Journal of Universal Computer Science, vol. 6, no. 1 (2000), 5-21
submitted: 20/10/99, accepted: 6/11/99, appeared: 28/1/00 © Springer Pub. Co.

Weak Inclusion Systems: Part Two[1]

Virgil Emil Căzănescu[2]
(Fundamentals of Computer Science, Faculty of Mathematics,
University of Bucharest, Romania
Email: vec@funinf.math.unibuc.ro.)

Grigore Roşu[3]
(Department of Computer Science & Engineering,
University of California at San Diego
Email: grosu@cs.ucsd.edu.)

Abstract: New properties and implications of inclusion systems are investigated in the present paper. Many properties of lattices, factorization systems and special practical cases can be abstracted and adapted to our framework, making the various versions of inclusion systems useful tools for computer scientists and mathematicians.
Key Words: Category theory, logic.
Category: F.4.1.

1 Introduction

Computing Science concepts often take birth observing and studying practical situations and phenomena encountered in the process of development of software systems. For this reason, well established mathematical concepts and tools seem not to be perfectly suitable for some computing aspects. A concrete case is modularization [GB92, DGS93, Roş99], whose purpose is to formalize and give semantics to operations on software modules, such as importing, aggregation, hiding, parameterization, etc. Most of the operations on modules involve the notion of *inclusion* as a *unique* interpretation of a module into another; the "uniqueness up to an isomorphism" does not reflect the intuition behind these operations. Therefore, categorical notions like (mono) subobjects and factorization systems are not proper for some areas of computing.

At the authors' knowledge, the first formal definition of a *factorization system* of a category was given by Herrlich and Strecker[4] [HS73] in 1973, and a first comprehensive study of factorization systems containing different equivalent definitions was done by Németi [Ném82] in 1982. However, the idea to form subobjects by factoring each morphism f as $e;m$, where e is an epimorphism and m is a monomorphism, seems to go back to Grothendieck [Gro57] in 1957, and was intensively used by Isbell [Isb64], Lambek [Lam66], Mitchell [Mit65], and many others. At our knowledge, Lambek was the first to explicitly state and prove a diagonal-fill-in lemma for factorization systems [Lam66] in 1966.

[1] C. S. Calude and Gh. Ştefănescu (eds.). *Automata, Logic, and Computability. Special issue dedicated to Professor Sergiu Rudeanu Festschrift.*
[2] The work of this author was done during his visit to the University of Reunion.
[3] Also Fundamentals of Computer Science, Faculty of Mathematics, University of Bucharest, Romania.
[4] They called it $\langle \mathcal{E}, \mathcal{M} \rangle$*-factorizable category.*

In general terms, this paper is a sequel to our paper *Weak Inclusion Systems* [CR97]. It develops the notion of *inclusion* in a categorical setting, emphasizing the idea of *unique* factorization. In [CR97], we defined the *weak inclusion systems* as a natural extension of *inclusion systems* and as an alternative to factorization systems. In this paper, we further explore weak inclusion systems' properties, our main goal being to present them as a real useful tool for computer scientists and, why not, for mathematicians in search of elegant, clear, and smooth proofs.

Section 2 just introduces some notations and basic categorical properties, and section 3 presents a bunch of easy but useful properties of weak inclusion systems. Section 4 introduces the notion of *complete weak inclusion system* as an alternative to inclusion systems. Section 5 explores relations between reachable and generated objects in a category. A criterion to say if a category has enough projectives is also given. Finally, section 6 is concerned with lifting (weak) inclusion systems to comma categories, functor categories, and categories of algebras and coalgebras of an endofunctor.

Acknowledgements We would like to thank to professor Sergiu Rudeanu for his remarks and suggestions on the application of inclusions systems in Birkhoff-like axiomatizability results for generalizations of equational logics, and to professor Joseph Goguen for his comments on previous versions of this paper.

2 Preliminaries

The reader is supposed to be familiar with the basics of category theory (e.g., see [Lan71, HS73]). In this section we present our formalism and remind the reader some notions used later in the paper.

Calligraphic letters denote categories and functors. If $\mathcal{A}$ is a category then $|\mathcal{A}|$ denotes its class of objects. The composition of morphisms is written in diagrammatic order, that is, if $f\colon A \to B$ and $g\colon B \to C$ are two morphisms, their composition is written $f;g\colon A \to C$. Sometimes, we use the word epic (monic, iso) instead of epimorphism (monomorphism, isomorphism). Some basic properties of epics and monics are supposed known, such as "$f;g$ is an epic implies g is an epic", etc.

An important notion in category theory is that of subobject of an object. To be more precise and to avoid confusion with another kind of subobject introduced later in the paper, we call it **mono subobject**. A mono subobject of an object A in a category $\mathcal{A}$ is a coset (an equivalence class) of the equivalence relation $\sim$ defined on monics of target A as follows: $m \sim m'$ if there exist two morphisms f and g such that $f;m = m'$ and $g;m' = m$ (actually, f and g are isomorphisms).

Inclusion systems are related to an old and useful concept in category theory, namely factorization systems (see [HS73] and also [Ném82]). There are many equivalent definitions of factorizations systems; we remind the reader the one we think is the closest to our approach:

Definition 1. A **factorization system** of a category $\mathcal{A}$ is a pair $\langle E, \mathcal{M}\rangle$, such that

- E and $\mathcal{M}$ are subcategories of epics and monics, respectively, in $\mathcal{A}$,
- all isomorphisms in $\mathcal{A}$ are both in E and $\mathcal{M}$, and

– every morphism f in $\mathcal{A}$ can be factored as $e;m$ with $e \in E$ and $m \in \mathcal{M}$ "uniquely up to isomorphism", that is, if $f = e';i'$ is another factorization of f then there is a unique isomorphism α such that $e;\alpha = e'$ and $\alpha;m' = m$.

$$\begin{tikzcd}
& \bullet \arrow[dr, "m"] \arrow[dd, dotted, "\alpha"] & \\
\bullet \arrow[ur, "e"] \arrow[dr, "e'"'] & & \bullet \\
& \bullet \arrow[ur, "m'"'] &
\end{tikzcd}$$

There also are many equivalent definitions of adjointness in the literature. Within this paper we adopt the following two:

Definition 2. A functor $\mathcal{F}\colon \mathcal{X} \to \mathcal{A}$ is a **left adjoint** of $\mathcal{U}\colon \mathcal{A} \to \mathcal{X}$ iff for each pair of objects $X \in |\mathcal{X}|$, $A \in |\mathcal{A}|$ there is a bijection $\mathcal{X}(X,\mathcal{U}(A)) \cong \mathcal{A}(\mathcal{F}(X), A)$ which is natural in X and A.

Definition 3. A functor $\mathcal{F}\colon \mathcal{X} \to \mathcal{A}$ is a **left adjoint** of $\mathcal{U}\colon \mathcal{A} \to \mathcal{X}$ iff there exists a natural transformation $\eta\colon 1_{\mathcal{X}} \Rightarrow \mathcal{F};\mathcal{U}$ having the universal property: for every $X \in |\mathcal{X}|$, $A \in |\mathcal{A}|$ and every $f\colon X \to \mathcal{U}(A)$ there is a unique $f^\natural\colon \mathcal{F}(X) \to A$ such that $\eta_X;\mathcal{U}(f^\natural) = f$. η is called the **unit** of adjunction. Given an object A in $\mathcal{A}$, let ϵ_A denote the morphism $1^\natural_{\mathcal{U}(A)}$.

Then $\epsilon\colon \mathcal{U};\mathcal{F} \Rightarrow 1_{\mathcal{A}}$ is also a natural transformation and it is called the **counit** of adjunction. It has the couniversal property: for every $g\colon \mathcal{F}(X) \to A$ there exists a unique $g^\flat\colon X \to \mathcal{U}(A)$ such that $\mathcal{F}(g^\flat);\epsilon_A = g$. Many useful properties of adjoint functors are known [Lan71, GB84a, GB84b], including the following:

Proposition 4. *Let $X,Y \in |\mathcal{X}|$ and $A,B \in |\mathcal{A}|$, and let $f\colon X \to \mathcal{U}(A)$, $g\colon \mathcal{F}(X) \to A$, $u\colon Y \to X$ and $h\colon A \to B$. Then*

1. $\eta^\natural_X = 1_{\mathcal{F}(X)}$ *and* $\epsilon^\flat_A = 1_{\mathcal{U}(A)}$,
2. $(f^\natural)^\flat = f$ *and* $(g^\flat)^\natural = g$,
3. $(u;f)^\natural = \mathcal{F}(u);f^\natural$ *and* $(g;h)^\flat = g^\flat;\mathcal{U}(h)$,
4. $(f;\mathcal{U}(h))^\natural = f^\natural;h$ *and* $(\mathcal{F}(u);g)^\flat = u;g^\flat$.

