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Connections Between MV_n Algebras and n-valued Lukasiewicz-Moisil Algebras - IV[1,2]

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Abstract: We introduce two chains of unary operations in the MV_n algebra of Revaz Grigolia; they will be used in establishing many connections between these algebras and n-valued Lukasiewicz-Moisil algebras (LM_n algebras for short). The study has four parts. It is by and large self-contained.
The main result of the first part is that MV_4 algebras coincide with LM_4 algebras. The larger class of "relaxed"-MV_n algebras is also introduced and studied. This class is related to the class of generalized LM_n pre-algebras.
The main results of the second part are that, for $n \geq 5$, any MV_n algebra is an LM_n algebra and that the canonical MV_n algebra can be identified with the canonical LM_n algebra.
In the third part, the class of good LM_n algebras is introduced and studied and it is proved that MV_n algebras coincide with good LM_n algebras.
In the present fourth part, the class of $\oplus$-proper LM_n algebras is introduced and studied. $\oplus$-proper LM_n algebras coincide (can be identified) with Cignoli's proper n-valued Lukasiewicz algebras. MV_n algebras coincide with $\oplus$-proper LM_n algebras ($n \geq 2$). We also give the construction of an LM_3 (LM_4) algebra from the odd (respectively even)-valued LM_n algebra ($n \geq 5$), which proves that LM_4 algebras are as much important than LM_3 algebras; MV_n algebras help to see this point.

Key Words: n-valued Lukasiewicz-Moisil algebra, MV_n algebra
Category: F.4.1.

8 $\oplus$-proper LM_n algebras

We have seen in [13] that MV_n algebras can be identified with good LM_n algebras. We shall see in this section that MV_n algebras can be also identified with $\oplus$-proper LM_n algebras, where the notion of $\oplus$-proper LM_n algebra is obtained from Cignoli's proper n-valued Lukasiewicz algebra, by slight changes.

Roberto Cignoli defined [5] the proper n-valued Lukasiewicz algebra starting from the Lukasiewiczian implication , $\rightarrow$, defined on $L_n = \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$ by

$$x \rightarrow y = \min(1, 1 \Leftrightarrow x + y);$$

therefore, in this section, I shall rename the proper n-valued Lukasiewicz algebra as " $\rightarrow$-proper LM_n algebra". The table of $\rightarrow$ in L_n is symmetric with respect to the second diagonal, therefore the $\rightarrow$-proper LM_n algebra was defined by

[1] C. S. Calude and G. Ştefănescu (eds.). *Automata, Logic, and Computability. Special issue dedicated to Professor Sergiu Rudeanu Festschrift.*

[2] The first 2 parts appeared in *Discrete Mathematics*, volumes 181 and 202, respectively. The 3rd part was submitted for publication; copies may be obtained from the author.

Cignoli in the following way (cf. ([2], 9.2.3 and 9.2.4)):
let

$$\begin{cases} S_n = \{(i,j) \in \mathbf{N}^2 \mid 3 \leq i \leq n \Leftrightarrow 2,\ 1 \leq j \leq n \Leftrightarrow 4,\ i > j\}, & \text{if } n \geq 5, \\ S_n = \emptyset, & \text{if } n < 5; \\ T_n = \{(i,j) \in \mathbf{N}^2 \mid 2 \leq i \leq n \Leftrightarrow 2,\ 1 \leq j \leq n \Leftrightarrow 3,\ i > j\}, & \text{if } n \geq 4, \\ T_n = \emptyset, & \text{if } n < 4, \end{cases} \tag{1}$$

let $\Rightarrow$ be the generalization of the residuation considered by Moisil (cf.([2], 4.3.2)):

$$x \Rightarrow y = y \vee \bigwedge_{j=1}^{n-1} \left((r_j x)^- \vee r_j y\right), \quad x, y \in A, \tag{2}$$

and let F_{ij}, $(i,j) \in S_n$, be a family of binary operations on A such that

$$r_k(F_{ij}(x,y)) = \begin{cases} 0, & k \leq i \Leftrightarrow j \\ d_i(x) \wedge d_j(y), & k > i \Leftrightarrow j, \end{cases} \tag{3}$$

for any $x, y \in A$, $(i,j) \in S_n$ and $k \in J = \{1, 2, \ldots, n \Leftrightarrow 1\}$, where if we put $r_0 x = 0$ and $r_n x = 1$ for any $x \in A$, then the unary operators d_i, $i = \overline{0, n \Leftrightarrow 1}$ are defined by:

$$d_i(x) = r_{n-i}x \wedge (r_{n-i-1}x)^-, \quad x \in A. \tag{4}$$

Definition 8.1 ([5], 2.1) A $\rightarrow$*-proper* LM_n *algebra* is a structure

$$\mathcal{A}^c = (\mathcal{A}, \Rightarrow, (F_{ij})_{(i,j) \in S_n}),$$

where $\mathcal{A} = (A, \vee, \wedge, ^-, (r_j)_{j \in J}, 0, 1)$ is an LM_n algebra, $\Rightarrow$ is a binary operation on A verifying (2) and F_{ij}, $(i,j) \in S_n$, are binary operations on A verifying (3), the unary operators d_i, $i = \overline{0, n \Leftrightarrow 1}$ being those from (4).

Example 8.2 ([5], 2.3) If we consider the canonical LM_n algebra, $\mathcal{L}_n = \mathcal{L}_n^{(LM_n)}$, then the structure

$$\mathcal{L}_n^c = \left(\mathcal{L}_n, \Rightarrow, (F_{ij})_{(i,j) \in S_n}\right)$$

is a $\rightarrow$-proper LM_n algebra, where

$$x \Rightarrow y = \begin{cases} 1, & x \leq y \\ y, & x > y, \end{cases} \tag{5}$$

$$F_{ij}\left(\frac{r}{n \Leftrightarrow 1}, \frac{s}{n \Leftrightarrow 1}\right) = \begin{cases} \frac{n-1-i+j}{n-1}, & (r,s) = (i,j) \\ 0, & (r,s) \neq (i,j) \end{cases} \tag{6}$$

and

$$d_j\left(\frac{i}{n \Leftrightarrow 1}\right) = \begin{cases} 0, & i \neq j, \\ 1, & i = j, \quad i \in \{0\} \cup J,\ j \in J. \end{cases} \tag{7}$$

$\mathcal{L}_n^c$ is called the *canonical* $\rightarrow$*-proper* LM_n *algebra*.

In every LM_n algebra we also have (cf. [5], again)

$$r_j x = \bigvee_{i=1}^{j} d_{n-i}(x), \quad j = \overline{1, n}. \tag{8}$$

Since we have started now from an MV_n algebra, i.e., a structure with the operation $\oplus$ instead of $\rightarrow$, I shall modify Cignoli's definition in order to obtain the $\oplus$-proper LM_n algebra, i.e., the proper LM_n algebra starting from the canonical addition, $\oplus$, defined on L_n by: $x \oplus y = \min(1, x + y) = x^- \rightarrow y$.

