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The Lattice Structure of Pseudo-Wajsberg Algebras[1]

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Abstract: We explore some properties related to the underlying lattice structure of pseudo-Wajsberg algebras. We establish a residuation result and we characterize the Boolean center of pseudo-Wajsberg algebras.

Key Words: Wajsberg algebra, MV algebra, pseudo-Wajsberg algebra, pseudo-MV algebra, lattice, residuated monoid, Boolean center.

Category: F.4.1.

1 Introduction

Pseudo-Wajsberg algebras are generalizations of Wajsberg algebras. The latter were introduced by Mordchaj Wajsberg in [10] and studied by Font, Rodriguez and Torrens in [7]. Pseudo-Wajsberg algebras were introduced in [2] (also [1]) with the explicit purpose of providing a concept categorically equivalent to that of pseudo-MV algebra , and which will have the same relationship with Wajsberg algebras as pseudo-MV algebras have with MV algebras. The concept of MV algebra is the most intensively studied algebraic counterpart of Lukasiewicz's calculus (see [3], [4], [5] and [6]), and pseudo-MV algebras are a non-commutative generalization of it, recently introduced by G. Georgescu and A. Iorgulescu in [8] and [9].

The desired categorical equivalence between pseudo-Wajsberg algebras and pseudo-MV algebras was established in [2] , but several other interesting properties of pseudo-Wajsberg algebras emerged, especially related to their order structure (and not necessarily connected to their presentation as pseudo-MV algebras). These properties justify a continuation of the study of this concept, and the present paper is a step towards this goal.

In section 2 we present the definition of pseudo-Wajsberg algebras, their relation to Wajsberg algebras, and some first properties of pseudo-Wajsberg algebras which derive gradually from the axioms. Almost all the results are presented without proofs, the material being contained in [2], to which we refer the reader for details. The presentation differs from that in [2] in certain points: we give here a more simple and direct proof of property (P10) (the equivalence of the two definitions of the order relation), we give several definitions of $\wedge$, and derive relations (A) and (B). The result ending this section, which establishes the fact that any pseudo-Wajsberg algebra has an underlying bounded lattice structure with two quasi-complements, has its proof in results contained in [2].

The rest of the paper is devoted to the presentation of new results.

[1] C. S. Calude and G. Ştefănescu (eds.). *Automata, Logic, and Computability. Special issue dedicated to Professor Sergiu Rudeanu Festschrift.*

In section 3 some further relations between the pseudo-Wajsberg algebra operations and the lattice operations are established. These relations are inspired from similar ones holding in Wajsberg algebras, most of them presented in [7], and are important in the sequel.

In section 4 another important result from [7] , a residuation result, is generalized for pseudo-Wajsberg algebras. Two theorems are proven, which together state that pseudo-Wajsberg algebras are categorically equivalent to (non-abelian) residuated bounded ordered monoids satisfying some supplementary conditions. The two implications of a pseudo-Wajsberg algebra appear as the right and left residuals of this underlying monoid structure. A result is also established which allows the expression of the lattice operation $\wedge$ in terms of the monoid multiplication.

Finally, in section 5 some equivalent characterizations of the Boolean center of a pseudo-Wajsberg algebra are given. The results are inspired by similar results in [9] , but the proofs are independent and rely entirely on properties of pseudo-Wajsberg lattices, making no use of their categorical equivalence to pseudo-MV algebras.

2 Pseudo-Wajsberg algebras: definitions and first results

Definition 2.1 *An algebra* $(A, \to, ^-, 1)$ *of type* $(2, 1, 0)$ *is called a Wajsberg algebra iff the following axioms hold:*

(Wajs1) $1 \to x = x$
(Wajs2) $(x \to y) \to y = (y \to x) \to x$
(Wajs3) $(x \to y) \to [(y \to z) \to (x \to z)] = 1$
(Wajs4) $(x^- \to y^-) \to (y \to x) = 1$

Definition 2.2 *An algebra* $< A, \to, \rightsquigarrow, ^-, ^\sim, 1 >$ *of arrity* $< 2, 2, 1, 1, 0 >$ *will be called a pseudo-Wajsberg algebra iff it satisfies axioms (W1) - (W6).*

(W1) (a) $1 \to x = x$
(b) $1 \rightsquigarrow x = x$
(W2) $(x \rightsquigarrow y) \to y = (y \rightsquigarrow x) \to x = (y \to x) \rightsquigarrow x = (x \to y) \rightsquigarrow y$
(W3) (a) $(x \to y) \to [(y \to z) \rightsquigarrow (x \to z)] = 1$
(b) $(x \rightsquigarrow y) \rightsquigarrow [(y \rightsquigarrow z) \to (x \rightsquigarrow z)] = 1$
(W4) $1^- = 1^\sim$ *(and let* 0 *denote this element).*
(W5) (a) $(x^- \rightsquigarrow y^-) \to (y \to x) = 1$
(b) $(x^\sim \to y^\sim) \to (y \rightsquigarrow x) = 1$
(W6) $(x \to y^-)^\sim = (y \rightsquigarrow x^\sim)^-$

We have the following result which gives us Wajsberg algebras as a particular case of pseudo-Wajsberg algebras.

Proposition 2.3 *A pseudo-Wajsberg algebra in which the two implications coincide is a Wajsberg algebra.*

Let us now derive the first properties of pseudo-Wajsberg algebras. The following are consequences of axioms (W1)-(W3).

Proposition 2.4 *In an algebra* $< A, \rightarrow, \rightsquigarrow, ^-, ^\sim, 1 >$ *which satisfies axioms (W1), (W2) and (W3), the following equalities and implications hold:*

(P1) (a) $x \rightarrow x = 1$
(b) $x \rightsquigarrow x = 1$
(P2) (a) If $x \rightarrow y = 1$ and $y \rightarrow x = 1$, then $x = y$.
(b) If $x \rightsquigarrow y = 1$ and $y \rightsquigarrow x = 1$, then $x = y$.
(c) If $x \rightarrow y = 1$ and $y \rightsquigarrow x = 1$, then $x = y$.
(P3) (a) $(x \rightarrow 1) \rightsquigarrow 1 = 1$
(b) $(x \rightsquigarrow 1) \rightarrow 1 = 1$
(P4) (a) $x \rightarrow 1 = 1$
(b) $x \rightsquigarrow 1 = 1$
(P5) (a) If $x \rightarrow y = 1$ and $y \rightarrow z = 1$, then $x \rightarrow z = 1$.
(b) If $x \rightsquigarrow y = 1$ and $y \rightsquigarrow z = 1$, then $x \rightsquigarrow z = 1$.
(P6) (a) $x \rightarrow (y \rightsquigarrow x) = 1$
(b) $x \rightsquigarrow (y \rightarrow x) = 1$
(P7) $x \rightarrow (y \rightsquigarrow z) = 1 \Longleftrightarrow y \rightsquigarrow (x \rightarrow z) = 1$
(P8) (a) $(x \rightarrow y) \rightsquigarrow [(z \rightarrow x) \rightarrow (z \rightarrow y)] = 1$
(b) $(x \rightsquigarrow y) \rightarrow [(z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y)] = 1$
(P9) $x \rightarrow (y \rightsquigarrow z) = y \rightsquigarrow (x \rightarrow z)$

The proofs can be found in [2].
Next follow some properties which are consequences of axioms (W1)-(W5).