3 Basic Definitions and Properties

It is well-known that a small category can be associated to any partially ordered set: there exists exactly one object A for each element a in the set and there exists a morphism from A to B, written $A \hookrightarrow B$, if and only if $a \leq b$. Furthermore, there is a bijection between partially ordered sets and small categories in which there is at most one morphism from A to B for every objects A and B (partiality), and if there is a morphism from A to B and a morphism from B to A then $A = B$ (anti-symmetry). The correspondents of infimum and supremum are the product and the coproduct, respectively. Generalizing all these to categories which are not required to be small, we get:

Definition 5. A category $\mathcal{I}$ is called a **category of inclusions** if and only if

- $\mathcal{I}(A, B)$ has at most one element, and
- $\mathcal{I}(A, B) \neq \emptyset$ and $\mathcal{I}(B, A) \neq \emptyset$ implies $A = B$.

for every pair of objects A and B. If $\mathcal{I}(A, B) \neq \emptyset$ then let $A \hookrightarrow B$ denote the unique morphism in $\mathcal{I}(A, B)$. It is called an **inclusion** and A is called a **subobject** of B. We say that $\mathcal{I}$ **has (finite) intersections** iff $\mathcal{I}$ has (finite) products and we say that $\mathcal{I}$ **has (finite) unions** iff $\mathcal{I}$ has (finite) coproducts. For every pair of objects A, B, let $A \cap B$ denote their product (also called their intersection) and let $A \cup B$ denote their coproduct (also called their union).

A small category of inclusions with finite intersections and finite unions corresponds to nothing else than a lattice [Rud63, Bir67, Grä71]. Consequently, many properties of lattices hold in categories of inclusions. The following are only a few:

Proposition 6. *For any category of inclusions $\mathcal{I}$ and any objects A, B and C in $|\mathcal{A}|$ (assume that $\mathcal{I}$ has finite intersections and/or finite unions whenever $\cap/\cup$ appear),*

1. $A \hookrightarrow A \cup B$ *and* $A \cap B \hookrightarrow A$,
2. $A \hookrightarrow B$ *implies* $A \cup B = B$ *and* $A \cap B = A$,
3. $A \cap (A \cup B) = A \cup (A \cap B) = A$,
4. $A \hookrightarrow B$ *implies* $A \cup C \hookrightarrow B \cup C$ *and* $A \cap C \hookrightarrow B \cap C$,
5. *The union and intersection are commutative, associative and idempotent,*
6. $(A \cap B) \cup (A \cap C) \hookrightarrow A \cap (B \cup C)$,
7. $A \cup (B \cap C) \hookrightarrow (A \cup B) \cap (A \cup C)$,

Definition 7. A category of inclusions $\mathcal{I}$ which is a subcategory of $\mathcal{A}$ having the same objects as $\mathcal{A}$ is called a **subcategory of inclusions** of $\mathcal{A}$ (alternatively, we can say that $\mathcal{A}$ **has inclusions** $\mathcal{I}$). $\mathcal{I}$ is a subcategory of **strong inclusions** of $\mathcal{A}$ (or $\mathcal{A}$ **has strong inclusions** $\mathcal{I}$) iff $\mathcal{I}$ is a subcategory of inclusions of $\mathcal{A}$, $\mathcal{I}$ has finite intersections and unions, and for every pair of objects A, B, their union in $\mathcal{I}$ is a pushout in $\mathcal{A}$ of their intersection in $\mathcal{I}$.

Example 1. We look at the following examples within the paper:

Set the category of sets and functions, in which the inclusions are the ordinary inclusions of sets. It is easy to see that these inclusions are strong for **Set**.

Top the category of topological spaces and continuous functions. The continuous inclusions form a subcategory of strong inclusions of **Top**: given A and B two topological spaces, their intersection is the set intersection of A and B together with the initial topology of its inclusions in A and B, and their union is the set union of A and B together with the final topology of the inclusions of A and B in their union.

Sign the category of many sorted algebraic signatures and morphisms of signatures. The signature inclusions form a subcategory of strong inclusions of **Sign**.

$\mathbf{Alg}_{\Sigma}$ the category of Σ-algebras and morphisms of Σ-algebras over a signature Σ. The inclusions of Σ-algebras form a subcategory of inclusions of $\mathbf{Alg}_{\Sigma}$, but it is not strong.

Alg$_{\Sigma,E}$ the full subcategory of **Alg**$_{\Sigma}$ containing all Σ-algebras that satisfy the Σ-equations E. Notice that depending on Σ and E, **Alg**$_{\Sigma,E}$ can be any category of important structures in mathematics or computer science: monoids, modules, groups, abelian groups, rings, commutative rings, etc. The inclusions in **Alg**$_{\Sigma,E}$ are not strong either.

We will not insist on the notion of strong inclusions in the present paper. However, strong inclusions together with *semiexactness* [DGS93] seem to play a major role in modularization. Abstract semantics is given for modularization in [Roş99], based on strong inclusions; no factorization (see Definition 8) is involved, which means that, perhaps, strong inclusions are good enough technical tools to handle complex modularization concepts.

Definition 8. $\langle \mathcal{I}, \mathcal{E} \rangle$ is a **weak inclusion system** of $\mathcal{A}$, or $\mathcal{A}$ **has a weak inclusion system** $\langle \mathcal{I}, \mathcal{E} \rangle$, iff $\mathcal{I}$ is a subcategory of inclusions of $\mathcal{A}$, $\mathcal{E}$ is a subcategory of $\mathcal{A}$ having the same objects as $\mathcal{A}$, and every morphism f in $\mathcal{A}$ has a *unique* factorization $f = e;i$ with $e \in \mathcal{E}$ and $i \in \mathcal{I}$. $\langle \mathcal{I}, \mathcal{E} \rangle$ is called an **inclusion system** if $\mathcal{E}$ contains only epics, and it is called a **regular inclusion system** if $\mathcal{E}$ contains only coequalizers.

Example 2. All structures in Example 1 have weak inclusion systems:

Set with $\mathcal{I}$ the set of inclusions and $\mathcal{E}$ the set of surjective functions. It is regular as each surjective function is a retract, so a coequalizer.

Top has two interesting weak inclusion systems (see [CR97]). One is $\langle \mathcal{I}_1, \mathcal{E}_1 \rangle$, where $\mathcal{I}_1$ is the set of continuous inclusions and $\mathcal{E}_1$ is the set of final continuous surjections, and the other one is $\langle \mathcal{I}_2, \mathcal{E}_2 \rangle$, where $\mathcal{I}_2$ is the set of initial continuous inclusions and $\mathcal{E}_2$ is the set of continuous surjections. $\langle \mathcal{I}_1, \mathcal{E}_1 \rangle$ is not an inclusion system as there are continuous surjective functions that are not final; $\langle \mathcal{I}_2, \mathcal{E}_2 \rangle$ is a regular inclusion system.

Alg$_{\Sigma}$ with inclusions of Σ-subalgebras and surjective morphisms of Σ-algebras is a regular inclusion system.

Alg$_{\Sigma,E}$ with inclusions of Σ-subalgebras satisfying E and surjective morphisms of Σ-algebras satisfying E is a regular inclusion system.

If $f : X \to Y$ is a morphism in $\mathcal{A}$, let $e_f; i_f$ denote its factorization and $f(X)$ denote the factorization object of f, that is, the target object of e_f. Moreover, we use the same notation, $f(A)$, for the factorization object of the morphism $A \hookrightarrow X; f$, where A is a subobject of X.

