation of the node and then returns the corresponding number via *Value*, it is easy to find the neighbours of a node, as indicated on the right side of Figure 10.

As indicated before, the proofs of the correctness of those algorithms rely mainly on the proof of theorem 6 which gives a lot of properties of the 01-assignment that are used in order to obtain the most straightforward computation as possible.

Notice that, for the sake of simplification, the latter algorithm uses a special kind of + operator. The addition is taken modulo 5 with also the convention that remainder 0 is written 5. Also $-$, when the orientation of the node is negative, is understood in the same way, modulo 5 and with the same convention relative to 0. Details are left to the reader.

Algorithm for Index_Father

```
list is Path(node)
ref := 4 − path.top
case path.top is
  0 => status := 2
  1 => status := 0
  2 => status := 1
esac
pop path; sign is 1;
while path is not void
loop
  sign is − sign;
  if status is 0 or 1
    then ref :=
          ref+sign.(2+path.top);
    else ref :=
          ref+sign.(1+path.top);
  fi
  status := path.top
  pop path
pool
Index_Father is ref
```

Algorithm for the neighbours

```
status := Status(node)
fa := Father(node)
i := Index_Father(node)
s := Sign(node)
co := Continuator (node)
  neigh(i) := fa
case status is
  0 => neigh(i+s.2) := co
        neigh(i+s.3) := co+1
        neigh(i+s.4) := co+2
        if status(fa−1) = 1
        then neigh(i+s.1) := fa−1
        else neigh(i+s.1) := co−1
        fi
        neigh(i+s.1) := co−1
  1 => neigh(i+s.2) := co
        neigh(i+s.3) := co+1
        neigh(i+s.4) := co+2
        neigh(i+s.1) := co−1
  2 => neigh(i+s.1) := co
        neigh(i+s.2) := co+1
        neigh(i+s.3) := co+2
        neigh(i+s.4) := fa+1
esac
```

Figure 10: Algorithms in order to use the 01-assignment.
Notice that in the case of a node that ends in 00, the algorithm for the neighbours distinguishes between the two possible cases for the left brother of the father.

5.2.2 A complexity estimation

As we say that the provided algorithms are *simple*, such a statement must be proved by a complexity analysis:

Theorem 7 − *The algorithms given in this section are linear in $log(n)$, where n is the number of the considered node, both in time and in space.*

The proof is straightforward from the examination of Figure 10 where the algorithms are displayed. It is easy to notice that the algorithm that gives the standard reprsentation of n is linear in $log(n)$. The algorithm that gives the path from the number of the node is also linear in $log(n)$ as it is trivially linear in the length of the standard representation. As the function *Index_Father* is a loop on the path and as the corpus of the loop has a time complexity that is bounded by a constant, the linearity is clear. As the computation of *Father* and *Continuator* are also linear in $log(n)$, the computation of the neighbours has also a linear complexity.

5.2.3 For the whole plane

In order to represent the whole pentagrid with trees constructed with the 01-assignment, we proceed in the same way as it was done in [4] for the standard Fibonacci tree. We remind the reader that we suggested in [4] to encode each quarter in a class of equal remainders modulo 4. In order to define the quarters, one of the diameters is called the vertical one, the other, the horizontal one. Then we may decide, as in [4], that multiples of 4 are devoted to the south-western quarter, that all remainders 1 go to the north-western one, that all remainders 2 go to the north-eastern one and that all remainders 3 go to the south-eastern one.

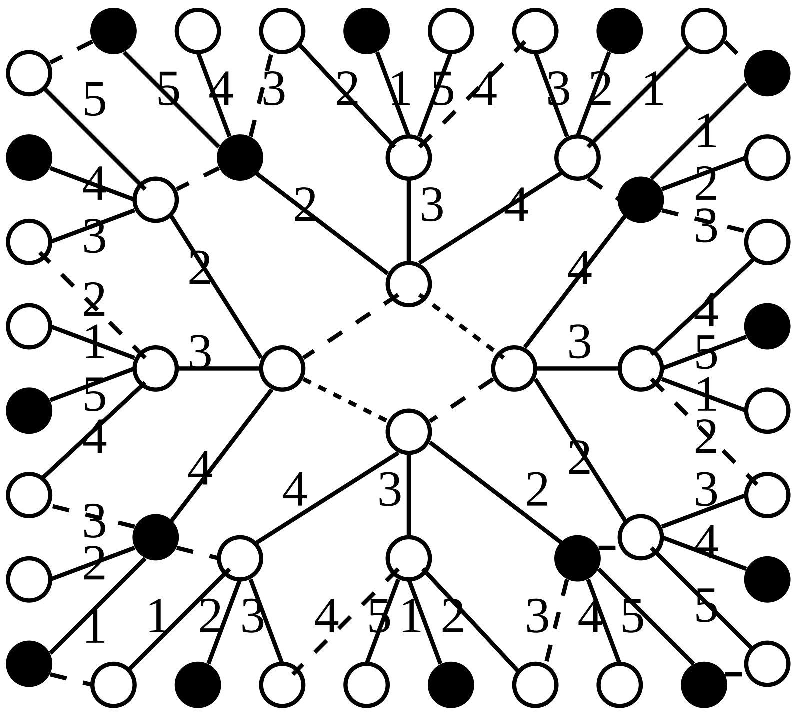


Figure 11: Connecting four Fibonacci trees associated to the 01-assignment.

We have now to indicate how we may connect four trees associated to the 01-assignment.