Proposition 2.5 *In an algebra* $< A, \rightarrow, \rightsquigarrow, ^-, ^\sim, 1 >$ *which satisfies axioms (W1)-(W5) the following equalities are true:*

(C1) (a) $(x^- \rightsquigarrow 0) \rightarrow x = 1$
(b) $(x^\sim \rightarrow 0) \rightsquigarrow x = 1$
(C2) $0 \rightarrow x = 1 = 0 \rightsquigarrow x$
(C3) (a) $x \rightarrow 0 = x^-$
(b) $x \rightsquigarrow 0 = x^\sim$
(C4) $(x^-)^\sim = x = (x^\sim)^-$
(C5) (a) $x^\sim \rightarrow y^\sim = y \rightsquigarrow x$
(b) $x^- \rightsquigarrow y^- = y \rightarrow x$
(C6) $x^\sim \rightarrow y = y^- \rightsquigarrow x$

Again the proofs can be found in [2].
Next, we prove by purely algebraic manipulations the following equivalence.

Lemma 2.6 *We have*

$$x \rightarrow y = 1 \quad \Longleftrightarrow \quad x \rightsquigarrow y = 1$$

Proof: Suppose $x \rightarrow y = 1$. We have

$$y = 1 \rightsquigarrow y = (x \rightarrow y) \rightsquigarrow y = (x \rightsquigarrow y) \rightarrow y$$

by (W1)(b), our hypothesis, and (W2). Replacing y in this form in $x \rightsquigarrow y$ we get:

$$x \rightsquigarrow y = x \rightsquigarrow [(x \rightsquigarrow y) \rightarrow y] = (x \rightsquigarrow y) \rightarrow (x \rightsquigarrow y) = 1$$

by (P9) and (P1)(a). Similarly for the reverse implication. □

In an algebra as above, let us define the following relation:

$$x \leq y \iff x \rightarrow y = 1 \iff x \rightsquigarrow y = 1$$

It is reflexive by (P1), antisymmetric by (P2) and transitive by (P5), and thus it defines a partial order on A, with greatest element 1 by (P4), and first element 0 by (C2).

The following result states that both negations reverse the order.

Lemma 2.7 *We have*

$$(C7) \quad x \leq y \iff y^- \leq x^- \iff y^\sim \leq x^\sim$$

Proof: Use (C5)(a) and (b) and the alternative definitions of the order. □

In every pseudo-Wajsberg algebra we also have the following equalities, which are equivalent to axiom (W6) in a straightforward manner using (C4).

(C8) (a) $(x \rightarrow y)^\sim = (y^\sim \rightsquigarrow x^\sim)^-$
(b) $(x \rightsquigarrow y)^- = (y^- \rightarrow x^-)^\sim$.

Some first relationships between the order relation and the implications are established next.

Proposition 2.8 *The following are true for every $x, y, z \in A$:*

(O1) (a) $x \rightarrow y \leq (y \rightarrow z) \rightsquigarrow (x \rightarrow z)$
(b) $x \rightsquigarrow y \leq (y \rightsquigarrow z) \rightarrow (x \rightsquigarrow z)$
(O2) (a) $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y)$
(b) $x \rightsquigarrow y \leq (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y)$
(O3) (a) $x \leq y \rightarrow x$
(b) $x \leq y \rightsquigarrow x$
(O4) $x \leq y \rightarrow z \iff y \leq x \rightsquigarrow z$
(O5) (a) $x \leq y \Longrightarrow y \rightarrow z \leq x \rightarrow z$
(b) $x \leq y \Longrightarrow y \rightsquigarrow z \leq x \rightsquigarrow z$
(O6) (a) $x \leq y \Longrightarrow z \rightarrow x \leq z \rightarrow y$
(b) $x \leq y \Longrightarrow z \rightsquigarrow x \leq z \rightsquigarrow y$

Proof: (O1) is a direct transcription of axiom (W3) in terms of the order $\leq$, and (O2) is a similar transcription of properties (P8). (O3) is immediate from (P6) and the definition of $\leq$, and (O4) from (P7).

(O5) follows from (W3) and (W1), again using the definition of $\leq$. In a similar manner (O6) follows from (P8) and (W1). □

Define now the binary operation

$$\begin{aligned} x \vee y := (x \rightarrow y) \rightsquigarrow y = (y \rightarrow x) \rightsquigarrow x = \\ = (x \rightsquigarrow y) \rightarrow y = (y \rightsquigarrow x) \rightarrow x. \end{aligned}$$

where the last equalities are given by axiom (W2).

Proposition 2.9 *$x \vee y$ is a supremum for x and y relative to $\leq$ on A.*

Proof. The proof can be found in [2], where two separate orders were introduced, consequently two separate operations, $\vee_1$ and $\vee_2$, were defined, using only the first and last equalities of (W2), and they were proven separately to be sups, thus implying the equality of the two orders. □

Define two binary operations on a pseudo-Wajsberg algebra A by:

$$x\wedge_1 y := [x \rightsquigarrow (x \to y)^{\sim}]^{-} = [(x \to y) \to x^{-}]^{\sim}$$

$$x\wedge_2 y := [y \to (y \rightsquigarrow x)^{-}]^{\sim} = [(y \rightsquigarrow x) \rightsquigarrow y^{\sim}]^{-}$$

where the last two equalities hold in view of (W6).

Also from [2] we give without proof the following results:

Proposition 2.10 *$x \wedge_1 y$ is an infimum for x and y on A.*

Proposition 2.11 *$x \wedge_2 y$ is an infimum for x and y on A.*

Corollary 2.12 *$x \wedge_1 y = x \wedge_2 y$.*

Since $x \wedge_1 y = x \wedge_2 y$ we will denote this operation by $\wedge$, and we have the following eight alternative definitions for it, where the first three equalities follow from the definitions and the equality $x \wedge_1 y = x \wedge_2 y$, and the last ones from the commutativity of the inf.

$$\begin{aligned} x \wedge y :&= [x \rightsquigarrow (x \to y)^{\sim}]^{-} = [(x \to y) \to x^{-}]^{\sim} = \\ &= [y \to (y \rightsquigarrow x)^{-}]^{\sim} = [(y \rightsquigarrow x) \rightsquigarrow y^{\sim}]^{-} = \\ &= [y \rightsquigarrow (y \to x)^{\sim}]^{-} = [(y \to x) \to y^{-}]^{\sim} = \\ &= [x \to (x \rightsquigarrow y)^{-}]^{\sim} = [(x \rightsquigarrow y) \rightsquigarrow x^{\sim}]^{-}. \end{aligned}$$

As a "by-product" of these equalities we also obtain

$$\begin{aligned} (A) \qquad (x \wedge y)^{-} &= (x \to y) \to x^{-} = y \to (y \rightsquigarrow x)^{-} = \\ &= (y \to x) \to y^{-} = x \to (x \rightsquigarrow y)^{-} \\ (B) \qquad (x \wedge y)^{\sim} &= x \rightsquigarrow (x \to y)^{\sim} = (y \rightsquigarrow x) \rightsquigarrow y^{\sim} = \\ &= y \rightsquigarrow (y \to x)^{\sim} = (x \rightsquigarrow y) \rightsquigarrow x^{\sim} \end{aligned}$$

where relations (A) are derived from those definitions of $\wedge$ expressed using $^{\sim}$, and (B) from those expressed using $^{-}$.