Notice that every category $\mathcal{A}$ admits a trivial weak inclusion system in which $\mathcal{I}$ contains only identities and $\mathcal{E} = \mathcal{A}$. The following fact contains properties of weak inclusion systems proved in [CR97]:

Proposition 9. *If $\langle \mathcal{I}, \mathcal{E} \rangle$ is a weak inclusion system of $\mathcal{A}$, $e \in \mathcal{E}$ and $i \in \mathcal{I}$, then*

1. *$\mathcal{I}$ contains only monics.*
2. *Each morphism in $\mathcal{I} \cap \mathcal{E}$ is an identity.*
3. **right-cancellable:** *If $f; i \in \mathcal{I}$ then $f \in \mathcal{I}$.*
4. *If $f; i \in \mathcal{E}$ then i is an identity and $f \in \mathcal{E}$.*
5. *If $f; g \in \mathcal{E}$ then $g \in \mathcal{E}$.*
6. *Any coequalizer is in $\mathcal{E}$.*

7. *Any retract is in $\mathcal{E}$.*
8. *All isomorphisms in $\mathcal{A}$ are in $\mathcal{E}$.*
9. **diagonal-fill-in:** *If $f;i = e;g$ then there is a* unique *morphism $h \in \mathcal{A}$ such that $e;h = f$ and $h;i = g$:*

$$\begin{tikzcd}
\bullet \arrow[r, "f"] \arrow[d, "e"'] & \bullet \arrow[d, hook, "i"] \\
\bullet \arrow[r, "g"'] \arrow[ur, dotted, "h"] & \bullet
\end{tikzcd}$$

The following proposition is also proved in [CR97] and it says that inclusions are preserved under pullbacks:

Proposition 10. *If $\mathcal{A}$ has pullbacks and a weak inclusion system $\langle \mathcal{I}, \mathcal{E} \rangle$, and if $i\colon B \hookrightarrow Y$ is an inclusion and $f\colon X \to Y$ is any morphism, then there is a unique pullback of the pair $\langle i, f \rangle$ such that the opposite arrow of i is an inclusion, too.*

The pullback object given by the proposition above is written $f^{-1}(B)$. An immediate consequence is that $f^{-1}(B)$ is a subobject of X.

Lemma 11. *Let $\langle \mathcal{I}, \mathcal{E} \rangle$ be a weak inclusion system of $\mathcal{A}$ and $f\colon X \to Y$ be a morphism in $\mathcal{A}$. Suppose that $\mathcal{A}$ has pullbacks whenever f^{-1} appears. Then*

1. *If $A \hookrightarrow A' \hookrightarrow X$ then $f(A) \hookrightarrow f(A') \hookrightarrow Y$,*
2. *If $B \hookrightarrow B' \hookrightarrow Y$ then $f^{-1}(B) \hookrightarrow f^{-1}(B') \hookrightarrow X$,*
3. *$(f;g)(A) = g(f(A))$ for every $g\colon Y \to Z$ and $A \hookrightarrow X$,*
4. *$(f;g)^{-1}(C) = f^{-1}(g^{-1}(C))$ for every $g\colon Y \to Z$ and $C \hookrightarrow Z$,*
5. *$A \hookrightarrow f^{-1}(f(A))$ and $\mathcal{E}(f^{-1}(f(A)), f(A)) \neq \emptyset$ for every $A \hookrightarrow X$,*
6. *$f(f^{-1}(B)) \hookrightarrow B$ for every $B \hookrightarrow Y$,*
7. *$A \hookrightarrow f^{-1}(B)$ iff $f(A) \hookrightarrow B$, for every $A \hookrightarrow X$ and $B \hookrightarrow Y$,*

Proof.

1. Let j be the inclusion $A \hookrightarrow A'$. Factor $A \hookrightarrow X; f$ as $e;i$ and $A' \hookrightarrow X; f$ as $e';i'$, and let $f(j)$ be the unique morphism given be the diagonal-fill-in lemma for the diagram $(j;e');i' = e;i$ in the picture below:

$$\begin{tikzcd}
A \arrow[rrr, "e"] \arrow[dr, hook] \arrow[dd, hook, "j"'] & & & f(A) \arrow[dl, hook, "i"] \arrow[dd, hook, dotted, "f(j)"] \\
& X \arrow[r, "f"] & Y & \\
A' \arrow[rrr, "e'"'] \arrow[ur, hook] & & & f(A') \arrow[ul, hook, "i'"']
\end{tikzcd}$$

Since $f(j);i' = i$, by the right-cancellable property, $f(j)$ is an inclusion, i.e., $f(A) \hookrightarrow f(A') \hookrightarrow Y$.

2. Let j be the inclusion $B \hookrightarrow B'$, let $\langle f^{-1}(B) \hookrightarrow X,\, u_f\rangle$ denote the pullback of $\langle B \hookrightarrow Y,\, f\rangle$, and let $\langle f^{-1}(B') \hookrightarrow X,\, v_f\rangle$ denote the pullback of $\langle B' \hookrightarrow Y,\, f\rangle$. Then define $f^{-1}(j)\colon\, f^{-1}(B) \to f^{-1}(B')$ as the unique morphism such that $f^{-1}(j); f^{-1}(B') \hookrightarrow X = f^{-1}(B) \hookrightarrow X$ and $f^{-1}(j); v_f = u_f; j$ (this is because $f^{-1}(B')$ is a pullback object):

$$
\begin{tikzcd}
 & X \arrow[r, "f"] & Y \\
 & f^{-1}(B') \arrow[u, hook] \arrow[r, "v_f"] & B' \arrow[u, hook] \\
f^{-1}(B) \arrow[uur, hook] \arrow[ur, hook, dotted, "f^{-1}(j)"'] \arrow[rr, "u_f"'] & & B \arrow[u, hook, "j"']
\end{tikzcd}
$$

   By the right-cancellable property, $f^{-1}(j)$ is an inclusion, that is, $f^{-1}(B) \hookrightarrow f^{-1}(B') \hookrightarrow X$.
3. It follows from the uniqueness of factorization for the morphism $A \hookrightarrow X; f; g$.
4. Let i_Z be the inclusion $C \hookrightarrow Z$, and let $\langle i_Y,\, u_g\rangle$, $\langle i_X,\, u_f\rangle$ and $\langle i'_X,\, u_{f;g}\rangle$ denote the pullbacks of the pairs $\langle i_Z,\, g\rangle$, $\langle i_Y,\, f\rangle$ and $\langle i_Z,\, f; g\rangle$, respectively, as in the diagram below:

$$
\begin{tikzcd}
 & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \\
 & f^{-1}(g^{-1}(C)) \arrow[u, hook, "i_X"'] \arrow[r, "u_f"] & g^{-1}(C) \arrow[u, hook, "i_Y"'] \arrow[r, "u_g"] & C \arrow[u, hook, "i_Z"'] \\
(f;g)^{-1}(C) \arrow[uur, hook, "i'_X"] \arrow[ur, hook, dotted, "i"] \arrow[ur, hook, dotted, "j"'] \arrow[urr, dotted, "v"] \arrow[urrr, "u_{f;g}"'] & & &
\end{tikzcd}
$$

   Since $\langle i_X,\, u_f; u_g\rangle$ is a cone of $\langle i_Z,\, f; g\rangle$, there is a unique j such that $j; u_{f;g} = u_f; u_g$ and $j; i'_X = i_X$. By the right-cancellable property, j is an inclusion. On the other hand, since $\langle i'_X; f,\, u_{f;g}\rangle$ is a cone of $\langle i_Z,\, g\rangle$ there is a unique v such that $v; u_g = u_{f;g}$ and $v; i_Y = i'_X; f$. Therefore, $\langle i'_X,\, v\rangle$ is a cone of $\langle i_Y,\, f\rangle$, so there is a unique i such that $i; u_f = v$ and $i; i_X = i'_X$. By the right-cancellable property, i is an inclusion. Consequently, $(f; g)^{-1}(C) = f^{-1}(g^{-1}(C))$.
5. Factor $A \hookrightarrow X; f$ as $e; i$, and let $\langle f^{-1}(f(A)) \hookrightarrow X,\, u_f\rangle$ be the pullback of

$\langle i, f\rangle$ as in the diagram below:

$$\begin{tikzcd}
& X \arrow[r, "f"] & Y \\
& A \arrow[u, hook] \arrow[r, "e"] \arrow[dl, dotted, hook, "j"'] & f(A) \arrow[u, hook, "i"'] \\
f^{-1}(f(A)) \arrow[uur, hook] \arrow[urr, "u_f"'] & &
\end{tikzcd}$$

By the pullback property, there is a unique morphism $j\colon A \to f^{-1}(f(A))$ such that $j;u_f = e$ and $j;f^{-1}(f(A)) \hookrightarrow X = A \hookrightarrow X$. By the right-cancellable property, j is an inclusion, and by 5. in Fact 9, $u_f \in \mathcal{E}$. Therefore, $A \hookrightarrow f^{-1}(f(A))$ and $\mathcal{E}(f^{-1}(f(A)), f(A)) \neq \emptyset$.