Since the table of canonical $\oplus$ is symmetric with respect to the principal diagonal (the operation $\oplus$ is commutative), we define:

$$\begin{cases} U_n = \{(i,j) \in J^2 \mid 1 \leq i \leq n \Leftrightarrow 4,\ 1 \leq j \leq n \Leftrightarrow 4;\ i + j < n \Leftrightarrow 1\}, & n \geq 5, \\ U_n = \emptyset, & n < 5; \\ V_n = \{(i,j) \in J^2 \mid 1 \leq i \leq n \Leftrightarrow 3,\ 1 \leq j \leq n \Leftrightarrow 3;\ i + j < n \Leftrightarrow 1\} & \\ \quad = \{(i,j) \in J^2 \mid 1 \leq j \leq n \Leftrightarrow 2 \Leftrightarrow i,\ 1 \leq i \leq n \Leftrightarrow 3\} & \\ \quad = \{(i,j) \in J^2 \mid 1 \leq i \leq n \Leftrightarrow 2 \Leftrightarrow j,\ 1 \leq j \leq n \Leftrightarrow 3\}, & n \geq 4, \\ V_n = \emptyset, & n < 4. \end{cases} \tag{9}$$

Then $\mid V_n \mid = 1 + 2 + \ldots + (n \Leftrightarrow 3) = \frac{(n-3)(n-2)}{2}$ and $V_n = U_n \cup \{(1, n \Leftrightarrow 3),\ (n \Leftrightarrow 3, 1)\}$.

Remark 8.3 We could take into account the commutativity of $\oplus$ and define a smaller set:

$$\begin{aligned} V_n' &= \{(i,j) \in J^2 \mid 1 \leq i \leq n \Leftrightarrow 3,\ 1 \leq j \leq n \Leftrightarrow 3,\ i + j < n \Leftrightarrow 1, j \geq i\} \\ &= \{(i,j) \in J^2 \mid i \leq j \leq n \Leftrightarrow 2 \Leftrightarrow i,\ 1 \leq i \leq [n/2] \Leftrightarrow 1\} \end{aligned}$$

with $\mid V_n' \mid = 1 + 2 + \ldots + (n \Leftrightarrow 4) = \frac{(n-4)(n-3)}{2}$.

Proposition 8.4 *For any $i, j \in J$ we have:*
(i) $(i,j) \in U_n \ \Leftrightarrow\ (n \Leftrightarrow 1 \Leftrightarrow i, j) \in S_n$
(ii) $(i,j) \in V_n \ \Leftrightarrow\ (n \Leftrightarrow 1 \Leftrightarrow i, j) \in T_n$.

Proof.

$$\begin{aligned} (i,j) \in U_n &\Leftrightarrow 1 \leq i \leq n \Leftrightarrow 4,\ 1 \leq j \leq n \Leftrightarrow 4,\ i + j < n \Leftrightarrow 1 \\ &\Leftrightarrow 4 \Leftrightarrow n \leq \Leftrightarrow i \leq \Leftrightarrow 1,\ 1 \leq j \leq n \Leftrightarrow 4,\ n \Leftrightarrow 1 \Leftrightarrow i > j \\ &\Leftrightarrow 3 \leq n \Leftrightarrow 1 \Leftrightarrow i \leq n \Leftrightarrow 2,\ 1 \leq j \leq n \Leftrightarrow 4,\ n \Leftrightarrow 1 \Leftrightarrow i > j \\ &\Leftrightarrow (n \Leftrightarrow 1 \Leftrightarrow i, j) \in S_n. \end{aligned}$$

Thus (i) holds. The proof of (ii) is similar. □

Lemma 8.5 *In any LM_n algebra*

$$d_{n-1-i}(x^-) = d_i(x), \quad i = \overline{0, n \Leftrightarrow 1}.$$

Proof. In L_n let $x = \frac{j}{n-1}$; then $d_{n-1-i}(x^-) = d_{n-1-i}(\frac{n-1-j}{n-1}) = \begin{cases} 1, & n \Leftrightarrow 1 \Leftrightarrow i = n \Leftrightarrow 1 \Leftrightarrow j \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} = d_i(x)$; then apply ([11], 2.8). □

Let $\mathcal{A}$ be an LM_n algebra. We shall define a binary operation on A, $\bigcirc$, by:

$$x \bigcirc y = x^- \Rightarrow y, \quad \text{i.e.} \quad x \bigcirc y = y \vee \bigwedge_{j=1}^{n-1} (r_{n-j}x \vee r_j y), \quad x, y \in A \tag{10}$$

and a family of binary operations, $(G_{ij})_{(i,j)\in U_n}$, by

$$G_{ij}(x, y) = F_{n-1-i,j}(x^-, y). \tag{11}$$

Then G_{ij} verifies, for every $(i,j) \in U_n$ and $k \in J$:

$$r_k(G_{ij}(x, y)) = \begin{cases} 0, & k \leq n \Leftrightarrow 1 \Leftrightarrow (i+j), \\ d_i(x) \wedge d_j(y), & k > n \Leftrightarrow 1 \Leftrightarrow (i+j). \end{cases} \tag{12}$$

Indeed, $r_k(G_{ij}(x, y)) =$

$$= r_k(F_{n-1-i,j}(x^-, y)) = \begin{cases} 0, & k \leq (n \Leftrightarrow 1 \Leftrightarrow i) \Leftrightarrow j, \\ d_{n-1-i}(x^-) \wedge d_j(y), & k > (n \Leftrightarrow 1 \Leftrightarrow i) \Leftrightarrow j \end{cases}$$

$$= \begin{cases} 0, & k \leq n \Leftrightarrow 1 \Leftrightarrow i \Leftrightarrow j, \\ d_i(x) \wedge d_j(y), & k > n \Leftrightarrow 1 \Leftrightarrow i \Leftrightarrow j, \end{cases}$$

by (11), (3) and Lemma 8.5.
Hence we can give the following

Definition 8.6 A $\oplus$-*proper* LM_n *algebra* is a structure

$$\mathcal{A}^a = (\mathcal{A}, \bigcirc, (G_{ij})_{(i,j)\in U_n}),$$

where $\mathcal{A}$ is an LM_n algebra, $\bigcirc$ is a binary operation on A verifying (10) and G_{ij}, $(i,j) \in U_n$, are binary operations on A verifying (12), d_i, $i = \overline{0, n \Leftrightarrow 1}$ being those from (4).

Let $\mathcal{A}$ be an LM_n algebra and let us consider the two kinds of proper LM_n algebras: $\mathcal{A}^c$ and $\mathcal{A}^a$. The two structures can be identified, namely we have the following

Theorem 8.7 *1) Let* $\mathcal{A}^c = (\mathcal{A}, \Rightarrow, (F_{ij})_{(i,j)\in S_n})$ *be a* $\rightarrow$*-proper* LM_n *algebra. Define*

$$\alpha(\mathcal{A}^c) = (\mathcal{A}, \bigcirc, (G_{ij})_{(i,j)\in U_n})$$

by $x \bigcirc y = x^- \Rightarrow y$, $G_{ij}(x, y) = F_{n-1-i,j}(x^-, y)$.
Then $\alpha(\mathcal{A}^c)$ *is a* $\oplus$*-proper* LM_n *algebra.*

2) Let $\mathcal{A}^a = (\mathcal{A}, \bigcirc, (G_{ij})_{(i,j)\in U_n})$ *be a* $\oplus$*-proper* LM_n *algebra. Define*

$$\beta(\mathcal{A}^a) = (\mathcal{A}, \Rightarrow, (F_{ij})_{(i,j)\in S_n})$$

by $x \Rightarrow y = x^- \bigcirc y$, $F_{ij}(x, y) = G_{n-1-i,j}(x^-, y)$.
Then $\beta(\mathcal{A}^a)$ *is a* $\rightarrow$*-proper* LM_n *algebra.*

3) The maps α, β *are mutually inverse.*

Proof. Obvious. □

This theorem allows us to extend all the results concerning $\rightarrow$-proper LM_n algebras to $\oplus$-proper LM_n algebras. In the sequel I shall present some of these results.