The connection on the rightmost branch of the tree is made by 2-nodes. For such nodes, there is a missing connection when we consider the tree for the quarter. That connection can be used to connect, say, the south-western tree with the south-eastern one. The connection corresponding to $i+4$ (or $i-4$ if the orientation is negative) is made with the leftmost node of the south-eastern tree which is *on the same level*. This is possible: leftmost nodes on a branch are 3-nodes that end in 00 and applying the rule of the left brother of their father, we connect their $i+1$ (reps. $i-1$) arc with the rightmost node of the other tree again on the same level: this corresponds to the fact that the pentagons that lie along an extremal branch of a tree are reflected on the other quarter through the same reflection that defines that border. As the reflection is perpendicular to the line, an extremal node must be connected to a node of the same level in the other tree, see Figure 11, above.

5.2.4 A remark

We already noticed that the endings 00 and 01 in the standard representation of the positive numbers occur quite often. We also already noticed that on another hand, the ending 10 is a bit less frequent. As we found 00-assignments for which the 2-nodes are exactly either all the nodes in 00, or all the nodes in 01, the question arises whether it is possible to find a 00-assignment with the same property for the nodes in 10?

The answer is no: assume that such an assignment exists. We know that each node contains its continuator among its sons. If a 2-node ends in 10, if its left son is its continuator, the right son cannot end in 10 and then no rule applies to this node. This already happens for the leftmost node of level 1 which is 10. Its left son is its continuator and so the process cannot be continued.

6 Conclusion

And so, we have now at our disposal a lot of Fibonacci trees which all allow us to locate cells of the pentagrid accurately. Our analysis proved that from the point of view of computer science, the better assignment is probably the 01-assignment.

This work is a direct continuation of [4] which opened a new way to locate cells on the pentagrid. As the quoted report, the method deals with a quarter of the hyperbolic plane.

A lot of questions remain open. We shall indicate two of them.

The first one concerns 00-assignments in general. We proved that there is a continuum of them. However, this does not indicate in some sense whether they are more numerous among all the assignments than the non 00-ones. As an example, we know that among the six fixed assignments, four ones are 00-assignments. Is it possible to say something more precise about such assignments than what was proved by theorem 5? Is there a probability for an assignment to be a 00-one and, if the answer is yes, what is that probability?

Another question deals with the other regular rectangular grids of the hyperbolic plane. It was indicated in [7] that the same construction based on the standard assignment can be generalised. Is it possible to say something more precise? At the time when this paper was under refereeing, works were going on this second line by the author and Gencho Skordev, as announced in [4]. They contain a generalisation which also gives a new picture for the pentagrid. This appeared in a preprint paper of the University of Bremen, see [9].

References

[1] C. Carathéodory. Theory of functions of a complex variable, vol.II, 177–184, Chelsea, New-York, 1954.

[2] M. Delorme and J. Mazoyer (eds.), Cellular automata, a parallel model, Kluwer Academic Publishers, **460**, 373pp., 1999.

[3] J. Gruska, Foundations of computing, International Thomson Computer Press, 716pp, 1997.

[4] Margenstern M., Cellular automata in the hyperbolic plane, Technical report, Publications du GIFM, I.U.T. of Metz, N°99-103, ISBN 2-9511539-3-7, 34p. 1999.

[5] Margenstern M., Cellular automata in the hyperbolic plane (II), Technical report, Publications du GIFM, I.U.T. of Metz, N°2000-101, ISBN 2-9511539-7-X, 40p. 2000.

[6] Margenstern M., Morita K., NP problems are tractable in the space of cellular automata in the hyperbolic plane. Technical report, Publications of the I.U.T. of Metz, 38p. 1998.

[7] Margenstern M., Morita K., A Polynomial Solution for 3-SAT in the Space of Cellular Automata in the Hyperbolic Plane, *Journal of Universal Computations and Systems*,

[8] Margenstern M., Morita K., NP problems are tractable in the space of cellular automata in the hyperbolic plane, to appear in *Theoretical Computer Science*.

[9] Margenstern M., Skordev G., Locating cells in regular grids of the hyperbolic plane for cellular automata, Technical report, N° 455, July 2000, Institut für Dynamische Systeme, Fachbereich Mathematik/Informatik/Technomathemtik, Universität Bremen, 2000, 38p.

[10] Maskit. B. On Poincaré's theorem for fundamental polygons. *Advances in Math.*, **7**, 219–230, (1971).

[11] H. Meschkowski, Noneuclidean geometry, translated by A. Shenitzer. Academic Press, New-York, 1964.

[12] Morita K., A simple construction method of a reversible finite automaton out of Fredkin gates, and its related model, *Transaction of the IEICE*, **E**, 978–984, 1990.

[13] Poincaré H., Théorie des groupes fuchsiens. *Acta Mathematica*, **1**, 1–62, (1882).

[14] A. Ramsay, R.D. Richtmyer, Introduction to Hyperbolic Geometry, Springer-Verlag, 1995, 287p.

[15] Robinson R.M. Undecidable tiling problems in the hyperbolic plane. *Inventiones Mathematicae*, **44**, 259-264, (1978).

[16] Zs. Róka, One-way cellular automata on Cayley Graphs, *Theoretical Computer Science*, **132**, 259–290, (1994).

[17] von Neuman J. Theory of self-reproducing automata. Ed. A. W. Burks, The University of Illinois Press, Urbana, (1966).