6. Let i be the inclusion $B \hookrightarrow Y$, let $\langle f^{-1}(B) \hookrightarrow X, u_f\rangle$ be the pullback of $\langle i, f\rangle$, and let $e';i'$ be the factorization of $f^{-1}(B) \hookrightarrow X;f$

$$\begin{tikzcd}
X \arrow[r, "f"] & Y & \\
f^{-1}(B) \arrow[u, hook] \arrow[r, "u_f"'] \arrow[drr, "e'"'] & B \arrow[u, hook, "i"'] & \\
& & f(f^{-1}(B)) \arrow[uul, hook, "i'"'] \arrow[ul, dotted, hook, "j"']
\end{tikzcd}$$

By the pullback property, there is a unique $j\colon f(f^{-1}(B)) \to B$ such that $e';j = u_f$ and $j;i = i'$. By the right-cancellable property, j is an inclusion, that is, $f(f^{-1}(B)) \hookrightarrow B$.

7. If $A \hookrightarrow f^{-1}(B)$ then by 1., $f(A) \hookrightarrow f(f^{-1}(B))$, and by 6., one gets $f(A) \hookrightarrow B$. On the other side, if $f(A) \hookrightarrow B$ then by 2., $f^{-1}(f(A)) \hookrightarrow f^{-1}(B)$, and by 5., $A \hookrightarrow f^{-1}(B)$.

Definition 12. If $\langle \mathcal{I}, \mathcal{E}\rangle$ is a weak inclusion system of $\mathcal{A}$ and $\mathcal{D}$ is a subcategory of $\mathcal{A}$ then let $\mathcal{I}^{\mathcal{D}}$ and $\mathcal{E}^{\mathcal{D}}$ denote the restrictions of $\mathcal{I}$ and $\mathcal{E}$, respectively, to $\mathcal{D}$. $Sub(A)$ is the full subcategory of $\mathcal{A}$ generated by all subobjects of an object A; it is called the **subobject category** of A and we write $\mathcal{I}_A$ and $\mathcal{E}_A$ instead of $\mathcal{I}^{Sub(A)}$ and $\mathcal{E}^{Sub(A)}$. As usual, a subcategory $\mathcal{D}$ of $\mathcal{A}$ is **closed under subobjects** iff $A \in |\mathcal{D}|$ whenever $A \hookrightarrow B$ and $B \in |\mathcal{D}|$.

Proposition 13. *Let $\langle \mathcal{I}, \mathcal{E}\rangle$ be a weak inclusion system of $\mathcal{A}$. Then*

1. *If $\mathcal{D}$ is a full subcategory of $\mathcal{A}$ closed under subobjects then $\langle \mathcal{I}^{\mathcal{S}}, \mathcal{E}^{\mathcal{S}}\rangle$ is a weak inclusion system of $\mathcal{S}$.*
2. *If A is an object in $\mathcal{A}$ then $\langle \mathcal{I}_A, \mathcal{E}_A\rangle$ is a weak inclusion system of $Sub(A)$.*

Proof. It is straightforward that $\mathcal{I}^{\mathcal{D}}$ is a subcategory of inclusions of $\mathcal{D}$. If f is a morphism in $\mathcal{D}$ then e_f and i_f belong to $\mathcal{D}$ because $\mathcal{D}$ is full and closed under subobjects, so f admits a factorization in $\langle \mathcal{I}^{\mathcal{D}}, \mathcal{E}^{\mathcal{D}} \rangle$. Furthermore, the factorization is unique as it is unique in $\mathcal{A}$. 2 follows from 1 observing that $Sub(A)$ is a full subcategory of $\mathcal{A}$ closed under subobjects.

Theorem 14. *If $\langle \mathcal{I}, \mathcal{E} \rangle$ is a weak inclusion system for $\mathcal{A}$ then every morphism $f\colon X \to Y$ yields a pair of adjoint functors, $f\colon \mathcal{I}_X \to \mathcal{I}_Y$ and $f^{-1}: \mathcal{I}_Y \to \mathcal{I}_X$, where $f\colon \mathcal{I}_X \to \mathcal{I}_Y$ is a left adjoint of $f^{-1}: \mathcal{I}_Y \to \mathcal{I}_X$.*

Proof. 1., 2., 3. and 4. in Lemma 11 say nothing else than $f\colon \mathcal{I}_X \to \mathcal{I}_Y$ and $f^{-1}: \mathcal{I}_Y \to \mathcal{I}_X$ are functors. An easy way to show that f is a left adjoint to f^{-1} is to use Definition 2: 7. in Lemma 11 says that for every $A \in \mathcal{I}_X$ and $B \in \mathcal{I}_Y$ there is a bijection $\mathcal{I}_X(A, f^{-1}(B)) \cong \mathcal{I}_Y(f(A), B)$; this bijection is natural in A and B because there is at most one inclusion between any two objects both in $\mathcal{I}_X$ and $\mathcal{I}_Y$.

Corollary 15. *Any category $\mathcal{A}$ admitting a weak inclusion system is an existential Lawvere doctrine.*

Proposition 16. *In the context of Proposition 14,*

1. $f; f^{-1}; f = f$ in **Cat**,
2. $f^{-1}; f; f^{-1} = f^{-1}$ in **Cat**, *and*
3. If f is an isomorphism and $f^*: Y \to X$ is its inverse, then $f^{-1} = f^*$ as functors $\mathcal{I}_Y \to \mathcal{I}_X$.

Proof. 1. It suffices to show that $f(f^{-1}(f(A))) = f(A)$ for every $A \hookrightarrow X$. Applying 1. in Lemma 11 for the inclusion $A \hookrightarrow f^{-1}(f(A))$ given by 5. in Lemma 11, we get $f(A) \hookrightarrow f(f^{-1}(f(A)))$. On the other hand, applying 6. in Lemma 11 for $B = f(A)$ we get $f(f^{-1}(f(A))) \hookrightarrow f(A)$.
2. Similar to 1..
3. Notice that $f; f^* = 1_{\mathcal{I}_X}$ and $f^*; f = 1_{\mathcal{I}_Y}$. Composing $f; f^{-1}; f = f$ in 1. with f^* on the left, we get $f^*; (f; f^{-1}; f) = f^*; f$, that is, $f^{-1}; f = 1_{\mathcal{I}_Y}$. Composing with f^* on the right we get $(f^{-1}; f); f^* = f^*$, that is, $f^{-1} = f^*$.

4 Inclusion Systems vs. Complete Weak Inclusion Systems

The notion of weak inclusion system is too general because it catches very uninteresting cases (for example, the case where $\mathcal{I}$ contains the identities and $\mathcal{E}$ contains all the morphisms). For this reason, stronger results usually require a stronger version of weak inclusion systems, such as inclusion systems.

This section presents an alternative of the inclusion system, called *complete* weak inclusion system, which does not require the morphisms in $\mathcal{E}$ be epimorphisms, still having much of the power of inclusion systems. For example, [Roş96] presents Birkhoff-like axiomatizability results for a categorical generalization of equational logic strongly based on inclusion systems; all the results in that paper could be very well done in a framework based on complete weak inclusion systems instead of inclusion systems.

Proposition 17. *The following assertions are equivalent in any category $\mathcal{A}$ admitting a weak inclusion system $\langle \mathcal{I}, \mathcal{E} \rangle$:*

1. *Each monomorphism in $\mathcal{E}$ is an isomorphism,*
2. *Each monomorphism is factored as an isomorphism and an inclusion,*
3. *Each mono subobject contains* exactly *one inclusion.*

Proof. 1. $\Rightarrow$ 2. Let m be a monic and let $e_m; i_m$ be its factorization. Then e_m is a monic, too, and by hypothesis it is an isomorphism.
2. $\Rightarrow$ 3. Each mono subobject contains at most one inclusion because if $i \sim i'$ then there exist f and g such that $f; i = i'$ and $g; i' = i$, so by the right cancellable property f and g are inclusions, that is, $i = i'$. To show that each mono subobject contains at least one inclusion, let m be a monic and consider $e_m; i_m$ its factorization; then e_m is an isomorphism. Thus $m \sim i_m$.
3. $\Rightarrow$ 1. Let m be a monic in $\mathcal{E}$ and let i be the unique inclusion such that $m \sim i$. Then there exist f and g such that $f; m = i$ and $g; i = m$. Since m is in $\mathcal{E}$, i is an identity (see 4. in Fact 9). Therefore $f; m = 1$; moreover, $(m; f); m = m$ implies $m; f = 1$ because m is a monic. Consequently, m is an isomorphism.

