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A Canonical Model Construction for Substructural Logics[1]

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Abstract: In this paper, we introduce a class of substructural logics, called *normal substructural logics*, which includes not only relevant logic, BCK logic, linear logic and the Lambek calculus but also weak logics with strict implication, and define Kripke-style semantics (Kripke frames and models) for normal substructural logics. Then we show a correspondence between axioms and properties on frames, and give a canonical construction of Kripke models for normal substructural logics.

Key Words: Substructural logics, linear logic, relevant logics, strict implication, Kripke-type semantics, canonical model.
Category: F.4.1

1 Introduction

Substructural logics [10] are logics with restricted structural rules in their sequent formulation. They include relevant logic taking its rise in philosophical problems on implication, BCK logic in set-theoretical paradoxes, linear logic in computer science, and the Lambek calculus of syntactic categories in linguistics (see [4]). Various kinds of semantics for substructural logics have been introduced: for examples, Kripke-style models with a ternary relation for relevant logic [9, 5], phase structures for linear logic [6], Kripke-style models using SO-monoids for BCK-logic [8], and algebraic models using FL-algebras for the Lambek calculus [7]. Došen [2, 3] introduced a very week substructural logic by the Hilbert-style system L and its extenstions, and gave a Kripke-style model (groupoid frames and models) and a canonical model construction for the logics.

In this paper, we introduce a class of substructural logics, called *normal substructural logics*, and show that they include not only the logics mentioned above but also weak logics with strict implication [1]. There are many normal substructural logics which are not extensions of Došen's logic. Then we introduce Kripke-style semantics (Kripke frames and models) for normal substructural logics in the spirit of [7], and show a correspondence between axioms and properties on frames. Finally, we give a canonical construction of Kripke models for normal substructural logics. The construction differs from one in [3].

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2 Normal substructural logics

We assume that our language $\mathcal{L}$ consists of the propositional variables $p, q, r, \ldots$, the logical constants 1, $\top$ and $\bot$, and the logical connectives $\vee$, $\wedge$, $*$, and $\supset$.

Definition 1. A *normal substructural logic* L is a set of formulas containing the following axioms:

$A1.\ 1,$	$A2.\ \ p \supset p,$
$A3.\ p \supset \top,$	$A4.\ \ \bot \supset p,$
$A5.\ p \wedge q \supset p,$	$A6.\ \ p \wedge q \supset q,$
$A7.\ p \supset p \vee q,$	$A8.\ \ q \supset p \vee q,$
$A9.\ (p \supset q) \wedge (p \supset r) \supset p \supset q \wedge r,$	$A10.\ p \supset q \supset p * q,$

and closed under the following rules:

$$\frac{\alpha \supset \beta \quad \alpha}{\beta}\ (\text{modus ponens}), \qquad \frac{\alpha}{[\beta_1/p_1, \ldots, \beta_n/p_n]\alpha}\ (\text{substitution}),$$

$$\frac{\beta \supset \gamma}{(\alpha \supset \beta) \supset \alpha \supset \gamma}\ (\text{prefixing}), \qquad \frac{\alpha \supset \beta}{(\beta \supset \gamma) \supset \alpha \supset \gamma}\ (\text{suffixing}),$$

$$\frac{\alpha \quad \beta}{\alpha \wedge \beta}\ (\text{adjunction}), \qquad \frac{\alpha}{1 \supset \alpha}\ (\text{necessitation}),$$

$$\frac{\alpha \supset \gamma \quad \beta \supset \gamma}{\alpha \vee \beta \supset \gamma}\ (\vee\text{-elimination}), \qquad \frac{\alpha \supset \beta \supset \gamma}{\alpha * \beta \supset \gamma}\ (\text{residuation}).$$

Remark. There are many normal substructural logics which are not extensions of Došen's logic, since

$$(p \supset r) \wedge (q \supset r) \supset p \vee q \supset r$$

is a theorem of the system L (see [2, (56)]) but not of the minimal normal substructural logic. On the other hand, since Došen's logic is not closed under modus ponens, it is not a normal substructural logic.

Proposition 2 (Došen's E^+). *The extension E^+ of Došen's logic is regarded as the $\{\vee, \wedge, *, \supset, 1\}$-fragment of the normal substructural logic DE^+ with the axiom:*

$$B1.\ (p \supset r) \wedge (q \supset r) \supset p \vee q \supset r.$$

Proof. Note that Došen used $\top$ for 1 in [2, 3]. The axioms and rules (1)-(5), (7)-(12), (38), (40) and (50) are easily shown from the axioms and rules of DE^+.