We have so far:

Theorem 2.13 *If $< A, \to, \rightsquigarrow, ^{-}, ^{\sim}, 0, 1 >$ is a pseudo-Wajsberg algebra then, with the definitions above for $\vee$ and $\wedge$, $< A, \vee, \wedge, 0, 1 >$ is a bounded lattice, with two quasi-complements, $^{-}$ and $^{\sim}$.*

3 Further properties of pseudo-Wajsberg lattices

We establish in this section relations between the implications, negations, and the lattice operations, $\vee$ and $\wedge$. The following properties are analogous to the ones proven for Wajsberg algebras in [7].

Theorem 3.1 *In a pseudo-Wajsberg algebra the following equalities are true for every x, y, z.*

(O7) $(x \vee y)^{-} = x^{-} \wedge y^{-}$ $\quad (x \vee y)^{\sim} = x^{\sim} \wedge y^{\sim}$
$(x \wedge y)^{-} = x^{-} \vee y^{-}$ $\quad (x \wedge y)^{\sim} = x^{\sim} \vee y^{\sim}$

(O8) *(a)* $(x \vee y) \to z = (x \to z) \wedge (y \to z)$
(b) $(x \vee y) \rightsquigarrow z = (x \rightsquigarrow z) \wedge (y \rightsquigarrow z)$

(O9) *(a)* $z \to (x \wedge y) = (z \to x) \wedge (z \to y)$
(b) $z \rightsquigarrow (x \wedge y) = (z \rightsquigarrow x) \wedge (z \rightsquigarrow y)$

(O10) *(a)* $(x \vee y) \to y = x \to y$
(b) $(x \vee y) \rightsquigarrow y = x \rightsquigarrow y$

(O11) *(a)* $x \to (x \wedge y) = x \to y$
(b) $x \rightsquigarrow (x \wedge y) = x \rightsquigarrow y$

(O12) *(a)* $(x \to y) \vee (y \to x) = 1$
(b) $(x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$

(O13) *(a)* $x \to (y \vee z) = (x \to y) \vee (x \to z)$
(b) $x \rightsquigarrow (y \vee z) = (x \rightsquigarrow y) \vee (x \rightsquigarrow z)$

(O14) *(a)* $(x \wedge y) \to z = (x \to z) \vee (y \to z)$
(b) $(x \wedge y) \rightsquigarrow z = (x \rightsquigarrow z) \vee (y \rightsquigarrow z)$

(O15) $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$

(O16) *(a)* $(x \wedge y) \to z = (x \to y) \to (x \to z) = (y \to x) \to (y \to z)$
(b) $(x \wedge y) \rightsquigarrow z = (x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z) = (y \rightsquigarrow x) \rightsquigarrow (y \rightsquigarrow z)$

(O17) *(a)* $z \to (x \vee y) = (x \to y) \rightsquigarrow (z \to y) = (y \to x) \rightsquigarrow (z \to x)$
(b) $z \rightsquigarrow (x \vee y) = (x \rightsquigarrow y) \to (z \rightsquigarrow y) = (y \rightsquigarrow x) \to (z \rightsquigarrow x)$

Proof. (O7)(the De Morgan laws for $^{-}$ and $^{\sim}$)

($\wedge^{-}$) On one hand we have:

$$(x \wedge y)^{-} = (x \to y) \to x^{-} = y \to (y \rightsquigarrow x)^{-}$$

from those definitions of $\wedge$ expressed using $^{\sim}$. On the other hand, we get from the definition of $\vee$ and (C5)(b)

$$x^{-} \vee y^{-} = (y^{-} \rightsquigarrow x^{-}) \to x^{-} = (x \to y) \to x^{-}$$

and from these two the equality $(x \wedge y)^{-} = x^{-} \vee y^{-}$ follows.

($\wedge^{\sim}$) On one hand we have:

$$(x \wedge y)^{\sim} = x \rightsquigarrow (x \to y)^{\sim} = (y \rightsquigarrow x) \rightsquigarrow y^{\sim}$$

from those definitions of $\wedge$ expressed using $^{-}$. On the other hand, we get from the definition of $\vee$ and (C5)(a)

$$x^{\sim} \vee y^{\sim} = (x^{\sim} \to y^{\sim}) \rightsquigarrow y^{\sim} = (y \rightsquigarrow x) \rightsquigarrow y^{\sim}$$

and from these two the equality $(x \wedge y)^{\sim} = x^{\sim} \vee y^{\sim}$ follows.

($\vee^{-}$) From the definition of $\vee$, (C5)(b), (C4) and the definition of $\wedge$ we get

$$(x \vee y)^{-} = [(x \to y) \rightsquigarrow y]^{-} = [(y^{-} \rightsquigarrow x^{-}) \rightsquigarrow y]^{-} = x^{-} \wedge y^{-}.$$

($\vee^{\sim}$) From the definition of $\vee$, (C5)(a), (C4) and the definition of $\wedge$ we get

$$(x \vee y)^{\sim} = [(y \rightsquigarrow x) \to x]^{\sim} = [(x^{\sim} \to y^{\sim}) \to x]^{\sim} = x^{\sim} \wedge y^{\sim}.$$

(O8)(a) We will prove the two inequalities which will lead to (O8)(a). Denote $t = (x \to z) \wedge (y \to z)$. Since $x \leq x \vee y$ and $y \leq x \vee y$, applying twice (O5)(a), we get:

$$(x \vee y) \to z \leq (x \to z) \quad \text{and} \quad (x \vee y) \to z \leq (y \to z)$$

from which, by the definition of t as an inf, it follows that

$$(x \vee y) \to z \leq t.$$

From $t \leq x \to z$ and $t \leq y \to z$, applying twice (O4) we have:

$$x \leq t \rightsquigarrow z \quad \text{and} \quad y \leq t \rightsquigarrow z$$

from which it follows that $x \vee y \leq t \rightsquigarrow z$, and, applying (O4) once again we get $t \leq (x \vee y) \to z$. (O8)(b) will be obtained in a similar manner, using (O5)(b) and (O4).