Examples 8.8 (i) Let $\mathcal{L}_n = \mathcal{L}_n^{(LM_n)}$ be the canonical LM_n algebra. Then the structure

$$\mathcal{L}_n^a = (\mathcal{L}_n\ ,\ \bigcirc\ , (G_{ij})_{(i,j)\in U_n})$$

is a $\oplus$-proper LM_n algebra, that I shall call *the canonical $\oplus$-proper LM_n algebra*, where

$$x \bigcirc y = x^- \Rightarrow y = \begin{cases} 1, & x^- \leq y \\ y, & x^- > y, \end{cases} \tag{13}$$

$$G_{ij}\left(\frac{r}{n-1}, \frac{s}{n-1}\right) = \begin{cases} \frac{i+j}{n-1}, & (r,s) = (i,j) \\ 0, & (r,s) \neq (i,j), \end{cases} \tag{14}$$

since

$$\begin{aligned} G_{n-1-i,j}(x^-, y) &= G_{n-1-i,j}\left(\frac{n-1-r}{n-1}, \frac{s}{n-1}\right) \\ &= \begin{cases} \frac{(n-1-i)+j}{n-1}, & (n \Leftrightarrow 1 \Leftrightarrow r, s) = (n \Leftrightarrow 1 \Leftrightarrow i, j) \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{n-1-i+j}{n-1}, & (r,s) = (i,j) \\ 0, & \text{otherwise} \end{cases} \\ &= F_{ij}\left(\frac{r}{n-1}, \frac{s}{n-1}\right) \end{aligned}$$

and $d_j,\ j \in J$ are those given by (7).

(ii) Take the canonical LM_5 algebra $\mathcal{L}_5$; $L_5 = \{0, 1/4, 2/4, 3/4, 1\}$ and its LM_5 subalgebras are: $S_{(1)} = L_5,\ S_{(2)} = \{0, 2/4, 1\} \simeq L_3,\ S_{(4)} = \{0,1\} \simeq L_2$ and $S = \{0, 1/4, 3/4, 1\}$. We have $J = \{1,2,3,4\}$ and hence

$$U_5 = \{(i,j) \in J^2 \mid 1 \leq i \leq 1,\ 1 \leq j \leq 1,\ i+j < 4\} = \{(1,1)\},$$

$$G_{11}\left(\frac{r}{4}, \frac{s}{4}\right) = \begin{cases} 2/4, & (r,s) = (1,1) \\ 0, & (r,s) \neq (1,1) \end{cases} = \begin{cases} 1/4 \oplus 1/4, & (r,s) = (1,1) \\ 0, & (r,s) \neq (1,1). \end{cases}$$

Then $(\mathcal{S}_{(1)}, \bigcirc|_{S_{(1)}}, G_{11}|_{S_{(1)}})$, $(\mathcal{S}_{(2)}, \bigcirc|_{S_{(2)}}, G_{11}|_{S_{(2)}})$ and $(\mathcal{S}_{(4)}, \bigcirc|_{S_{(4)}}, G_{11}|_{S_{(4)}})$ are $\oplus$-proper LM_5 subalgebras of $\mathcal{L}_5^a$, hence they are $\oplus$-proper LM_5 algebras, while $(\mathcal{S},\ \bigcirc|_S\ , G_{11}|_S)$ is not a $\oplus$-proper LM_5 subalgebra of $\mathcal{L}_5^a$, since $2/4 \notin S$, and therefore it is not a $\oplus$-proper LM_5 algebra.

Remarks 8.9 (i) For $n \in \{2,3,4\}$, $U_n = \emptyset$, therefore in these cases any LM_n algebra is $\oplus$-proper. Recall that in these cases any LM_n algebra is $\rightarrow$-proper too.

(ii) For $n \geq 5$, not any LM_n algebra is $\oplus$-proper, since not any LM_n subalgebra of $\mathcal{L}_n^{(LM_n)}$ is a $\oplus$-proper LM_n subalgebra of $\mathcal{L}_n^a$ (see Examples 8.8(ii)).

(iii) For $n \geq 4$ we can extend the definition of G_{ij} for any $(i,j) \in V_n$ by setting (see ([5], 2.2)):

$$\begin{cases} G_{1,n-3}(x,y) = d_1(x) \wedge d_{n-3}(y) \wedge x^- \\ G_{n-3,1}(x,y) = d_{n-3}(x) \wedge d_1(y) \wedge y^- \end{cases} \tag{15}$$

Then G_{ij} satisfies the condition (12) for any $(i,j) \in V_n$.

Proposition 8.10 $G_{1,n-3}(x,y) = G_{n-3,1}(y,x), \quad x, y \in A.$

Proof. Obvious, by (15) and (4). □

Example 8.11 Let n=5 and let us consider the canonical $\oplus$-proper LM_5 algebra, $\mathcal{L}_5^a$; then we have:

$$G_{12}(x,y) = d_1(x) \wedge d_2(y) \wedge x^- = \begin{cases} 1 \wedge 1 \wedge 3/4, & (x,y) = (1/4, 2/4) \\ 0, & (x,y) \neq (1/4, 2/4) \end{cases}$$

$$= \begin{cases} 3/4, & (x,y) = (1/4, 2/4) \\ 0, & (x,y) \neq (1/4, 2/4) \end{cases} = \begin{cases} 1/4 \oplus 2/4, & (x,y) = (1/4, 2/4) \\ 0, & (x,y) \neq (1/4, 2/4). \end{cases}$$

Let $\mathcal{A}$ be an LM_n algebra and let us consider the Boolean center of $\mathcal{A}$:

$$C(A) = \{x \in A \mid r_j x = x, \text{ for every } j \in J\}.$$

Lemma 8.12 *Let* $\mathcal{A}^a = (\mathcal{A}, \bigcirc, (G_{ij})_{(i,j) \in U_n})$ *be a* $\oplus$*-proper-*LM_n *algebra,* x, $y \in A$ *and* a, $b \in C(A)$. *Then the following properties hold:*
(1) $G_{ij}(x,y) = G_{ji}(y,x)$,
(2) $G_{ij}(x \vee a, y \wedge b) = G_{ij}(x,y) \wedge a^- \wedge b$,
(3) $G_{ij}(x \wedge a, y \vee b) = G_{ij}(x,y) \wedge a \wedge b^-$,
(4) $G_{ij}(x,b) = G_{ij}(a,y) = 0$.