The fact above presents conditions under which the inclusions given by a weak inclusion system give a complete and independent system of representatives of the mono subobjects. The following definition introduces formally the notion of complete weak inclusion system:

Definition 18. A (weak) inclusion system $\langle \mathcal{I}, \mathcal{E} \rangle$ verifying the equivalences in the proposition above is called a **complete** (weak) inclusion system.

Example 3. Among the structures in Example 2, only **Top** with $\langle \mathcal{I}_2, \mathcal{E}_2 \rangle$ is not complete because there are monics which are continuous surjections but which are not isomorphisms.

The following propositions show relations between inclusion systems and complete weak inclusion systems:

Proposition 19. *Any regular inclusion system is both an inclusion system and a complete weak inclusion system (i.e., it is a complete inclusion system).*

Proof. It follows immediately from the fact that any monomorphism which is a coequalizer, actually is an isomorphism.

Definition 20. We let *Mono* and *Epi* denote the subcategories of $\mathcal{A}$ containing all monics and epics of $\mathcal{A}$, respectively. A category in which any morphism that is both an epic and a monic is an isomorphism is called a **balanced category**.

Proposition 21. *Let $\langle \mathcal{I}, \mathcal{E} \rangle$ be a weak inclusion system of $\mathcal{A}$. Then*

1. *If $\mathcal{A}$ is balanced then $Epi \subseteq \mathcal{E}$,*
2. *If $\mathcal{A}$ is balanced and $\mathcal{E} = Epi$ then $\langle \mathcal{I}, \mathcal{E} \rangle$ is complete,*
3. *If $\mathcal{A}$ is balanced and $\langle \mathcal{I}, \mathcal{E} \rangle$ is an inclusion system then $\langle \mathcal{I}, \mathcal{E} \rangle$ is complete,*
4. *If $\langle \mathcal{I}, \mathcal{E} \rangle$ is a complete weak inclusion system and $\langle E, \mathcal{M} \rangle$ is a factorization system of $\mathcal{A}$ then $\mathcal{E} \subseteq E$,*

5. *If $\langle \mathcal{I}, \mathcal{E} \rangle$ is a complete weak inclusion system and $\mathcal{A}$ has a factorization system then $\langle \mathcal{I}, \mathcal{E} \rangle$ is an inclusion system,*
6. *If $\mathcal{A}$ is balanced and has a factorization system then $\langle \mathcal{I}, \mathcal{E} \rangle$ is an inclusion system iff $\langle \mathcal{I}, \mathcal{E} \rangle$ is a complete weak inclusion system,*
7. *If $\langle \mathcal{I}, \mathcal{E} \rangle$ is a complete inclusion system then $\langle \mathcal{E}, Mono \rangle$ is a factorization system of $\mathcal{A}$.*

Proof. 1. Let g be any epic and let $e_g;i_g$ be its factorization. Then i_g is an epic, too. Since i_g is a monic (1. in Fact 9) and $\mathcal{A}$ is balanced, i_g is an iso. Hence i_g is an identity (see 8. and 2. in Fact 9), so $g = e_g \in \mathcal{E}$.
2. Let m be a monic in $\mathcal{E} = Epi$. Then m is an iso as $\mathcal{A}$ is balanced.
3. It is an immediate consequence of 1. and 2..
4. Let $e \in \mathcal{E}$ and let $e';m'$ be a factorization of e in $\langle E, \mathcal{M} \rangle$. By 5. in Fact 9 m' is in $\mathcal{E}$, and since $\langle \mathcal{I}, \mathcal{E} \rangle$ is complete, m' is an isomorphism. So $e = e';m'$ is in E.
5. Immediately from 4..
6. It follows from 3. and 5..
7. Every morphism f admits a factorization in $\langle \mathcal{E}, Mono \rangle$, namely $e_f;i_f$, because i_f is a mono (1. in Fact 9). Now, let us consider two factorizations $e;m = e';m'$ and let $m = e_m;i_m$ and $m' = e_{m'};i_{m'}$ be factorizations in $\langle \mathcal{I}, \mathcal{E} \rangle$; then $(e;e_m);i_m = (e';e_{m'});i_{m'}$, so $i_m = i_{m'}$ (the factorization is unique in $\langle \mathcal{I}, \mathcal{E} \rangle$). Since $\langle \mathcal{I}, \mathcal{E} \rangle$ is complete, e_m and $e_{m'}$ are isomorphisms. Consequently, there exists an isomorphism $\alpha = e_m;e_{m'}^{-1}$ such that $e;\alpha = e'$ and $\alpha;m' = m$. Since $\mathcal{E} \subseteq Epi$, we conclude that $\langle \mathcal{E}, Mono \rangle$ is a factorization system.

Example 4. Looking to the categories in Example 2, it can be easily seen that **Set** and $\mathbf{Alg}_\Sigma$ are balanced. **Top** is not balanced. $\mathbf{Alg}_{\Sigma,E}$ can be either balanced or not. For example, groups form a balanced category but rings do not: the inclusion of integers in rationals is both a monic and an epic without being an isomorphism.

5 Reachability and Projectivity

Definition 22. Let $\mathcal{I}$ be a subcategory of inclusions of $\mathcal{A}$. An object A is **$\mathcal{I}$-reachable** iff it has no proper subobjects, i.e., $B \hookrightarrow A$ implies $B = A$. If $\mathcal{A}$ admits an initial object I and $\mathcal{E}$ is a class of morphisms in $\mathcal{A}$, then A is **$\mathcal{E}$-generated** iff the unique morphism $\alpha_A : I \to A$ is in $\mathcal{E}$.

Proposition 23. *If $\mathcal{A}$ admits an initial object and $\langle \mathcal{I}, \mathcal{E} \rangle$ is a weak inclusion system of $\mathcal{A}$ then an object is $\mathcal{I}$-reachable if and only if it is $\mathcal{E}$-generated.*

Proof. Let A be a an $\mathcal{I}$-reachable object and let $e;i$ be the factorization of the unique morphism $\alpha_A : I \to A$. Since A is $\mathcal{I}$-reachable, i is an identity. Hence α_A is equal to $e \in \mathcal{E}$. Conversely, if A is $\mathcal{E}$-generated and B is a subobject of A (let i denote the inclusion $B \hookrightarrow A$) then $\alpha_B;i = \alpha_A \in \mathcal{E}$, so i is an identity (see 4. in Fact 9).

It is well-known that a category $\mathcal{A}$ admits an initial object iff $\mathcal{U}\colon \mathcal{A} \to \{*\}$ has a left adjoint, where $\{*\}$ is the category having one object and one morphism and $\mathcal{U}$ takes every object/morphism to the object/morphism of $\{*\}$. Generalizing that, from now on in this section, let $\mathcal{U}\colon \mathcal{A} \to \mathcal{X}$ be a functor having a left adjoint $\mathcal{F}\colon \mathcal{X} \to \mathcal{A}$.

Definition 24. Let $\mathcal{I}$ and $\mathcal{E}$ be classes of morphisms in $\mathcal{A}$. An object A in $\mathcal{A}$ is $(\mathcal{U},\mathcal{I})$**-reachable** iff there exists no proper subobject B of A such that $\mathcal{U}(B \hookrightarrow A)$ is an isomorphism. A is $(\mathcal{U},\mathcal{E})$**-generated** iff there exist some free objects F and some morphisms $e\colon F \to A$ in $\mathcal{E}$.

Theorem 25. *Let $\langle \mathcal{I}, \mathcal{E}\rangle$ be a weak inclusion system of $\mathcal{A}$. Then*

1. *A is $(\mathcal{U},\mathcal{E})$-generated iff $\epsilon_A \in \mathcal{E}$,*
2. *A is $(\mathcal{U},\mathcal{I})$-reachable iff A is $(\mathcal{U},\mathcal{E})$-generated.*

Proof. 1. If $\epsilon_A\colon \mathcal{F}(\mathcal{U}(A)) \to A$ is in $\mathcal{E}$ then obviously A is $(\mathcal{U},\mathcal{E})$-generated as $\mathcal{F}(\mathcal{U}(A))$ is a free object. Conversely, if A is $(\mathcal{U},\mathcal{E})$-generated then there exist an X in $|\mathcal{X}|$ and a morphism $e\colon \mathcal{F}(X) \to A$ in $\mathcal{E}$. Since $e = \mathcal{F}(e^\flat);\epsilon_A$, by 5. in Fact 9, ϵ_A is in $\mathcal{E}$.