(O9)(b) We have

$$(z \rightsquigarrow x) \wedge (z \rightsquigarrow y) = (x^{\sim} \to z^{\sim}) \wedge (y^{\sim} \to z^{\sim}) = (x^{\sim} \vee y^{\sim}) \to z^{\sim} =$$

$$= z \rightsquigarrow (x^{\sim} \vee y^{\sim})^{-} = z \rightsquigarrow (x \wedge y)$$

by (C5)(a), (O8)(a), (C5)(b), (O7) and (C4). Similarly for (O9)(a), using (C5)(b), (O8)(b), (C5)(a), (O7) and (C4).

(O10)(a) We have

$$(x \vee y) \to y = (x \to y) \wedge (y \to y) = (x \to y) \wedge 1 = x \to y$$

by taking in (O8)(a) $z = y$, and by (P1)(a). For (O10)(b) use (O8)(b) with $z = y$ and (P1)(b).

(O11)(a) We have

$$x \to (x \wedge y) = (x \to x) \wedge (x \to y) = 1 \wedge (x \to y) = x \to y$$

by taking $z = x$ in (O9)(a). Similarly, by taking $z = x$ in (O9)(b), we obtain (O11)(b).

(O12)(a) We calculate:

$$\begin{aligned}
(x \to y) \to (y \to x) &= [(x \vee y) \to y] \to [(x \vee y) \to x) \\
&= [y^{-} \rightsquigarrow (x \vee y)^{-}] \to [x^{-} \rightsquigarrow (x \vee y)^{-}] \\
&= x^{-} \rightsquigarrow [[y^{-} \rightsquigarrow (x \vee y)^{-}] \to (x \vee y)^{-}] \\
&= x^{-} \rightsquigarrow [y^{-} \vee (x \vee y)^{-}] \\
&= [y^{-} \vee (x \vee y)^{-}]^{\sim} \to x \\
&= [y \wedge (x \vee y)] \to x \\
&= y \to x
\end{aligned}$$

by (O10)(a), (C5)(b), (P9), the definition of $\vee$, (C4), (C5)(b), (O7), (C4) and the obvious fact that $y \wedge (x \vee y) = y$. Replacing what we just obtained in the definition of $\vee$, we have

$$(x \to y) \vee (y \to x) = [(x \to y) \to (y \to x)] \rightsquigarrow (y \to x) = (y \to x) \rightsquigarrow (y \to x) = 1.$$

Similarly for (O12)(b).

(O13)(a) We will prove the equality by a double inequality. First, from $y \leq y \vee z$ and $z \leq y \vee z$, by (O6)(a) we get

$$x \to y \leq x \to (y \vee z) \quad \text{and} \quad x \to z \leq x \to (y \vee z)$$

from which it follows that

$$(x \to y) \vee (x \to z) \leq x \to (y \vee z).$$

On the other hand, we have

$$\begin{aligned} 1 &= (z \to y) \vee (y \to z) \\ &= [(z \vee y) \to y] \vee [(z \vee y) \to z] \\ &\leq [(x \to (z \vee y)) \to (x \to y)] \vee [(x \to (z \vee y)) \to (x \to z)] \\ &\leq (x \to (z \vee y)) \to [(x \to y) \vee (x \to z)] \end{aligned}$$

from (O12)(a), twice (O10)(a), twice (O2)(a), and the previously proven inequality. It follows that

$$(x \to (z \vee y)) \to [(x \to y) \vee (x \to z)] = 1$$

which means precisely

$$x \to (z \vee y) \leq (x \to y) \vee (x \to z).$$

Similarly for (O13)(b).

(O14)(b) We have

$$\begin{aligned} (x \wedge y) \rightsquigarrow z &= ((x \wedge y)^{\sim})^{-} \rightsquigarrow (z^{\sim})^{-} = z^{\sim} \to (x \wedge y)^{\sim} = \\ &= z^{\sim} \to (x^{\sim} \vee y^{\sim}) = (z^{\sim} \to x^{\sim}) \vee (z^{\sim} \to y^{\sim}) = \\ &= (x \rightsquigarrow z) \vee (y \rightsquigarrow z) \end{aligned}$$

by (C4), (C5)(b), (O7), (O13)(a) and (C5)(a). Similarly for (O14)(a).

(O15) (distributivity of $\vee$ over $\wedge$) We calculate

$$\begin{aligned} (x \wedge y) \vee z &= [(x \wedge y) \rightsquigarrow z] \to z = [(x \rightsquigarrow z) \vee (y \rightsquigarrow z)] \to z = \\ &= [(x \rightsquigarrow z) \to z] \wedge [(y \rightsquigarrow z) \to z] = \\ &= (x \vee z) \wedge (y \vee z) \end{aligned}$$

from the definition of $\vee$, (O14)(b), (O8)(a), and again the definition of $\vee$.

(O16)(b) We have a first equality by calculating:

$$\begin{aligned} (x \wedge y) \rightsquigarrow z &= ((x \wedge y)^{\sim})^{-} \rightsquigarrow (z^{\sim})^{-} = z^{\sim} \to (x \wedge y)^{\sim} = \\ &= z^{\sim} \to (x^{\sim} \vee y^{\sim}) = z^{\sim} \to [(y^{\sim} \to x^{\sim}) \rightsquigarrow x^{\sim}] = \\ &= (y^{\sim} \to x^{\sim}) \rightsquigarrow (z^{\sim} \to x^{\sim}) = (x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z) \end{aligned}$$

from (C4), (C5)(b), (O7), the definition of $\vee$, (P9), and (C5)(a). The second equality follows from the commutativity of $\vee$. Similarly for (O16)(a).

(O17)(a) We have a first equality by calculating:

$$\begin{aligned} z \to (x \vee y) &= (z^-)^\sim \to ((x \vee y)^-)^\sim = (x \vee y)^- \rightsquigarrow z^- = \\ &= (x^- \wedge y^-) \rightsquigarrow z^- = [(y^- \rightsquigarrow x^-) \rightsquigarrow (y^-)^\sim]^- \rightsquigarrow z^- = \\ &= z \to [(y^- \rightsquigarrow x^-) \rightsquigarrow y] = (y^- \rightsquigarrow x^-) \rightsquigarrow (z \to y) = \\ &= (x \to y) \rightsquigarrow (z \to y) \end{aligned}$$

from (C4), (C5)(a), (O7), the definition of $\wedge$, (C5)(b), (C4), (P9), and (C5)(b). The second equality follows from the commutativity of $\wedge$. Similarly for (O17)(b). $\square$

Remark. As we noticed, relations (O7) are the De Morgan laws for $^-$ and $^\sim$. Moreover, relation (O15) states the distributivity of $\vee$ over $\wedge$, from which the distributivity of $\wedge$ over $\vee$ follows. We have thus

Theorem 3.2 *If A is a pseudo-Wajsberg algebra, then its underlying bounded lattice has two De Morgan algebra structures on it, $< A, \vee, \wedge, ^-, 0, 1 >$ and $< A, \vee, \wedge, ^\sim, 0, 1 >$.*

4 A residuation result

Let $< A, \cdot, \leq, 1 >$ be a partially ordered monoid (not necessarily abelian). This means that $\cdot$ is increasing in each argument, i.e., $x \leq y$ implies $a \cdot x \cdot b \leq a \cdot y \cdot b$ for every $a, b \in A$.