Proof. By ([2], 9.2.8), we get
(1) $G_{ij}(x,y) = F_{n-1-i,j}(x^-, y) = F_{n-1-j,i}(y^-, x) = G_{ji}(y,x)$,
(2) $G_{ij}(x \vee a, y \wedge b) = F_{n-1-i,j}((x \vee a)^-, y \wedge b) =$
$= F_{n-1-i,j}(x^-, y) \wedge a^- \wedge b = G_{ij}(x,y) \wedge a^- \wedge b$,
(3) $G_{ij}(x \wedge a, y \vee b) = F_{n-1-i,j}((x \wedge a)^-, y \vee b) =$
$= F_{n-1-i,j}(x^-, y) \wedge a \wedge b^- = G_{ij}(x,y) \wedge a \wedge b^-$,
(4) $G_{ij}(x,b) = F_{n-1-i,j}(x^-, b) = 0, \quad G_{ij}(a,y) = F_{n-1-i,j}(a^-, y) = 0.$ □

Proposition 8.13 *Any* $\oplus$*-proper* LM_n *algebra is isomorphic to a subdirect product of a family of* $\oplus$*-proper* LM_n *subalgebras of the canonical* $\oplus$*-proper* LM_n *algebra,* $\mathcal{L}_n^a$.

Proof. By ([2], 9.2.11)). □

Definition 8.14 (See([5], (3.1)) or ([2], 9.2.12))
If $\mathcal{A}^a = \left(\mathcal{A}, \bigcirc, (G_{ij})_{(i,j) \in U_n}\right)$ is a $\oplus$-proper LM_n algebra, define

$$\Psi^a(\mathcal{A}^a) = (A, \oplus, \cdot, ^-, 0, 1),$$

where $\oplus$ is defined by

$$x \oplus y = (x \bigcirc y) \vee x \vee \bigvee_{(i,j) \in V_n} G_{ij}(x,y) \tag{16}$$

and $x \cdot y = (x^- \oplus y^-)^-$.

Proposition 8.15 *If* $\mathcal{L}_n^a$ *is the canonical* $\oplus$*-proper* LM_n *algebra, then* $\Psi^a(\mathcal{L}_n^a)$ *is the canonical* MV_n *algebra,* $\mathcal{L}_n = \mathcal{L}_n^{(MV_n)}$.

Proof. By ([2], 9.2.15), ([11], 1.11) and since $x \oplus y = x^- \rightarrow y$. □

Theorem 8.16 *If $\mathcal{A}^a$ is a $\oplus$-proper LM_n algebra, then $\Psi^a(\mathcal{A}^a)$ is an MV_n algebra.*

Proof. By Proposition 8.13, Proposition 8.15 and the converse of ([11], 1.12). □

Proposition 8.17 *(See ([2], 9.2.14)) In every $\oplus$-proper LM_n algebra $\mathcal{A}^a$ the following properties hold:*
(i) $r_1(x \oplus y) = r_1(x \bigcirc y)$,
(ii) $x \bigcirc y = r_1(x \oplus y) \vee y$,
(iii) If $a \in C(A)$, *then* $x \oplus a = x \vee a$,
(iv) If $b \in C(A)$, *then* $b \oplus x = b \vee x$,
(v) $0 \oplus x = x$,
(vi) $x \oplus y = 1$ *iff* $x^- \leq y$.

Proof. For every $(i,j) \in V_n$ we have $i + j < n \Leftrightarrow 1 \Leftrightarrow n \Leftrightarrow 1 \Leftrightarrow (i+j) > 0 \Leftrightarrow (n \Leftrightarrow 1) \Leftrightarrow (i+j) \geq 1$. Hence $r_1 G_{ij}(x,y) = 0$, by (5). But $r_1 x \wedge x^- = 0 \leq y$, therefore we get $r_1 x \leq r_1(x \bigcirc y)$. Consequently, $r_1(x \oplus y) = r_1(x \bigcirc y) \vee r_1 x = r_1(x \bigcirc y)$, and thus (i) holds. Since $x \bigcirc y = y \vee \bigwedge_{i=1}^{n-1}(r_{n-i}x \vee r_i y)$, then $y \vee r_1(x \oplus y) = y \vee r_1(x \bigcirc y) = y \vee r_1 y \vee \bigwedge_{i=1}^{n-1}(r_{n-i}x \vee r_i y) = y \vee \bigwedge_{i=1}^{n-1}(r_{n-i}x \vee r_i y) = x \bigcirc y$ and thus (ii) holds. The remaining of the proof is routine. □

Definition 8.18 If $\mathcal{A} = (A, \oplus, \cdot, ^-, 0, 1)$ is an MV_n algebra, define

$$\Phi^a(\mathcal{A}) = (\Phi(\mathcal{A}), \bigcirc, (G_{ij})_{(i,j)\in U_n}),$$

where $\Phi(\mathcal{A})$ is defined by ([12], 5.19), $\bigcirc$ is defined by (10) and (see ([5], (3.11)) or ([2], 9(2.27)))

$$G_{ij}(x,y) = (x \oplus y) \wedge d_i(x) \wedge d_j(y),\ (i,j) \in U_n,\ x, y \in A,$$

with $d_i,\ i = \overline{0, n \Leftrightarrow 1}$ given by (4).

Then we have the following

Theorem 8.19 *(1) If $\mathcal{A}$ is an MV_n algebra, then $\Phi^a(\mathcal{A})$ is a $\oplus$- proper LM_n algebra.*
(2) The maps Φ^a and Ψ^a are mutually inverse.

Proof. To prove (1), $G_{ij}(x,y) = F_{n-1-i,j}(x^-, y) = (x^- \rightarrow y) \wedge d_{n-1-i}(x^-) \wedge d_j(y) = (x \oplus y) \wedge d_i(x) \wedge d_j(y)$, by ([2], 9(2.27)).
(2) is obvious. □

By Theorems 8.16 and 8.19, MV_n algebras are identified with $\oplus$-proper LM_n algebras. Since, by [13], MV_n algebras can also be identified with good LM_n algebras, it follows that we have the following

Corollary 8.20 *Good LM_n algebras can be identified with $\oplus$-proper LM_n algebras.*

Remark 8.21 For $n \in \{2,3,4\}$, LM_n algebras can be identified with good LM_n algebras and with $\oplus$-proper LM_n algebras, therefore they can be identified with MV_n algebras, as we have already seen in [11].

9 The construction of LM_3 (LM_4) algebra from the odd (respectively even)-valued LM_n algebra, $n \geq 5$.

Let $\mathcal{L}_n^{(MV_n)} = (L_n, \oplus, \cdot, ^-, 0, 1)$ be the canonical MV_n algebra $(n \geq 5)$ and $\mathcal{L}_n = (L_n, \vee, \wedge, ^-, (s_j)_{j \in J}, (s'_j)_{j \in J}, 0, 1)$ be the canonical $g.LM_n$ pre-algebra constructed by ([11], 3.9). The first result is that the determination principle is not verified in some points (i.e. $\mathcal{L}_n$ is a proper pre-algebra):