2. Firstly, assume that A is $(\mathcal{U},\mathcal{E})$-generated and let $B \hookrightarrow A$ such that $\mathcal{U}(B \hookrightarrow A)$ is an isomorphism. Since $\epsilon\colon \mathcal{U};\mathcal{F} \Rightarrow 1_{\mathcal{A}}$ is a natural transformation, $\epsilon_B;(B \hookrightarrow A) = \mathcal{F}(\mathcal{U}(B \hookrightarrow A));\epsilon_A$, and since $\mathcal{F}(\mathcal{U}(B \hookrightarrow A))$ is an isomorphism and $\mathcal{E}$ contains all isomorphisms, $\epsilon_B;(B \hookrightarrow A)$ is in $\mathcal{E}$. By 5. in Fact 9, $B \hookrightarrow A$ is in $\mathcal{E}$, so by 2. in Fact 9, $B = A$. Therefore A is $(\mathcal{U},\mathcal{I})$-reachable.

Conversely, let A be $(\mathcal{U},\mathcal{I})$-reachable and factor ϵ_A as $e;i$. Then by Fact 4, $1_{\mathcal{U}(A)} = \epsilon_A^\flat = e^\flat;\mathcal{U}(i)$. Let B be the factorization object of ϵ_A, i.e., the target of e.

$$\begin{tikzcd}
\mathcal{U}(A) \arrow[d, shift left, "e^\flat"] & \mathcal{F}(\mathcal{U}(A)) \arrow[r, "\epsilon_A"] \arrow[d, shift left, "\mathcal{F}(e^\flat)"] \arrow[dr, "e"] & A \\
\mathcal{U}(B) \arrow[u, shift left, "\mathcal{U}(i)"] & \mathcal{F}(\mathcal{U}(B)) \arrow[r, "\epsilon_B"'] \arrow[u, shift left, "\mathcal{F}(\mathcal{U}(i))"] & B \arrow[u, hook, "i"']
\end{tikzcd}$$

Then

$$\begin{aligned}
\epsilon_B;i &= \mathcal{F}(\mathcal{U}(i));\epsilon_A && (\epsilon \text{ is natural})\\
&= \mathcal{F}(\mathcal{U}(i);e^\flat;\mathcal{U}(i));\epsilon_A && (\text{ as } e^\flat;\mathcal{U}(i) = 1_{\mathcal{U}(A)})\\
&= \mathcal{F}(\mathcal{U}(i);e^\flat);(\mathcal{F}(\mathcal{U}(i));\epsilon_A) &&\\
&= \mathcal{F}(\mathcal{U}(i);e^\flat);\epsilon_B;i && (\epsilon \text{ is natural})
\end{aligned}$$

Since i is a monic, $\epsilon_B = \mathcal{F}(\mathcal{U}(i);e^\flat);\epsilon_B$, so $\mathcal{U}(i);e^\flat = \epsilon_B^\flat = 1_{\mathcal{U}(B)}$. Therefore $e^\flat;\mathcal{U}(i) = 1_{\mathcal{U}(A)}$ and $\mathcal{U}(i);e^\flat = 1_{\mathcal{U}(B)}$, that is, $\mathcal{U}(i)$ is an isomorphism. Since A is $(\mathcal{U},\mathcal{I})$-reachable, i is an identity, so $\epsilon_A = e \in \mathcal{E}$, which means that A is $(\mathcal{U},\mathcal{E})$-generated.

Definition 26. Given a class $\mathcal{E}$ of morphisms in $\mathcal{A}$, then an object P is **$\mathcal{E}$-projective** iff for every morphism $e\colon A \to B$ in $\mathcal{E}$ and every morphism $f\colon P \to B$ in $\mathcal{A}$, there are some $g\colon P \to A$ such that $g;e = f$

$$\begin{tikzcd} A \arrow[rr, "e\in\mathcal{E}"] & & B \\ & P \arrow[ul, "g\in\mathcal{A}"] \arrow[ur, "f\in\mathcal{A}"'] & \end{tikzcd}$$

$\mathcal{A}$ **has enough $\mathcal{E}$-projectives** iff for every object A there are some $\mathcal{E}$-projective objects P and some morphisms $e\colon P \to A$ in $\mathcal{E}$.

$\mathcal{E}$ contains exactly the surjective morphisms in many practical examples. From now on in this section, let $\mathcal{U}\colon \mathcal{A} \to \mathcal{X}$ be a functor having a left adjoint $\mathcal{F}\colon \mathcal{X} \to \mathcal{A}$. In some situations free and projective objects conflate, such as in groups or abelian groups, and in most situations free objects are projective. However, there are situations in which free objects are not necessary projective, such as the discrete Housdorff spaces in the category of Housdorff spaces and continue functions[5]. On the other hand, projective objects are not necessary free because, for example, there can be built an adjunction for each category with initial object such that the initial object is the only free object, but of course there could be more projective objects in that category.

In this section we show some conditions under which free objects are projective, and also conditions under which a category has enough $\mathcal{E}$-projectives.

Proposition 27. *If $\mathcal{E}$ is a class of morphisms in $\mathcal{A}$ then*

1. *$X \in |\mathcal{X}|$ is $\mathcal{U}(\mathcal{E})$-projective if and only if $\mathcal{F}(X)$ is $\mathcal{E}$-projective,*
2. *If every object in $\mathcal{X}$ is $\mathcal{U}(\mathcal{E})$-projective then every free object in $\mathcal{A}$ is $\mathcal{E}$-projective.*

Proof. Suppose that X is $\mathcal{U}(\mathcal{E})$-projective, let $e\colon A \to B$ be any morphism in $\mathcal{E}$, and let $f\colon \mathcal{F}(X) \to B$ be any morphism in $\mathcal{A}$.

$$\begin{tikzcd} A \arrow[rr, "e"] & & B \\ & \mathcal{F}(X) \arrow[ul, dotted, "g^{\natural}"] \arrow[ur, "f"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{U}(A) \arrow[rr, "\mathcal{U}(e)"] & & \mathcal{U}(B) \\ & X \arrow[ul, dotted, "g"] \arrow[ur, "f^{\flat}"'] & \end{tikzcd}$$

Since X is $\mathcal{U}(\mathcal{E})$-projective, there exist some $g\colon X \to \mathcal{U}(A)$ such that $g;\mathcal{U}(e) = f^{\flat}$. By Fact 4, $f = (f^{\flat})^{\natural} = (g;\mathcal{U}(e))^{\natural} = g^{\natural};e$. Therefore $\mathcal{F}(X)$ is $\mathcal{E}$-projective.

On the other hand, suppose that $\mathcal{F}(X)$ is $\mathcal{E}$-projective, let $e\colon A \to B$ be any morphism in $\mathcal{E}$, and let $f\colon X \to \mathcal{U}(B)$ be any morphism in $\mathcal{X}$.

$$\begin{tikzcd} A \arrow[rr, "e"] & & B \\ & \mathcal{F}(X) \arrow[ul, dotted, "g"] \arrow[ur, "f^{\natural}"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{U}(A) \arrow[rr, "\mathcal{U}(e)"] & & \mathcal{U}(B) \\ & X \arrow[ul, dotted, "g^{\flat}"] \arrow[ur, "f"'] & \end{tikzcd}$$

[5] This is because the epimorphisms are exactly the functions for which the image of the source is dense in the target.

Since $\mathcal{F}(X)$ is $\mathcal{E}$-projective, there are some $g\colon \mathcal{F}(X) \to A$ such that $g;e = f^{\natural}$. By Fact 4, $f = (f^{\natural})^{\flat} = (g;e)^{\flat} = g^{\flat};\mathcal{U}(e)$, that is, X is $\mathcal{U}(\mathcal{E})$-projective.

2. follows immediately from 1..