Definition 4.1 *We say that A is* right residuated *iff for every $a, b \in A$ there exists a greatest element x such that $a \cdot x \leq b$. We will denote this element by $a \to b$.*

We say that A is left residuated *iff for every $a, b \in A$ there exists a greatest element y such that $y \cdot a \leq b$. We will denote this element by $a \rightsquigarrow b$.*

We say that A is residuated *iff it is left and right residuated.*

The condition for A to be residuated can be expressed in a compact manner by stating that $\to$ and $\rightsquigarrow$, as defined above, are total binary operations on A, which satisfy the following equivalent conditions:

$$x \cdot y \leq z \iff y \leq x \to z \iff x \leq y \rightsquigarrow z.$$

If A is abelian, the left residual (if it exists) coincides with the right residual.

We recall from [7] the following two results, which give us a categorical equivalence between Wajsberg algebras and residuated ordered bounded abelian monoids, in which the residual satisfies a certain supplementary condition.

Theorem 4.2 *Let $< A, \to, ^-, 1 >$ be a Wajsberg algebra. Define the binary operation*

$$x \cdot y := (x \to y^-)^-.$$

Then $< A, \cdot, \leq, 1 >$ is a residuated ordered abelian monoid, with lower bound 0 *and upper bound* 1*, having $\to$ as the residual of $\cdot$.*

Theorem 4.3 *Let $< A, \cdot, \leq, 1 >$ be a residuated ordered abelian monoid, having 1 as upper bound, having a lower bound, 0, and $\rightarrow$, the residual of $\cdot$, satisfies the condition*

(Wajs2) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$.

Then, defining

$$x^{-} := x \rightarrow 0$$

we have that $< A, \rightarrow, ^{-}, 1 >$ is a Wajsberg algebra.

In the following, we will generalize these results for pseudo-Wajsberg algebras. Let us first prove:

Proposition 4.4 *Let $< A, \rightarrow, \rightsquigarrow, ^{-}, ^{\sim}, 1 >$ be a pseudo-Wajsberg algebra. Define the binary operation*

$$x \cdot y := (y \rightarrow x^{-})^{\sim} = (x \rightsquigarrow y^{\sim})^{-}$$

where the second equality is given by axiom (W6). Then we have:

1. The operation $\rightarrow$ is the right residual of $\cdot$, i.e. we have

$$x \cdot y \leq z \iff y \leq x \rightarrow z$$

2. The operation $\rightsquigarrow$ is the left residual of $\cdot$, i.e. we have

$$x \cdot y \leq z \iff x \leq y \rightsquigarrow z$$

Proof. For assertion (1), let us make the following calculations:

$$\begin{aligned}(x \cdot y) \rightarrow z = (y \rightarrow x^{-})^{\sim} \rightarrow z = z^{-} \rightsquigarrow (y \rightarrow x^{-}) = \\ = y \rightarrow (z^{-} \rightsquigarrow x^{-}) = y \rightarrow (x \rightarrow z)\end{aligned}$$

by applying the first definition of $\cdot$, (C5)(b), (C4), (P9) and again (C5)(b). From this and the definition of $\leq$ we have the desired equivalence.

For assertion (2), let us note that property (O4) states precisely

$$y \leq x \rightarrow z \iff x \leq y \rightsquigarrow z. \quad \square$$

Theorem 4.5 *Let $< A, \rightarrow, \rightsquigarrow, ^{-}, ^{\sim}, 1 >$ be a pseudo-Wajsberg algebra. Define the binary operation*

$$x \cdot y := (y \rightarrow x^{-})^{\sim} = (x \rightsquigarrow y^{\sim})^{-}$$

where the second equality is given by axiom (W6). Then $< A, \cdot, \leq, 1 >$ is a residuated ordered monoid, with lower bound 0 and upper bound 1, having $\rightarrow$ as the right residual, and $\rightsquigarrow$ as the left residual of $\cdot$.

Proof. By the preceding result, we have the residuation property. Also, in view of the already known facts about pseudo-Wajsberg algebras and their order structure, all that remains to be proven is the monoid structure of $< A, \cdot, 1 >$ and the monotony of $\cdot$.

To prove that 1 is the neutral element, we calculate and obtain:

$$\begin{aligned}x \cdot 1 = (1 \rightarrow x^{-})^{\sim} = (x^{-})^{\sim} = x \\ 1 \cdot x = (1 \rightsquigarrow x^{\sim})^{-} = (x^{\sim})^{-} = x\end{aligned}$$

by the definitions of $\cdot$, (W1), and (C4).

To prove associativity, calculate:

$$\begin{aligned}[(x\cdot y)\cdot z]^- &= z \to (x\cdot y)^- = z\to (y\to x^-) = z\to ((x^-)^- \rightsquigarrow y^-) = \\ &= (x^-)^- \rightsquigarrow (z\to y^-) = (x^-)^- \rightsquigarrow (y\cdot z)^- = (y\cdot z)\to x^- = \\ &= [x\cdot(y\cdot z)]^-\end{aligned}$$

where we have used twice the definition of $\cdot$ which uses $^\sim$, then (C5)(b), (P9), again the definition of $\cdot$, (C5)(b) and the definition of $\cdot$. Then apply $^\sim$ and (C4).

For the monotony of $\cdot$ with respect to multiplication to the right: from $x \le y$ follows by (C7) that $y^- \le x^-$, from which, by (O6)(a)

$$a \to y^- \le a \to x^- \quad \text{for every} \quad a \in A$$

from which, applying $^\sim$ and (C7) again we get

$$(a \to x^-)^\sim \le (a \to y^-)^\sim \quad \text{for every} \quad a \in A$$

which means precisely

$$x \cdot a \le y \cdot a \quad \text{for every} \quad a \in A.$$

Similar calculations for multiplication to the left. □

Theorem 4.6 *Let $< A, \cdot, \le, 1 >$ be a residuated ordered monoid, with lower bound 0 and upper bound 1, having $\to$ as the right residual, and $\rightsquigarrow$ as the left residual of $\cdot$. Define the unary operations*

$$x^- := x \to 0$$

$$x^\sim := x \rightsquigarrow 0$$

and suppose the residuals and the unary operations satisfy the supplementary conditions:

(W2) $(x \rightsquigarrow y) \to y = (y \rightsquigarrow x) \to x = (y \to x) \rightsquigarrow x = (x \to y) \rightsquigarrow y$

(W6) $(x \to y^-)^\sim = (y \rightsquigarrow x^\sim)^-$

Then $< A, \to, \rightsquigarrow, ^-, ^\sim, 1 >$ is a pseudo-Wajsberg algebra.

Moreover, if on this pseudo-Wajsberg algebra we define the binary operation

$$x \star y := (y \to x^-)^\sim = (x \rightsquigarrow y^\sim)^-$$

which, according to theorem 4.5, makes $< A, \star, \le, 1 >$ an ordered bounded residuated monoid, we have that the two monoid structures coincide, i.e. $x\cdot y = x \star y$.