Proposition 9.1 *(i) If* $n = 2k + 1$ $(k \geq 2)$ *, then*

$$L_n = \left\{0, \frac{1}{2k}, \frac{2}{2k}, \ldots, \frac{k \Leftrightarrow 1}{2k}, \frac{\mathbf{k}}{\mathbf{2k}} = \mathbf{C}, \frac{k+1}{2k}, \ldots, \frac{2k \Leftrightarrow 1}{2k}, 1\right\}$$

and there exist $x_1 = \frac{k}{2k} = C$ *(C is the "center" point of* L_n*) and* $x_2 = \frac{k+1}{2k}, \ldots x_k = \frac{2k-1}{2k}$ *(all in the second half of* L_n*), all distinct and such that:*

$$s_j x_1 = s_j x_2 = \ldots = s_j x_k, \qquad \textit{for every } j \in J;$$

(ii) If $n = 2k$ $(k \geq 3)$ *, then*

$$L_n = \left\{0, \frac{1}{2k \Leftrightarrow 1}, \frac{2}{2k \Leftrightarrow 1}, \ldots, \frac{k \Leftrightarrow 1}{2k \Leftrightarrow 1}, \frac{k}{2k \Leftrightarrow 1}, \ldots, \frac{2k \Leftrightarrow 2}{2k \Leftrightarrow 1}, 1\right\}$$

and there exist $x_1 = \frac{k}{2k-1}$, $x_2 = \frac{k+1}{2k-1}, \ldots, x_{k-1} = \frac{2k-2}{2k-1}$ *(all in the second half of* L_n*), all distinct and such that:*

$$s_j x_1 = s_j x_2 = \ldots = s_j x_{k-1}, \qquad \textit{for every } j \in J.$$

Proof. First we prove (i) in four steps:

1. $s_1 x_1 = s_1 x_2 = \ldots = s_1 x_k = 0$; indeed, $s_1 x_i = x_i^{n-1} = 0$, by ([11], 1.14), for $i = \overline{1, k}$.

2. $s_2 x_1 = 1$; indeed, $s_2 x_1 = (2x_1)^{n-1}$ and $2x_1 = \min(1, 2x_1) = \min(1, \frac{2k}{2k}) = 1$, hence $s_2 x_1 = 1^{n-1} = 1$, by ([11], 1.14).

3. $s_2 x_2 = s_2 x_3 = \ldots = s_2 x_k = 1$; indeed, since $x_1 < x_2 < \ldots < x_k$, it follows that $s_2 x_1 \leq s_2 x_2 \leq \ldots \leq s_2 x_k$, by ([11], 3.8); we also have $s_2 x_1 = 1$.

4. $s_j x_1 = s_j x_2 = \ldots = s_j x_k = 1$, for every $j = \overline{3, n \Leftrightarrow 1}$, by 3. and by the axiom (G5) from [11]. Thus (i) holds. The proof of (ii) is similar. □

Corollary 9.2 *(i) If* $n = 2k + 1$ $(k \geq 2)$*, there exist* $y_1 = \frac{1}{2k}, y_2 = \frac{2}{2k}, \ldots, y_{k-1} = \frac{k-1}{2k}$ *(in the first half of* L_n*) and* $y_k = \frac{k}{2k} = \mathbf{C}$ *(C is the "center" of* L_n*), all distinct and such that:*

$$s'_j y_1 = s'_j y_2 = \ldots = s'_j y_k = 0, \quad \textit{for every } j = \overline{1, n \Leftrightarrow 2} \textit{ and}$$
$$s'_{n-1} y_1 = s'_{n-1} y_2 = \ldots = s'_{n-1} y_k = 1 \ ;$$

(ii) If $n = 2k$ $(k \geq 3)$*, there exist* $y_1 = \frac{1}{2k-1}, y_2 = \frac{2}{2k-1}, \ldots, y_{k-1} = \frac{k-1}{2k-1}$ *(all in the first half of* L_n*), all distinct and such that:*

$$s'_j y_1 = s'_j y_2 = \ldots = s'_j y_{k-1} = 0, \quad \textit{for every } j = \overline{1, n \Leftrightarrow 2} \textit{ and}$$
$$s'_{n-1} y_1 = s'_{n-1} y_2 = \ldots = s'_{n-1} y_{k-1} = 1 \ .$$

Proof. (i) follows by Proposition 9.1, since $y_1 = x_k^-, y_2 = x_{k-1}^-, \ldots, y_k = x_1^- = x_1$ and by ([11], (G4)). (ii) follows by Proposition 9.1, since $y_1 = x_{k-1}^-, y_2 = x_{k-2}^-, \ldots, y_{k-1} = x_1^-$ and by ([11], (G4)). □

Remarks 9.3 (i) If $n = 2k+1$ $(k \geq 2)$, we put $X = \{x_1, x_2, \ldots, x_k\}$, $Y = \{y_1, y_2, \ldots, y_k\}$; then $X, Y \subset L_n$, $X \cap Y = \{C\}$, $L_n = \{0\} \cup Y \cup X \cup \{1\}$, Y and X are chains and $y \leq x$ for every $y \in Y$ and $x \in X$.

(ii) If $n = 2k$ $(k \geq 3)$, we put $X = \{x_1, x_2, \ldots, x_{k-1}\}$, $Y = \{y_1, y_2, \ldots, y_{k-1}\}$; then $X, Y \subset L_n$, $X \cap Y = \emptyset$, $L_n = \{0\} \cup Y \cup X \cup \{1\}$, Y and X are chains and $y < x$ for every $y \in Y$ and $x \in X$.

I shall now put together all the elements of L_n for which s_j or s'_j coincide, for every $j \in J$, to obtain an algebra verifying the determination principle.

Definition 9.4 For $n = 2k$ $(k \geq 3)$, let us define the relation S on the canonical $g.LM_n$ pre-algebra $\mathcal{L}_n$ by:

$$xSy \text{ if and only if either } 1)\ s_j x = s_j y, \quad \text{for every } j \in J \quad \text{or}$$
$$2)\ s'_j x = s'_j y, \quad \text{for every } j \in J.$$

Remark that if $x, y \neq 0, 1$ in the above definition, then 1) means that $x, y \in X$ and 2) means that $x, y \in Y$, by Proposition 9.1, Corollary 9.2 and Remarks 9.3.

Proposition 9.5 *The relation S is an equivalence relation on L_n, which verifies, for every $x, y, u, v \in L_n$, $j \in J$, the property: if xSy and uSv, then*
a) $x^- S y^-$,
b) $(x \vee u) S (y \vee v)$,
c) $(x \wedge u) S (y \wedge v)$,
d) one of the following holds

$$(i)\ (s_j x) S (s_j y), \quad \textit{for every } j \in J \quad \textit{or}$$
$$(ii)\ (s'_j x) S (s'_j y), \quad \textit{for every } j \in J.$$

Proof. The reflexivity and the symmetry are immediate. To prove the transitivity, suppose xSy and ySz. By Proposition 9.1, Corollary 9.2 and Remarks 9.3, the element y cannot be in the same time in X and in Y, so there are only two posibilities: either $s_j x = s_j y$, $j \in J$ and $s_j y = s_j z$, $j \in J$, hence xSz, or $s'_j x = s'_j y$, $j \in J$ and $s'_j y = s'_j z$, $j \in J$, hence xSz again. Thus S is an equivalence relation. To prove now a), let $x, y \in L_n$ such that xSy. If 1) holds, then $(s_j x)^- = (s_j y)^-$, $j \in J \iff s'_{n-j}(x^-) = s'_{n-j}(y^-)$, $j \in J$, i.e. $x^- S y^-$. If 2) holds, the proof is similar. Thus a) holds. To prove b), let xSy and uSv. There are four cases: (I) $s_j x = s_j y$, $j \in J$ and $s_j u = s_j v$, $j \in J$, which mean, by Proposition 9.1 and Remarks 9.3, that $x, y, u, v \in X$. Then $s_j(x \vee u) = s_j x \vee s_j u = s_j y \vee s_j v = s_j(y \vee v)$, for every $j \in J$, by ([11], (G1)); hence $(x \vee u) S (y \vee v)$. (II) $s_j x = s_j y$, $j \in J$ and $s'_j u = s'_j v$, $j \in J$, which mean, by Proposition 9.1, Corollary 9.2 and Remarks 9.3, that $x, y \in X$ and $u, v \in Y$. Then $u, v < x, y$ and hence $x \vee u = x$ and $y \vee v = y$. Then $s_j(x \vee u) = s_j x = s_j y = s_j(y \vee v)$, $j \in J$, hence $(x \vee u) S (y \vee v)$. (III)