Theorem 28. *If $\langle \mathcal{I}, \mathcal{E}\rangle$ is a weak inclusion system of $\mathcal{A}$ such that every object in $\mathcal{X}$ is $\mathcal{U}(\mathcal{E})$-projective, then $\mathcal{A}$ has enough projectives whenever every object in $\mathcal{A}$ is $(\mathcal{U},\mathcal{I})$-reachable.*

Notice that $\mathcal{X}$ is **Set** in most practical situations, and that testing if an object is $(\mathcal{U},\mathcal{I})$-reachable is easy. Therefore, Theorem 28 can be viewed as an easy criterion to say if a category has enough projectives.

Example 5. All categories in Example 2 have enough projectives.

6 Weak Inclusion Systems for Complex Categories

In this section, (weak) inclusion systems are built for comma categories, functor categories and categories of algebras and coalgebras, from (weak) inclusion systems for the categories involved.

6.1 Comma Categories

Given $\mathcal{F}\colon \mathcal{C} \to \mathcal{A}$ and $\mathcal{G}\colon \mathcal{D} \to \mathcal{A}$, the comma category $(\mathcal{F}/\mathcal{G})$ has triples $\langle C, f\colon \mathcal{F}(C) \to \mathcal{G}(D), D\rangle$ as objects, where C and D are objects in $\mathcal{C}$ and $\mathcal{D}$, respectively, and f is a morphism in $\mathcal{A}$. A morphism from $\langle C, f\colon \mathcal{F}(C) \to \mathcal{G}(D), D\rangle$ to $\langle C', f'\colon \mathcal{F}(C') \to \mathcal{G}(D'), D'\rangle$ in $(\mathcal{F}/\mathcal{G})$ is a pair $(c\colon C \to C', d\colon D \to D')$ of morphisms in $\mathcal{C}$ and $\mathcal{D}$, respectively, such that $f;\mathcal{G}(d) = \mathcal{F}(c);f'$.

Proposition 29. *Let $\langle \mathcal{I}_{\mathcal{C}}, \mathcal{E}_{\mathcal{C}}\rangle$, $\langle \mathcal{I}_{\mathcal{D}}, \mathcal{E}_{\mathcal{D}}\rangle$ and $\langle \mathcal{I}_{\mathcal{A}}, \mathcal{E}_{\mathcal{A}}\rangle$ be weak inclusion system for $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{A}$, respectively, and let $\mathcal{F}\colon \mathcal{C} \to \mathcal{A}$ and $\mathcal{G}\colon \mathcal{D} \to \mathcal{A}$ be functors such that $\mathcal{F}$ preserves the $\mathcal{E}_{\mathcal{C}}$-morphisms and $\mathcal{G}$ preserves the inclusions. Then*

1. *$\langle \mathcal{I}, \mathcal{E}\rangle$ is a weak inclusion system for the comma category $(\mathcal{F}/\mathcal{G})$, where*

$$\mathcal{I} = \{(i_{\mathcal{C}}, i_{\mathcal{D}}) \in (\mathcal{F}/\mathcal{G}) \mid i_{\mathcal{C}} \in \mathcal{I}_{\mathcal{C}},\ i_{\mathcal{D}} \in \mathcal{I}_{\mathcal{D}}\}$$
$$\mathcal{E} = \{(e_{\mathcal{C}}, e_{\mathcal{D}}) \in (\mathcal{F}/\mathcal{G}) \mid e_{\mathcal{C}} \in \mathcal{E}_{\mathcal{C}},\ e_{\mathcal{D}} \in \mathcal{E}_{\mathcal{D}}\}$$

2. *$\langle \mathcal{I}, \mathcal{E}\rangle$ is an inclusion system if $\langle \mathcal{I}_{\mathcal{C}}, \mathcal{E}_{\mathcal{C}}\rangle$ and $\langle \mathcal{I}_{\mathcal{D}}, \mathcal{E}_{\mathcal{D}}\rangle$ are inclusion systems.*

Proof. 1. It is straightforward that $\mathcal{I}$ and $\mathcal{E}$ can be organized as subcategories of $(\mathcal{F}/\mathcal{G})$ having the same objects as $(\mathcal{F}/\mathcal{G})$; also, it can be easily seen that $\mathcal{I}$ is a partial order. It remains to show that every morphism (c, d) can be factored uniquely as a morphism in $\mathcal{E}$ composed with a morphism in $\mathcal{I}$.

$$\begin{tikzcd}
C \arrow[dd, "c"'] \arrow[dr, "e_c"] & \\
& c(C) \arrow[dl, hook, "i_c"] \\
C' &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}(C) \arrow[rrr, "f"] \arrow[dd, "\mathcal{F}(c)"'] \arrow[dr, "\mathcal{F}(e_c)"] & & & \mathcal{G}(D) \arrow[dd, "\mathcal{G}(d)"] \arrow[dl, "\mathcal{G}(e_d)"'] \\
& \mathcal{F}(c(C)) \arrow[r, dotted, "h"] \arrow[dl, hook, "\mathcal{F}(i_c)"] & \mathcal{G}(d(D)) \arrow[dr, hook, "\mathcal{G}(i_d)"'] & \\
\mathcal{F}(C') \arrow[rrr, "f'"] & & & \mathcal{G}(D')
\end{tikzcd}
\qquad
\begin{tikzcd}
& D \arrow[dd, "d"] \arrow[dl, "e_d"'] \\
d(D) \arrow[dr, hook, "i_d"'] & \\
& D'
\end{tikzcd}$$

We claim that (c, d) can be uniquely factored as $(e_c, e_d); (i_c, i_d)$. Let us show that $(e_c, e_d); (i_c, i_d)$ is a correct factorization of (c, d), i.e., (e_c, e_d) and (i_c, i_d) are morphisms in $(\mathcal{F}/\mathcal{G})$ and $(c, d) = (e_c, e_d); (i_c, i_d)$. Since $\mathcal{F}$ and $\mathcal{G}$ are morphisms of weak inclusive categories, by diagonal-fill-in lemma there exists a unique $h\colon \mathcal{F}(c(C)) \to \mathcal{G}(d(D))$ such that $f; \mathcal{G}(e_d)) = \mathcal{F}(e_c); h$ and $h; \mathcal{G}(i_d) = \mathcal{F}(i_c); f'$. This certifies that (e_c, e_d) and (i_c, i_d) are morphisms in $(\mathcal{F}/\mathcal{G})$, from $\langle C, f, D\rangle$ to $\langle c(C), h, d(D)\rangle$ and from $\langle c(C), h, d(D)\rangle$ to $\langle C', f', D'\rangle$, respectively. Since $(e_c, e_d); (i_c, i_d) = (e_c; i_c, e_d, i_d) = (c, d)$, we deduce that it is a factorization of (c, d). Now, suppose that $(e_\mathcal{C}, e_\mathcal{D}); (i_\mathcal{C}, i_\mathcal{D})$ is another factorization of (c, d). Then $e_\mathcal{C}; i_\mathcal{C} = c$ and $e_\mathcal{D}; i_\mathcal{D} = d$, and because of the uniqueness of factorizations in $\mathcal{C}$ and $\mathcal{D}$, we deduce that $(e_\mathcal{C}, e_\mathcal{D}) = (e_c, e_d)$ and $(i_\mathcal{C}, i_\mathcal{D}) = i_c, i_d)$. Therefore, $\langle \mathcal{I}, \mathcal{E}\rangle$ is a weak inclusion system for $(\mathcal{F}/\mathcal{G})$.

2. It follows from the fact that $(e_\mathcal{C}, e_\mathcal{D})$ is an epic whenever $e_\mathcal{C}$ and $e_\mathcal{D}$ are epics.

6.2 Functor Categories

The category $\mathcal{C}^\mathcal{D}$ has functors $\mathcal{D} \to \mathcal{C}$ as objects and natural transformations between them as morphisms.