Proof. Remember that the residuation property states that

$$x\cdot y \le z \iff y \le x \to z \iff x \le y \rightsquigarrow z.$$

Also, from the definitions of the residuals we have two easy consequences:

$$\begin{array}{ll} (R1) & x\cdot(x\to y)\le y \\ (R2) & (y \rightsquigarrow z)\cdot y \le z \end{array}$$

Let us prove next some intermediate results, relations (1) and (2).

$$(1) \quad x \leq y \quad \text{iff} \quad x \to y = 1 \quad \text{iff} \quad x \rightsquigarrow y = 1$$

From the residuation property and the fact that 1 is the neutral element of the monoid we have:
$x \leq y \Longleftrightarrow x \cdot 1 \leq y \Longleftrightarrow 1 \leq x \to y$
$\Longleftrightarrow 1 \cdot x \leq y \Longleftrightarrow 1 \leq x \rightsquigarrow y$
and 1 being the greatest element of A, the proof of (1) is finished.

Let us now prove
(2) If $x \leq y$ then $z \to x \leq z \to y$
$z \rightsquigarrow x \leq z \rightsquigarrow y$
By definition $z \to x$ is the greatest b such that $z \cdot b \leq x$, but $x \leq y$, so by the transitivity of $\leq$ and the definition of $z \to y$ it follows that $z \to x \leq z \to y$. An analogous straightforward proof for the second assertion.

Note that from relation (1) it follows that in a residuated ordered bounded monoid $< A, \cdot, \leq, 1 >$ as the one given by the theorem, properties (P1), (P2), (P3), (P4), (P5), (P6), (P7) and (C2) are valid for any $x, y, z \in A$.

To prove that $< A, \to, \rightsquigarrow, ^-, ^\sim, 1 >$ is a pseudo-Wajsberg algebra, we have to show that (W1), (W3), (W4) and (W5) hold (since (W2) and (W6) are supposed to be true).

For (W1)(a) let us estimate $(1 \to x) \rightsquigarrow x$. We have

$$(1 \to x) \rightsquigarrow x = (x \to 1) \rightsquigarrow 1 = 1$$

from (W2) and (P3), and, applying (1), we get $1 \to x \leq x$. On the other hand, from $1 \cdot x \leq x$ and the residuation property we get $x \leq 1 \to x$. The same for (b).

(W3)(a) is equivalent by (1) to

$$(x \to y) \leq (y \to z) \rightsquigarrow (x \to z)$$

which by the residuation property is equivalent to

$$(x \to y) \cdot (y \to z) \leq (x \to z)$$

which, again by the residuation property, is equivalent to

$$x \cdot (x \to y) \cdot (y \to z) \leq z.$$

By applying twice (R1) and using the monotony of $\cdot$ we get

$$x \cdot (x \to y) \cdot (y \to z) \leq y \cdot (y \to z) \leq z$$

which is precisely what we desired. The same proof for (W3)(b), using (R2).

Note that because we already have axioms (W1), (W2) and (W3), relations (P8) and (P9) will also hold.

(W4)(a) is equivalent to $1 \to 0 = 0$, But by (R1) we have $1 \cdot (1 \to 0) \leq 0$, which implies $(1 \to 0) \leq 0$, and 0 is the smallest element of A. For (b) use (R2).

(W5)(a) is equivalent by (1) to

$$x^- \rightsquigarrow y^- \leq y \to x.$$

We will actually prove equality (which is (C5)(b)). We have

$$\begin{aligned} x^- \rightsquigarrow y^- &= (x \to 0) \rightsquigarrow (y \to 0) = y \to [(x \to 0) \rightsquigarrow 0] = \\ &= y \to [(0 \to x) \rightsquigarrow x] = y \to (1 \rightsquigarrow x) = \\ &= y \to x \end{aligned}$$

by the definition of $^-$, (P9), (W2), (C2)(a) and (W1)(b). This ends the proof of the fact that $< A, \to, \rightsquigarrow, ^-, ^\sim, 1 >$ is a pseudo-Wajsberg algebra.

Let us now define on it the binary operation

$$x \star y := (y \to x^-)^\sim = (x \rightsquigarrow y^\sim)^-$$

and let us prove $x \cdot y = x \star y$ by double inequality.

First, $x \cdot y \leq x \star y = (x \rightsquigarrow y^\sim)^-$ is equivalent by the residuation property to $y \leq x \to (x \rightsquigarrow y^\sim)^-$ and the right-hand side is

$$x \to (x \rightsquigarrow y^\sim)^- = (x \wedge y^\sim)^- = x^- \vee y$$

by (1) and (O7), so the relation to be proven becomes $y \leq x^- \vee y$, and is thus true.

The second inequality to be proven has the following equivalent forms:

$$\begin{aligned} (y \to x^-)^\sim \leq x \cdot y &\Longleftrightarrow (x \cdot y)^- \leq y \to x^- \\ &\Longleftrightarrow (x \cdot y)^- \rightsquigarrow (y \to x^-) = 1 \\ &\Longleftrightarrow y \to [((x \cdot y)^-) \rightsquigarrow x^-] = 1 \\ &\Longleftrightarrow y \to (x \to x \cdot y) = 1 \\ &\Longleftrightarrow y \leq x \to x \cdot y \\ &\Longleftrightarrow x \cdot y \leq x \cdot y \end{aligned}$$

equivalences given by (C7), (C4), (1), (P9), (C5)(b), (1) and the residuation property, and the last relation is trivially true. This ends the proof of the equality $x \cdot y = x \star y$. □

Before ending this section let us give the following result which allows us to express the $\wedge$ in a pseudo-Wajsberg algebra in a more compact manner, using the underlying monoid structure given by the multiplication operation $\cdot$.

Lemma 4.7 *We have*

$$x \wedge y = x \cdot (x \to y) = (x \rightsquigarrow y) \cdot x.$$

Proof. By the definitions of $\cdot$ and $\wedge$ we have:

$$\begin{aligned} x \cdot (x \to y) &= [x \rightsquigarrow (x \to y)^\sim]^- = x \wedge y \\ (x \rightsquigarrow y) \cdot x &= [x \to (x \rightsquigarrow y)^-]^\sim = x \wedge y \end{aligned}$$

□

5 The Boolean center of a pseudo-Wajsberg algebra

For L, a bounded distributive lattice, we denote by $B(L)$ the Boolean algebra of complemented elements of L. We recall that the lattice complement, if it exists, is unique.

For a pseudo-Wajsberg algebra $< A, \rightarrow, \rightsquigarrow, ^-, ^\sim, 1 >$, we consider its underlying structure of bounded distributive lattice $< A, \vee, \wedge, 0, 1 >$, and we denote by $B(A)$ the Boolean algebra of its complemented elements. We call $B(A)$ the *Boolean center* of A, and elements of it will be called *Boolean elements* of A.