$s'_j x = s'_j y$, $j \in J$ and $s'_j u = s'_j v$, $j \in J$, which mean, by Corollary 9.2 and Remarks 9.3, that $x, y, u, v \in Y$. Then $s'_j(x \vee u) = s'_j x \vee s'_j u = s'_j y \vee s'_j v = s'_j(y \vee v)$, hence $(x \vee u)S(y \vee v)$. (IV) $s'_j x = s'_j y$, $j \in J$ and $s_j u = s_j v$, $j \in J$, which mean that $x, y \in Y$ and $u, v \in X$. Hence $x, y < u, v$ and then $x \vee u = u$, $y \vee v = v$. It follows $(x \vee u)S(y \vee v)$ and thus b) holds. The proof of c) is similar. Finally, to prove d), if xSy means 1) and $k \in J$, then $s_j(s_k x) = s_k x = s_k y = s_j(s_k y)$, by ([11], (G9)); hence $(s_k x)S(s_k y)$ for every $k \in J$. If xSy means 2) and $k \in J$, then $s_j(s'_k x) = s'_k x = s'_k y = s_j(s'_k y)$, by ([11], (G10)), hence $(s'_k x)S(s'_k y)$, for every $k \in J$. Thus d) holds. □

Theorem 9.6 *If $n = 2k$ $(k \geq 3)$, then the structure:*

$$(L_n/S\,,\vee\,,\wedge\,,^-\,,R_1\,,R_2\,,R_3\,,\hat{0}\,,\hat{1})$$

is an LM_4 algebra, isomorphic to the canonical LM_4 algebra, where $L_n/S = \{\hat{0} < \widehat{y_1} < \widehat{x_1} < \hat{1}\}$, with $\widehat{y_1} = Y$, $\widehat{x_1} = X$, $\hat{0} = \{0\}$, $\hat{1} = \{1\}$, $\hat{x} \vee \hat{y} = \widehat{x \vee y}$, $\hat{x} \wedge \hat{y} = \widehat{x \wedge y}$, $(\hat{x})^- = \widehat{(x^-)}$ and R_1, R_2, R_3 are defined by the table:

$\hat{x}$	$\hat{0}$	$\widehat{y_1}$	$\widehat{x_1}$	$\hat{1}$
R_1	$\widehat{s'_1 0} = \hat{0}$	$\widehat{s'_1 y_1} = \hat{0}$	$\widehat{s_1 x_1} = \hat{0}$	$\widehat{s_1 1} = \hat{1}$
R_2	$\widehat{s'_{n-2} 0} = \hat{0}$	$\widehat{s'_{n-2} y_1} = \hat{0}$	$\widehat{s_2 x_1} = \hat{1}$	$\widehat{s_2 1} = \hat{1}$
R_3	$\widehat{s'_{n-1} 0} = \hat{0}$	$\widehat{s'_{n-1} y_1} = \hat{1}$	$\widehat{s_{n-1} x_1} = \hat{1}$	$\widehat{s_{n-1} 1} = \hat{1}$

Proof. Obvious, by Remarks 9.3 and Proposition 9.5 (see also ([12], Figure 1)). □

Definition 9.7 For $n = 2k + 1$ $(k \geq 2)$, let us define the relation H on the canonical $g.LM_n$ pre-algebra $\mathcal{L}_n$ by:

xHy if and only if either 1) $s_j x = s_j y$, for every $j \in J$ or
2) $s'_j x = s'_j y$, for every $j \in J$ or
3) $s_j x = (s'_{n-j} y)^-$, for every $j \in J$ or
4) $s'_j x = (s_{n-j} y)^-$, for every $j \in J$.

Remark that if $x, y \neq 0, 1$ in the above definition, then 1) means that $x, y \in X$, 2) means that $x, y \in Y$, 3) means that $x \in X, y \in Y$ and 4) means that $x \in Y, y \in X$, by Proposition 9.1, Corollary 9.2 and Remarks 9.3.

Proposition 9.8 *The relation H is an equivalence relation on L_n, which verifies, for every $x, y, u, v \in L_n$, $j \in J$, the property: if xHy and uHv, then*
a) $x^- H y^-$,
b) $(x \vee u)H(y \vee v)$,

c) $(x \wedge u)H(y \wedge v)$,
d') one of the following holds

$$\begin{array}{ll} (i)\ (s_j x)H(s_j y), & \textit{for every } j \in J \quad \textit{or} \\ (ii)\ (s'_j x)H(s'_j y), & \textit{for every } j \in J \quad \textit{or} \\ (iii)\ (s_j x)H(s'_{n-j} y)^-, & \textit{for every } j \in J \quad \textit{or} \\ (iv)\ (s'_j x)H(s_{n-j} y)^-, & \textit{for every } j \in J. \end{array}$$

Proof. The reflexivity is immediate. Let xHy. If 1) or 2) holds, then yHx; if 3) holds, then $(s_j x)^- = s'_{n-j} y$, for every $j \in J$, hence $s'_i y = (s_{n-i} x)^-$, for every $i \in J$, i.e. yHx; if 4) holds, the proof is similar. Thus H is symmetric. To prove the transitivity, suppose xHy and yHz. If $x, y, z \in X$ or if $x, y, z \in Y$, then it is obvious that xHz. If $x, y \in X$ and $z \in Y$, i.e. $s_j x = s_j y$, $j \in J$ and $s_j y = (s'_{n-j} z)^-$, for every $j \in J$,then $s_j x = (s'_{n-j} z)^-$, for every $j \in J$, hence xHz. If $x \in X$ and $y, z \in Y$, i.e. $s_j x = (s'_{n-j} y)^-$, for every $j \in J$ and $s'_j y = s'_j z$, for every $j \in J$, then $s_j x = (s'_{n-j} z)^-$, for every $j \in J$, i.e. xHz again. The proof is similar for the other cases. Thus H is an equivalence relation. To prove now a) we use ([11], 3.4(iii)). To prove b), let xHy and uHv. There are eight cases: (I) $x, y, u, v \in Y$, (II) $x, y, u, v \in X$, (III) $u, v \in Y$, $x, y \in X$, (IV) $x, y \in Y$, $u, v \in X$, (V) $x, u \in Y$, $y, v \in X$, (VI) $y, u \in Y$, $x, v \in X$, (VII) $x, v \in Y$, $y, u \in X$, and (VIII) $y, v \in Y$, $x, u \in X$. If, for instance, we are in the case (V), i.e. $s'_j x = (s_{n-j} y)^-$ and $s'_j u = (s_{n-j} v)^-$, then $x \vee u \in Y$ and $y \vee v = \max(y, v) \in X$, $y \wedge v = \min(y, v) \in X$, hence $s'_j(x \vee u) = s'_j x \vee s'_j u = (s_{n-j} y)^- \vee (s_{n-j} v)^- = (s_{n-j} y \wedge s_{n-j} v)^- = (s_{n-j}(y \wedge v))^- = (s_{n-j}(y \vee v))^-$, for every $j \in J$, hence $(x \vee u)H(y \vee v)$. The proof is similar for the other cases. Thus b) holds. The proof for c) is similar. To prove d'), suppose that xHy means 3) for instance and let $k \in J$. Then $s_j(s_k x) = s_k x = (s'_{n-k} y)^- = (s'_{n-j}(s'_{n-k} y))^-$, for every $j \in J$, i.e. $(s_k x)H(s'_{n-k} y)$, by ([11], (G9), (G10), 3.10). The proof is similar in the cases (1), (2), (4). □