Proposition 30. *Let $\langle \mathcal{I}_\mathcal{C}, \mathcal{E}_\mathcal{C}\rangle$ be a weak inclusion system for $\mathcal{C}$ and let $\mathcal{D}$ be any category. Then*

1. *$\langle \mathcal{I}, \mathcal{E}\rangle$ is a weak inclusion system for the functor category $\mathcal{C}^\mathcal{D}$, where*

$$\mathcal{I} = \{i\colon F \Rightarrow G \mid (\forall F, G \in |\mathcal{C}^\mathcal{D}|)(\forall D \in |\mathcal{D}|)\ i_D \in \mathcal{I}\}$$
$$\mathcal{E} = \{e\colon F \Rightarrow G \mid (\forall F, G \in |\mathcal{C}^\mathcal{D}|)(\forall D \in |\mathcal{D}|)\ e_D \in \mathcal{E}\}$$

2. *$\langle \mathcal{I}, \mathcal{E}\rangle$ is an inclusion system whenever $\langle \mathcal{I}_\mathcal{C}, \mathcal{E}_\mathcal{C}\rangle$ is an inclusion system.*

Proof. 1. It is straightforward that $\mathcal{I}$ and $\mathcal{E}$ are subcategories of $\mathcal{C}^\mathcal{D}$ with the same objects as $\mathcal{C}^\mathcal{D}$, and that $\mathcal{I}$ is a partial order. It remains to show that every natural transformation can be uniquely factored as $e; i$ with e in $\mathcal{E}$ and i in $\mathcal{I}$. Let $\sigma\colon F \Rightarrow G$ be a natural transformation and $\alpha\colon D \to D'$ be a morphism in $\mathcal{D}$.

$$\begin{tikzcd}
D \arrow[d, "\alpha"'] & F(D) \arrow[r, "e_{\sigma_D}"'] \arrow[rr, bend left, "\sigma_D"] \arrow[d, "F(\alpha)"'] & H_D \arrow[r, hook, "i_{\sigma_D}"'] \arrow[d, "H_\alpha"] & G(D) \arrow[d, "G(\alpha)"] \\
D' & F(D') \arrow[r, "e_{\sigma_{D'}}"] \arrow[rr, bend right, "\sigma_{D'}"'] & H_{D'} \arrow[r, hook, "i_{\sigma_{D'}}"] & G(D')
\end{tikzcd}$$

Denote by H_D and $H_{D'}$ the objects $\sigma_D(F(D))$ and $\sigma_{D'}(F(D'))$, respectively, and by H_α the unique morphism given by the diagonal-fill-in lemma, such that $e_\sigma; H_\alpha = F(\alpha); e_{\sigma_{D'}}$ and $H_\alpha; i_{\sigma_{D'}} = i_{\sigma_D}; G(\alpha)$. The reader may check that

$H\colon \mathcal{D} \to \mathcal{C}$ defined by $H(D) = H_D$ and $H(\alpha) = H_\alpha$ is a functor. Thus, we got the factorization $\sigma = e_\sigma; i_\sigma$, where $e_\sigma\colon F \Rightarrow H$ and $i_\sigma\colon H \Rightarrow G$ are the natural transformations $e_\sigma = \{e_{\sigma_D} \mid D \in |\mathcal{D}|\}$ and $i_\sigma = \{i_{\sigma_D} \mid D \in |\mathcal{D}|\}$. The uniqueness of this factorization comes from the uniqueness of factorizations of each σ_D with $D \in |\mathcal{D}|$.

2. It follows from the fact that $e\colon F \Rightarrow G$ is an epic in $\mathcal{C}^{\mathcal{D}}$ whenever each $e_D\colon F(D) \to G(D)$ is an epic in $\mathcal{C}$, for each D in $\mathcal{D}$.

6.3 Algebras and Coalgebras

Given a functor $\mathcal{F}\colon \mathcal{A} \to \mathcal{A}$, a pair (α_A, A) is an $\mathcal{F}$**-algebra** iff $\alpha_A\colon \mathcal{F}(A) \to A$ is a morphism in $\mathcal{A}$. Giving two $\mathcal{F}$-algebras (α_A, A) and (α_B, B), $f\colon A \to B$ is a morphism of $\mathcal{F}$-algebras iff $\mathcal{F}(f);\alpha_B = \alpha_A; f$. $\mathcal{F}$-algebras together with morphisms of $\mathcal{F}$-algebras give a category, $Alg(\mathcal{F})$. Dually, (A, α_A) is an $\mathcal{F}$**-coalgebra** iff $\alpha_A\colon A \to \mathcal{F}(A)$, and $f\colon A \to B$ is a morphism of $\mathcal{F}$-coalgebras iff $f;\alpha_B = \alpha_A;\mathcal{F}(f)$. The category of $\mathcal{F}$-coalgebras is written $CoAlg(\mathcal{F})$.

Proposition 31. *Let $\langle \mathcal{I}, \mathcal{E}\rangle$ be a weak inclusion system of $\mathcal{A}$. Then $\langle \mathcal{I}, \mathcal{E}\rangle$ is a weak inclusion system of*

1. *$CoAlg(\mathcal{F})$ if $\mathcal{F}$ preserves inclusions,*
2. *$Alg(\mathcal{F})$ if $\mathcal{F}$ preserves $\mathcal{E}$-morphisms.*

Proof. We prove only 1., 2. being dual. Let $f\colon (\alpha_A, A) \to (\alpha_B, B)$ be a morphism in $CoAlg(\mathcal{F})$ and let $e_f; i_f$ be its factorization in $\mathcal{A}$, with $e_f\colon A \to f(A)$ and $i_f\colon f(A) \hookrightarrow B$. Since $\mathcal{F}$ preserves inclusions we get that $\mathcal{F}(i_f)$ is in $\mathcal{I}$, so by the diagonal-fill lemma there is a unique morphism, let us denote it $\alpha_{f(A)}$, from $f(A)$ to $\mathcal{F}(f(A))$ such that $e_f;\alpha_{f(A)} = \alpha_A;\mathcal{F}(e_f)$ and $i_f;\alpha_B = \alpha_{f(A)};\mathcal{F}(i_f)$. Therefore $e_f\colon (A, \alpha_A) \to (f(A), \alpha_{f(A)})$ and $i_f\colon (f(A), \alpha_{f(A)}) \to (B, \alpha_B)$ are morphisms of $\mathcal{F}$-algebras and they give a factorization of f in $CoAlg(\mathcal{F})$. Obviously, this factorization is unique.

References

[Bir67] Garrett Birkhoff. *Lattice Theory*, volume 25 of *American Mathematical Societ Colloquium Publications*. American Mathematical Society, New York, N.Y., third edition edition, 1967.

[CR97] Virgil Emil Căzănescu and Grigore Roşu. Weak inclusion systems. *Mathematical Structures in Computer Science*, 7(2):195–206, 1997.

[DGS93] Răzvan Diaconescu, Joseph Goguen, and Petros Stefaneas. Logical support for modularization. In Gerard Huet and Gordon Plotkin, editors, *Logical Environments*, pages 83–130. Cambridge, 1993.

[GB84a] Joseph Goguen and Rod Burstall. Some fundamental algebraic tools for the semantics of computation, part 1: Comma categories, colimits, signatures and theories. *Theoretical Computer Science*, 31(2):175–209, 1984.

[GB84b] Joseph Goguen and Rod Burstall. Some fundamental algebraic tools for the semantics of computation, part 2: Signed and abstract theories. *Theoretical Computer Science*, 31(3):263–295, 1984.

[GB92] Joseph Goguen and Rod Burstall. Institutions: Abstract model theory for specification and programming. *Journal of the Association for Computing Machinery*, 39(1):95–146, January 1992.

[Grä71] George Grätzer. *Lattice theory*. W.H. Freeman and Company, San Francisco, 1971.

[Gro57] Alexandre Grothendieck. Sur quelques points d'algébre homologique. *Tôhoku Mathematical Journal*, 2:119–221, 1957.

[HS73] Horst Herrlich and George Strecker. *Category Theory.* Allyn and Bacon, 1973.

[Isb64] J. R. Isbell. Subobjects, adequacy, completeness and categories of algebras. *Rozprawy Matematyczne*, 36:1–33, 1964.

[Lam66] Joachim Lambek. *Completions of Categories.* Springer-Verlag, 1966. Lecture Notes in Mathematics, Volume 24.

[Lan71] Saunders Mac Lane. *Categories for the Working Mathematician.* Springer, 1971.

[Mit65] B. Mitchell. *Theory of categories.* Academic Press, New York, 1965.

[Ném82] István Németi. On notions of factorization systems and their applications to cone-injective subcategories. *Periodica Mathematica Hungarica*, 13(3):229–335, 1982.

[Roş96] Grigore Roşu. Axiomatizability in inclusive equational logic, 1996. Submitted to publication.

[Roş99] Grigore Roşu. Abstract semantics for module composition, 1999. In preparation.

[Rud63] Sergiu Rudeanu. *Axiomele laticilor şi ale algebrelor booleene*[6] Romanian Academy Press, 1963.

[6] Lattice and Boolean Algebra Axioms.