In this section, we will characterize the elements of $B(A)$ in terms of pseudo-Wajsberg algebra operations, and also in terms of its canonically associated monoid structure, following similar results in [9].

We begin by proving two easy inequalities, and their consequences.

Lemma 5.1 *In a pseudo-Wajsberg lattice we have that for every $x, y \in A$ the following inequalities hold:*
(M1) $x \vee y \leq x^\sim \rightarrow y$
(M2) $(y \rightarrow x^-)^\sim \leq x \wedge y$

Proof. (M1) By (O3)(a) $x^\sim \leq y^\sim \rightarrow x^\sim$ is true, and by (C5)(a) this is equivalent to $x^\sim \leq x \rightsquigarrow y$. From this, by (O5)(a) follows $(x \rightsquigarrow y) \rightarrow y \leq x^\sim \rightarrow y$, which is precisely the desired inequality.

(M2) By (O3)(a) we have $y \leq x \rightarrow y$, and by (C5)(b) this is equivalent to $y \leq y^- \rightsquigarrow x^-$. This implies, by (O5)(a), $(y^- \rightsquigarrow x^-) \rightarrow x^- \leq y \rightarrow x^-$. From the definition of $\vee$ and (O7) this is equivalent to $(x \wedge y)^- \leq y \rightarrow x^-$, to which, applying $^\sim$ and using (C7) and (C4), we get $x \wedge y \geq (y \rightarrow x^-)^\sim$, which is precisely (M2). □

Remark: We also have the following inequalities, equivalent to (M1) and (M2) respectively, which are consequences of the commutativity of $\vee$ (respectively $\wedge$) and of equalities (C6) and, respectively, (W6):
(M1') $x \vee y \leq y^\sim \rightarrow x$
(M1") $x \vee y \leq y^- \rightsquigarrow x$
(M1"') $x \vee y \leq x^- \rightsquigarrow y$
(M2') $(x \rightarrow y^-)^\sim \leq x \wedge y$
(M2") $(x \rightsquigarrow y^\sim)^- \leq x \wedge y$
(M2"') $(y \rightsquigarrow x^\sim)^- \leq x \wedge y$

Corollary 5.2 *(1) If $x \vee y = 1$ then we have* $\begin{array}{ll} x^\sim \leq y & y^- \leq x \\ y^\sim \leq x & x^- \leq y \end{array}$

(2) If $x \wedge y = 0$ then we have $\begin{array}{ll} y \leq x^- & x \leq y^\sim \\ x \leq y^- & y \leq x^\sim \end{array}$

Proof. (1) 1 being the greatest element, from $x \vee y = 1$ follows, by (M1) and (M1')

$x^\sim \rightarrow y = 1 \iff x^\sim \leq y \iff y^- \leq x$
$y^\sim \rightarrow x = 1 \iff y^\sim \leq x \iff x^- \leq y$

where the first equivalences come from the definition of $\leq$ and the second by (C7) and (C4).

(2) 0 being the smallest element, from $x \wedge y = 0$ follows, by (M2) and (M2')

$$(y \to x^-)^\sim = 0 \iff y \to x^- = 1 \iff y \le x^- \iff x \le y^\sim$$
$$(x \to y^-)^\sim = 0 \iff x \to y^- = 1 \iff x \le y^- \iff y \le x^\sim$$

where the equivalences are given by the application of $^-$ and (W4), the definition of $\le$ and (C7), (C4). □

Corollary 5.3 *If x has a lattice complement, y, then $y = x^- = x^\sim$.*

Proof: Suppose there exists an y such that $x \vee y = 1$ and $x \wedge y = 0$. By the previous result, from the first condition it follows we have the following group of four inequalities

(I) $x^\sim \le y \quad y^- \le x$
$\quad y^\sim \le x \quad x^- \le y$

and from the second condition, we have another group

(II) $y \le x^- \quad x \le y^\sim$
$\quad x \le y^- \quad y \le x^\sim$

From $x^\sim \le y$ in (I) and $y \le x^\sim$ in (II) we obtain $y = x^\sim$. From $x^- \le y$ in (I) and $y \le x^-$ in (II) we get $y = x^-$. □

Proposition 5.4 *The following conditions are equivalent:*

(1) $x^\sim \to x = x$	*(1')* $x^- \rightsquigarrow x = x$
(2) $(x \to x^-)^\sim = x$	*(2')* $(x \rightsquigarrow x^\sim)^- = x$
(3) $x \to x^- = x^-$	*(3')* $x \rightsquigarrow x^\sim = x^\sim$
(4) $x \vee x^\sim = 1$	*(4')* $x \wedge x^- = 0$
(5) $x \wedge x^\sim = 0$	*(5')* $x \vee x^- = 1$
(6) $x \cdot x = x$	

Proof. Almost all the horizontal equivalences are immediate:

(1) $\iff$ (1') by replacing y with x in (C6) $x^\sim \to y = y^- \rightsquigarrow x$.

(2) $\iff$ (2') by replacing y with x in (W6) $(x \to y^-)^\sim = (y \rightsquigarrow x^\sim)^-$.

(4) $\iff$ (4') and (5) $\iff$ (5') by (O7), (C4) and (W4).

(2) $\iff$ (3) and (2') $\iff$ (3') by applying $^-$ in one direction and $^\sim$ in the other, and using (C4).

We now have a group of four equivalent relations, (2), (2'), (3), (3').

(1) $\Longrightarrow$ (4): $x \vee x^\sim = (x^\sim \to x) \rightsquigarrow x = x \rightsquigarrow x = 1$, by the definition of $\vee$, the hypothesis, and (P1).

(4) $\Longrightarrow$ (1): $x = 1 \to x = (x \vee x^\sim) \to x = (x \to x) \wedge (x^\sim \to x) = x^\sim \to x$, by (W1), the hypothesis, (O8)(a) and (P1).

(1') $\Longrightarrow$ (5'): $x \vee x^- = (x^- \rightsquigarrow x) \to x = x \to x = 1$, by the definition of $\vee$, the hypothesis, and (P1).

(5') $\Longrightarrow$ (1'): $x = 1 \rightsquigarrow x = (x \vee x^-) \rightsquigarrow x = (x \rightsquigarrow x) \wedge (x^- \rightsquigarrow x) = x^- \rightsquigarrow x$, by (W1), the hypothesis, (O8)(b) and (P1).

We now have a second group of equivalent relations, namely (1), (1'), (4), (4'), (5), (5'). We establish next the equivalence between the two groups.

(4') $\Longrightarrow$ (3): $x^- = x \to 0 = x \to (x \wedge x^-) = (x \to x) \wedge (x \to x^-) = x \to x^-$, by (C3)(a), the hypothesis, (O9)(a) and (P1).

(3) $\Longrightarrow$ (4'): $x \wedge x^- = [(x \to x^-) \to x^-]^\sim = (x^- \to x^-)^\sim = 1^\sim = 0$, by the definition of $\wedge$, the hypothesis, (P1) and (W4).