Theorem 9.9 *If* $n = 2k + 1$ $(k \geq 2)$, *then the structure:*

$$(L_n/H\,, \vee\,, \wedge\,, ^-\,, R_1\,, R_2\,, \hat{0}\,, \hat{1})$$

is an LM_3 *algebra, isomorphic to the canonical* LM_3 *algebra,*
where $L_n/H = \left\{\hat{0} < \hat{C} < \hat{1}\right\}$, *with* $\hat{C} = Y \cup X$, $Y \cap X = \{C\}$, $\hat{0} = \{0\}$, $\hat{1} = \{1\}$, $\hat{x} \vee \hat{y} = \widehat{x \vee y}$, $\hat{x} \wedge \hat{y} = \widehat{x \wedge y}$, $(\hat{x})^- = \widehat{(x^-)}$ *and* R_1, R_2 *are defined by the table:*

$\hat{x}$	$\hat{0}$	$\hat{C}$	$\hat{1}$
R_1	$\widehat{s'_1 0} = \hat{0}$	$\widehat{s'_1 C} = \hat{0}$	$\widehat{s_1 1} = \hat{1}$
R_2	$\widehat{s'_{n-1} 0} = \hat{0}$	$\widehat{s'_{n-1} C} = \hat{1}$	$\widehat{s_{n-1} 1} = \hat{1}$

Proof. Obvious, by Remarks 9.3 and Proposition 9.8 (see also [12], Figure 2). □

We remark now that the relations S and H can be embedded in more simply relations, with the same results:

Proposition 9.10 *Let $\mathcal{L}_n$ be the canonical g.LM_n pre-algebra.*
(i) If $n = 2k$ $(k \geq 3)$, let us define the relation S' for every $x, y \in L_n$ by:

$$xS'y \quad \textit{if and only if} \quad (s_1x = s_1y,\ s_2x = s_2y \textit{ and } s_{n-1}x = s_{n-1}y).$$

Then the following hold:
a) $S \subset S'$;
b) S' is a congruence relation of the LM_4 pre-algebra

$$(L_n\,, \vee, \wedge, ^-, s_1\,, s_2\,, s_{n-1}\,, 0, 1);$$

c) The structure $(L_n/S'\,, \vee, \wedge, ^-, R_1\,, R_2\,, R_3\,, \hat{0}, \hat{1})$ is an LM_4 algebra, isomorphic to the canonical LM_4 algebra, where $R_1\hat{x} = \widehat{s_1x}$, $R_2\hat{x} = \widehat{s_2x}$, $R_3\hat{x} = \widehat{s_{n-1}x}$, .

(i') If $n = 2k+1$ $(k \geq 2)$, let us define the relation H' for every $x, y \in L_n$ by:

$$xH'y \quad \textit{if and only if} \quad (s_1x = s_1y \textit{ and } s_{n-1}x = s_{n-1}y).$$

Then the following hold:
a') $H \subset H'$;
b') H' is a congruence relation of the LM_3 pre-algebra

$$(L_n\,, \vee, \wedge, ^-, s_1\,, s_{n-1}, 0, 1);$$

c') The structure $(L_n/H'\,, \vee, \wedge, ^-, R_1\,, R_2\,, \hat{0}, \hat{1})$ is an LM_3 algebra, isomorphic to the canonical LM_3 algebra, where $R_1\hat{x} = \widehat{s_1x}$, $R_2\hat{x} = \widehat{s_{n-1}x}$, .

Proof. Routine. □

I shall generalize now the two constructions from Proposition 9.10 to arbitrary even, respectively odd - valued LM_n algebras, by ([12], 5.13).

Proposition 9.11 *If $n = 2k$ $(k \geq 3)$, let $(A\,, \vee, \wedge, ^-, (r_j)_{j\in J}\,, 0, 1)$ be an arbitrary LM_n algebra. Let us define the relation S'' on A by (see Proposition 9.10(i) and ([12], 5.10(ii), 5.13)):*

$$x\,S''\,y \quad \textit{if and only if} \quad (r_1x = r_1y,\ r_kx = r_ky \textit{ and } r_{n-1}x = r_{n-1}y).$$

Then S'' is a congruence relation of the LM_4 pre-algebra

$$(A\,, \vee, \wedge, ^-, r_1\,, r_k\,, r_{n-1}\,, 0, 1).$$

Proof. Routine. □

Theorem 9.12 *If $n = 2k$ $(k \geq 3)$, then the structure:*

$$(\,A/S''\,,\ \vee\,,\ \wedge\,,\ ^-\,,\ R_1\,,\ R_2\,,\ R_3\,,\ \hat{0}\,,\ \hat{1}\,)$$

is an LM_4 algebra, where $R_1\hat{x} = \widehat{r_1x}\,$, $R_2\hat{x} = \widehat{r_kx}\,$, $R_3\hat{x} = \widehat{r_{n-1}x}$.

Proof. To prove that $(A/S'', \vee, \wedge)$ is a distributive lattice we need to prove, by [22], that $\hat{x} \wedge (\hat{x} \vee \hat{y}) = \hat{x}$ and $\hat{x} \wedge (\hat{y} \vee \hat{z}) = (\hat{z} \wedge \hat{x}) \vee (\hat{y} \wedge \hat{x})$, which is simply routine. It is routine also to prove that $(A/S'', \vee, \wedge, ^-, \hat{1})$ is a De Morgan algebra and that the axioms (L1)-(L5) from [11] are verified. We verify now the axiom (L6) from [11]:

$$\begin{aligned}
& R_j\hat{x} = R_j\hat{y}, \text{ for } j = \overline{1,3} \\
& \Longleftrightarrow ((r_1x)S''(r_1y),\ (r_kx)S''(r_ky) \text{ and } (r_{n-1}x)S''(r_{n-1}y)) \\
& \Longrightarrow (r_1x = r_1y,\ r_kx = r_ky \text{ and } r_{n-1}x = r_{n-1}y) \\
& \Longleftrightarrow xS''y \Longleftrightarrow \hat{x} = \hat{y}.
\end{aligned}$$

□

Proposition 9.13 *If $n = 2k + 1$ $(k \geq 2)$, let $(A, \vee, \wedge, ^-, (r_j)_{j \in J}, 0, 1)$ be an arbitrary LM_n algebra. Let us define the relation H'' on A by (see Proposition 9.10(i'), ([12], 5.10(ii'), 5.13)):*

$$x\,H''\,y \quad \text{if and only if} \quad (r_1x = r_1y \text{ and } r_{n-1}x = r_{n-1}y).$$

Then H'' is a congruence relation of the LM_3 pre-algebra

$$(A, \vee, \wedge, ^-, r_1, r_{n-1}, 0, 1).$$

Proof. Routine. □

Theorem 9.14 *If $n = 2k + 1$ $(k \geq 2)$, then the structure:*

$$(\,A/H'', \vee, \wedge, ^-, R_1, R_2, \hat{0}, \hat{1}\,)$$

is an LM_3 algebra, where $R_1\hat{x} = \widehat{r_1x}$, $R_2\hat{x} = \widehat{r_{n-1}x}$.

