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The Automorphism Group of a Hypercube[1]

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Abstract: We present explicitly in this expository note the automorphism group of the hypercube Q_d of dimension d as a permutation group acting on its 2^d nodes. This group $\Gamma(Q_d)$ acts on the node set V_d of Q_d and thus has degree 2^d. It is expressed as the binary operation called exponentiation which combines the two symmetric groups S_2 (of degree and order 2) and S_d (of degree d and order $d!$). Specifically,

$$\Gamma(Q_d) = [S_2]^{S_d}.$$

has order $2^d d!$.

Key Words: Automorphism group, hypercube, permutation graph.
Category: F.4.1.

1 Introduction

For completeness, in this mainly expository note, we include some definitions that can be found in the books [4,6].

A *graph* $G = (V, E)$ consists of a finite nonempty set V of n vertices or nodes and a subset E of the set of all unordered pairs of distinct nodes, called *edges*. We may write $V = \{v_1, v_2, \dots v_n\}$ and $E = \{e_1, e_2, \dots e_m\}$. Two nodes of G are *adjacent* if they are joined by an edge.

An *isomorphism* between two graphs G, G' is a 1-1 correspondence between V and V' that preserves adjacency. An *automorphism* of G is an isomorphism of G with itself. Hence each automorphism of G is a permutation of V (but not conversely, unless $G = K_m$, the complete graph, or its complement, the graph with n nodes and no edges).

It is well known that the set $\Gamma(G)$ of all automorphisms of a graph G forms a permutation group of degree $n = \mid V \mid$ acting on the object set $V = V(G)$.

2 Two Binary Operations on Permutation Groups

Consider two permutation groups A, B acting on object sets X, Y with orders a, b, and degrees d, e, respectively. Their *composition* or *wreath product*, written $A[B]$ and called "A around B" (see Polya [9]) acts on the cartesian product $X \times Y$ as follows. For each permutation $\alpha \in A$ and any sequence $(\beta_1, \beta_2, \dots, \beta_d)$

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of d (not necessarily distinct) permutations in B, there is a unique permutation $\gamma \in A \times B$ such that for each ordered pair (x_j, y_j) in $X \times Y$, we have

$$\gamma(x_i, y_j) = (\alpha x_i, \beta_i y_j).$$

Frucht [1] made the following observation on the automorphism group $\Gamma(G)$ of a graph G consisting of k copies of a connected graph H:

$$\Gamma(G) = S_k[\Gamma(H)].$$

The symmetric group S_k serves to permute the k copies of H.

3 The Exponentiation

The exponentiation group written $[B]^A$ of B raised to the A is the permutation group acting on Y^X, the set of all functions from X into Y, as follows:

> For each $\alpha \in A$ and each sequence $(\beta_1, \beta_2, \ldots, \beta_d)$ of permutations in B, we obtain the permutation $\gamma \in [B]^A$ which takes a function $f \in Y^X$ to the function γf defined for all $x_i \in X$ by
>
> $$\gamma f(x_i) = \beta_i f(\alpha x_i).$$

By these definitions, the two *abstract* groups $A[B]$ and $[B]^A$ are isomorphic.

4 Graphs with These Two Operations

It is well known [4, p. 165] that a graph G and its complement $\bar{G}$ have isomorphic automorphism groups.

The complement of the octahedron O_3 is the graph consisting of three disjoint edges, written $3K_2$. Similarly, the complement of the d-dimensional hyperoctahedron O_d is the graph dK_2. By the result of Frucht [1], $\Gamma(O_d) = S_d[S_2]$.

For the definition of the cartesian product $G \times H$ of two graphs, see [4, p. 163]. One of the many equivalent ways [5] to define a hypercube Q_d uses the recursive cartesian product equations:

$$Q_1 = K_2,\ Q_{d+1} = Q_d \times K_2.$$

Thus $Q_3 = K_2 \times K_2 \times K_2$.

Theorem 1. *The automorphism group $\Gamma(Q_d)$ of the d-dimensional hypercube Q_d is the exponentiation group $[S_2]^{S_d}$.*

The proof was extensively generalized by Palmer and Robinson [8]. Earlier, Slepian [11] in 1953 developed painstakingly the cycle index (see [9]) of this exponentiation group in order to enumerate the types of Boolean functions of a given number of variables. And still earlier, Polya [10] in 1940 did the same thing but only for $d \leq 8$. Neither of these specified the group of the theorem explicitly.

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