Finally, we also have

(2) $\iff$ (6): trivial, because $x \cdot x = (x \to x^-)^\sim$. □

Proposition 5.5 *An element x of a pseudo-Wajsberg algebra A is a Boolean element, iff any one of the equivalent conditions of the previous proposition holds.*

Proof: The equivalent conditions (4) or (5) state the $x^\sim$ is a lattice complement for x, so $x \in B(A)$. For the reverse implication use 5.3. □

Proposition 5.6 *If $x \in B(A)$ and $y \in A$ then*

$$(1) \quad x \vee y = x^\sim \to y$$
$$(2) \quad x \wedge y = (y \to x^-)^\sim$$

Proof: (1) From (M1) we have $x \vee y \leq x^\sim \to y$. For the other inequality we have:

$$\begin{aligned}(x^\sim \to y) \rightsquigarrow (x \vee y) &= [(x^\sim \to y) \rightsquigarrow x] \vee [(x^\sim \to y) \rightsquigarrow y] \\ &\geq x \vee x^\sim \vee y \geq x \vee x^\sim = 1\end{aligned}$$

from (O13)(b), (O3)(b) and $x \in B(A)$.

(2) From (M2) we have $x \wedge y \geq (y \to x^-)^\sim$. For the other inequality we have:

$$\begin{aligned}(x \wedge y) \rightsquigarrow (y \to x^-)^\sim &= [x \rightsquigarrow (y \to x^-)^\sim] \vee [y \rightsquigarrow (y \to x^-)^\sim] \\ &= [(y \to x^-)^{\sim\sim} \to x^\sim] \vee (x^- \wedge y)^\sim \\ &\geq x^\sim \vee x \vee y^\sim \geq x \vee x^\sim = 1\end{aligned}$$

from (O14)(b), (C5)(a), the definition of $\wedge$, (O7), (O3)(a) and $x \in B(A)$. □

Corollary 5.7 *If $x, y \in B(A)$ then*

(1) $\quad x^\sim \to y \quad = \quad x \vee y \quad = y^\sim \to x$
(2) $\quad (y \to x^-)^\sim = \quad x \wedge y \quad = (x \to y^-)^\sim$

Proof: Immediate from the commutativity of $\vee$ and $\wedge$ and the previous result. □

Corollary 5.8 *1. $B(A)$ is a sublattice of $< A, \vee, \wedge, 0, 1 >$.*

2. $B(A)$ is a pseudo-Wajsberg subalgebra of $< A, \to, \rightsquigarrow, ^-, ^\sim, 1 >$.

3. For any C, pseudo-Wajsberg subalgebra of $< A, \to, \rightsquigarrow, ^-, ^\sim, 1 >$, whose underlying lattice structure is a Boolean algebra, we have $C \subseteq B(A)$.

4. $B(A)$ is a submonoid of the residuated bounded monoid $< A, \cdot, \leq, 1 >$.

5. For any C, submonoid of the residuated bounded monoid $< A, \cdot, \leq, 1 >$, in which multiplication is idempotent, we have $C \subseteq B(A)$.

6. $B(A)$ is a Wajsberg subalgebra of A.

Proof: 1. $B(A)$ is closed with respect to $^-$ and $^\sim$, immediate from 5.4. For $x, y \in B(A)$ we will prove that $x \vee y \in B(A)$, using for instance (4) in 5.4.

$$(x \vee y) \vee (x \vee y)^\sim = (x \vee y) \vee (x^\sim \wedge y^\sim) = [(x \vee y) \vee x^\sim] \wedge [(x \vee y) \vee y^\sim] = 1$$

from (O7), (O15), associativity and commutativity of $\vee$ and $x, y \in B(A)$. Closure to $\wedge$ follows by (O7) from closure to $\vee$ and $^-$.

2. For $x, y \in B(A)$ we will prove that $x \to y \in B(A)$:

$$x \to y = (x^-)^\sim \to y = x^- \vee y \in B(A)$$

from (C4), 5.7, and closure of $B(A)$ to $^-$ and $\vee$. Since $B(A)$ is closed to $^\sim$ and $\rightarrow$, we also have closure to $\rightsquigarrow$ by

$$x \rightsquigarrow y = y^\sim \rightarrow x^\sim \in B(A).$$

3. Obvious.

4. Since the monoid multiplication is defined in terms of the operations of pseudo-Wajsberg algebra, and $B(A)$ is closed to these, closure to $\cdot$ follows. Moreover, $\cdot$ on $B(A)$ is idempotent (see condition (6) in 5.4, and also 5.5) and commutative because it coincides with $\wedge$ (see (2) in 5.7).

5. Obvious.

6. We have to show that $\rightarrow$ and $\rightsquigarrow$ coincide on $B(A)$. Let $x, y \in B(A)$. By (C4) and Corollary 5.7 we have

$$x \rightarrow y = (x^-)^\sim \rightarrow y = x^- \vee y$$

and by (C5)(a) and Corollary 5.7 we have

$$x \rightsquigarrow y = y^\sim \rightarrow x^\sim = y \vee x^\sim.$$

From $x \in B(A)$ we have $x^- = x^\sim$, so the right-hand sides of the above equalities coincide, which leads to $x \rightarrow y = x \rightsquigarrow y$. □

References

[1] R. Ceterchi, On algebras with implications, categorically equivalent to pseudo-MV algebras, in: "Information Technology", *Proceedings of the 4th International Symposium on Economic Informatics, May 1999*, INFOREC Printing House, Bucharest, Romania, 1999, p.912-917 Bucharest.

[2] R. Ceterchi, Pseudo-Wajsberg algebras (to appear in *Mult. Val. Logic*) (also at http://funinf.math.unibuc.ro/~logic).

[3] C.C. Chang, Algebraic analysis of many valued logics, *Trans. Amer. Math. Soc.* 88 (1958) p. 467-490.

[4] C.C. Chang, A new proof of the completeness of the Lukasiewicz axioms, *Trans. Amer. Math. Sos.* 93 (1959) p. 74-80.

[5] R. Cignoli and D. Mundici, An elementary proof of Chang's completeness theorem for the infinite-valued calculus of Lukasiewicz, *Studia Logica*, 58 (1997) 79-97.

[6] R. Cignoli, I. M. L. D'Ottaviano, D. Mundici, Algebraic Foundations of many-valued Reasoning (to appear).

[7] J.M. Font, A.J. Rodriguez, A. Torrens, Wajsberg algebras, *Stochastica*, Vol. VIII, Nr. 1 (1984) p.5-31.

[8] G. Georgescu, A. Iorgulescu, Pseudo-MV Algebras: a Noncommutative Extension of MV Algebras, *The Proceedings of the Fourth International Symposium on Economic Informati cs, Bucharest, Romania*, 961-968, 1999.

[9] G. Georgescu, A. Iorgulescu, Pseudo-MV algebras, (to appear in *Mult. Val. Logic*).

[10] M. Wajsberg, Beiträge zum Metaaussagenkalkül, *Monat. Math. Phys.* 42 (1935) p. 240.