Proof. Routine. □

Remarks 9.15 1.) If $n = 2k + 1$ $(k \geq 2)$, let $\mathcal{A}$ be an LM_n algebra. We can generalize Proposition 9.13 and Theorem 9.14 (and also Proposition 9.10(i')) for the relations H_j'', $j = \overline{1,k}$, where for any $x, y \in A$:

$$x\,H_j''\,y \quad \text{if and only if} \quad (r_jx = r_jy \text{ and } r_{n-j}x = r_{n-j}y).$$

H_j'' is a congruence relation of the LM_3 pre-algebra $(A, \vee, \wedge, r_j, r_{n-j}, 0, 1)$ and $H_1'' = H''$. See [2], pg.349.

2.) In [2], pg.349, the relations H'' and H_j'', $j = 1, 2, \ldots, [\frac{n}{2}]$, are defined for any LM_n algebra and any n, **odd** or **even**, which is possible indeed; but it is now clear why the adequate case when the relations H'' and H_j'' must be considered is the case: n be an **odd** number!

3.) All this study have proved that LM_4 algebras are as much important as LM_3 algebras and MV_n algebras have helped us to see that.

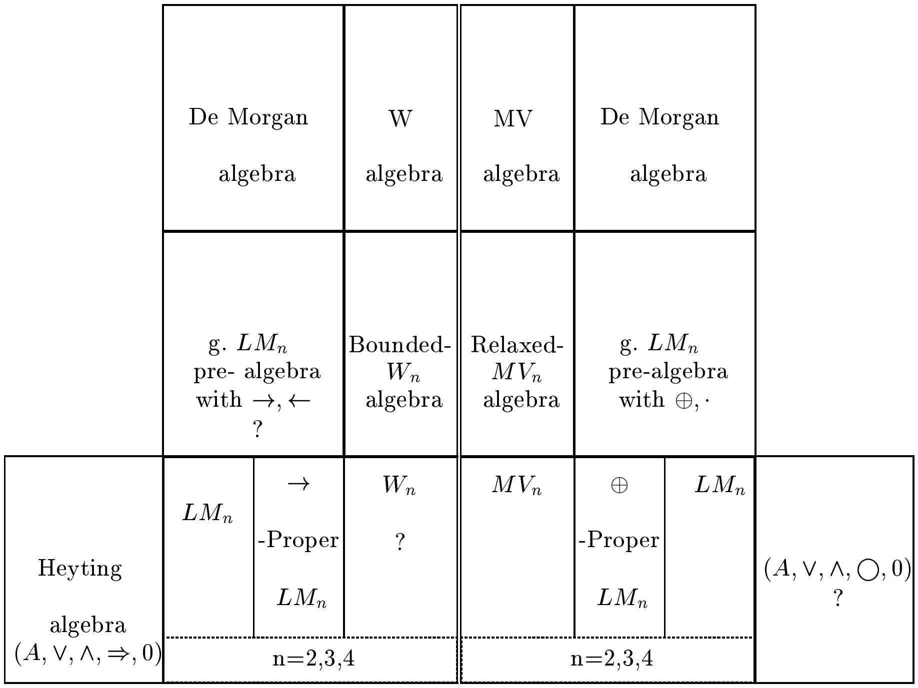

Figure 1:

10 Final remarks and open problems

(i) In the canonical LM_2 algebra (the canonical MV_2 algebra, the Boolean algebra) $\mathcal{L}_2$, with $L_2 = \{0, 1\}$, both operations $\oplus$ and $\bigcirc$ coincide with the operation $\vee$.

(ii) If we consider the set $\mathcal{O}^{(l)} = \{\vee, \wedge, \rightarrow, ^-, 0, 1\}$ of logical operators, then there exist some basis of it, as for example: the canonical base $\mathcal{B}_1 = \{\vee, \wedge, ^-, 0, 1\}$, $\mathcal{B}_2 = \{\vee, \wedge, ^-\}$, $\mathcal{B}_3 = \{\vee, ^-\}$, $\mathcal{B}_4 = \{\wedge, ^-\}$, $\mathcal{B}_5 = \{\vee, ^-, 0\}$, $\mathcal{B}_6 = \{\wedge, ^-, 1\}$, $\mathcal{B}_7 = \{\rightarrow, ^-\}$, $\mathcal{B}_8 = \{\rightarrow, ^-, 1\}$, $\mathcal{B}_9 = \{\rightarrow, 0\}$. The Boolean algebra is usually defined by using the canonical base, $\mathcal{B}_1$. The De Morgan algebra is the structure which generalizes the Boolean algebra (i.e. uses the same base). If we consider the set $\mathcal{O}^{(a)} = \{\vee, \wedge, \oplus, \cdot, ^-, 0, 1\}$ of operators, then we can say, analogously, that there are different basis of it. The MV algebra was defined by Chang [3] as a structure $(A, \oplus, \cdot, ^-, 0, 1)$, i.e. by using the base $\{\oplus, \cdot, ^-, 0, 1\}$, and it was defined equivalently, in [6], as a structure $(A, \oplus, ^-, 0)$, i.e. by using the base $\{\oplus, ^-, 0\}$ of operators. It is proved in [21] that the MV algebra is isomorphic to the Wajsberg algebra (W algebra, for short), which is a structure $(A, \to, ^-, 1)$, i.e. defined by using the base $\{\to, ^-, 1\}$ of operators.

(iii) Our relaxed-MV_n algebras [11] are isomorphic to the n-bounded W algebras (bounded-W_n algebra, for short). One open problem is to define W_n algebras (a bounded-W_n algebra with an axiom corresponding to the axiom (M13) from [11]) (see ([12] 5.26)) and to establish the connection with $\to$-Proper LM_n algebras.

(iv) Let us define in a W-algebra the operation $\leftarrow$ by q setting:

$$x \leftarrow y = (x^- \to y^-)^-.$$

Thus $x \leftarrow y = x^- \cdot y$ and $x \cdot y = x^- \leftarrow y$. Then another open problem is to define g.LM_n algebras with $\to$, $\leftarrow$.

(v) A general view of all mentioned structures and related structures is given in the table presented in the Figure 1, where " ? " means that the structure must be defined and studied. The table has two sides, the left one and the right one. One side is the image in a kind of a "mirror" of the other side. The left side contains the structures related to the operation $\to$, while the right side contains the structures related to the operation $\oplus$; the left side is related to the logic, while the right side is related to the algebra.

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