For the rule (6), we see

$$\dfrac{\dfrac{\alpha\supset\alpha_1 \qquad \dfrac{\dfrac{\alpha_1\supset\beta_1\supset\alpha_1*\beta_1 \qquad \dfrac{\dfrac{\beta\supset\beta_1}{(\beta_1\supset\alpha_1*\beta_1)\supset\beta\supset\alpha_1*\beta_1}\,(\text{suff.})}{(\alpha_1\supset\beta_1\supset\alpha_1*\beta_1)\supset\alpha_1\supset\beta\supset\alpha_1*\beta_1}\,(\text{pref.})}{\alpha_1\supset\beta\supset\alpha_1*\beta_1}\,(\text{m.p.})}{(\alpha\supset\alpha_1)\supset\alpha\supset\beta\supset\alpha_1*\beta_1}\,(\text{pref.})}{\alpha\supset\beta\supset\alpha_1*\beta_1}\,(\text{m.p.})}{\alpha*\beta\supset\alpha_1*\beta_1}\,(\text{residuation}) \quad .$$

Conversely, it is straightforward to see that the axioms and the rules for $\vee$, $\wedge$, $*$, $\supset$, 1 of DE^+ are derivable in the system E^+ in [2]. □

Proposition 3 (Relevant logic). *The positive fragment B^+ of the basic relevant logic is a normal substructural logic with the axiom for DE^+ and the axiom:*

$$B2.\ p\wedge(q\vee r)\supset(p\wedge q)\vee(p\wedge r),$$

and the following its extensions are normal substructural logics: T^+ with the axioms for B^+ and the axioms:

$B3.\ (q\supset r)\supset(p\supset q)\supset p\supset r,$ $\qquad$ $B4.\ (p\supset q)\supset(q\supset r)\supset p\supset r,$
$B5.\ (p\supset p\supset q)\supset p\supset q;$

E^+ with the axioms for T^+ and the axiom

$$B6.\ (1\supset p)\supset p;$$

$S4^+$ with the axioms for E^+ and the axiom

$$B7.\ p\supset 1;$$

R^+ with the axioms for E^+ and the axiom

$$B8.\ p\supset(p\supset q)\supset q.$$

Proof. See [5, 9]. □

Definition 4. Let Γ be a finite sequence of formulas, and let θ be a formula. Then the formula $\Gamma\supset\theta$ is inductively defined by

1. $\langle\rangle\supset\alpha := \alpha$,
2. $\gamma,\Gamma\supset\alpha := \gamma\supset\Gamma\supset\alpha$.

Here $\langle\rangle$ denotes the empty sequence.

Lemma 5. *Let L be a normal substructural logic. Then*

1. *L is closed under the rule:*

$$\frac{\alpha \supset \beta \quad \beta \supset \gamma}{\alpha \supset \gamma}\ (cut),$$

2. *L is closed under the rule:*

$$\frac{\alpha \supset \beta}{(\Gamma \supset \alpha) \supset \Gamma \supset \beta}\ (prefixing^*),$$

3. *$\top \supset \Gamma \supset \top \in L$,*
4. *$(\Gamma \supset \alpha) \supset \Gamma \supset \alpha \vee \beta, (\Gamma \supset \beta) \supset \Gamma \supset \alpha \vee \beta \in L$,*
5. *$(\Gamma \supset \alpha \supset \Delta \supset \theta) \supset \Gamma \supset \alpha \wedge \beta \supset \Delta \supset \theta, (\Gamma \supset \beta \supset \Delta \supset \theta) \supset \Gamma \supset \alpha \wedge \beta \supset \Delta \supset \theta \in L$,*
6. *$(\Gamma \supset \alpha) \wedge (\Gamma \supset \beta) \supset \Gamma \supset \alpha \wedge \beta \in L$,*
7. *if $B1 \in L$, then $(\Gamma \supset \alpha \supset \Delta \supset \theta) \wedge (\Gamma \supset \beta \supset \Delta \supset \theta) \supset \Gamma \supset \alpha \vee \beta \supset \Delta \supset \theta \in L$,*
8. *if $B3 \in L$, then $(\alpha \supset \beta) \supset (\Gamma \supset \alpha) \supset \Gamma \supset \beta \in L$,*
9. *if $B3 \in L$, then L is closed under the rule:*

$$\frac{\Gamma \supset \alpha}{(\alpha \supset \beta) \supset \Gamma \supset \beta}\ (suffixing^+)$$

   where Γ must be nonempty,

10. *if $B3, B6 \in L$, then L is closed under the rule:*

$$\frac{\Gamma \supset \alpha}{(\alpha \supset \beta) \supset \Gamma \supset \beta}\ (suffixing^*).$$

Proof. (1). Straightforward.

(2). By induction on the length of Γ. For induction step,

$$\cfrac{\cfrac{\alpha \supset \beta}{(\Gamma \supset \alpha) \supset \Gamma \supset \beta}\ \text{(induction hypothesis)}}{(\gamma \supset \Gamma \supset \alpha) \supset \gamma \supset \Gamma \supset \beta}\ \text{(pref.)}\quad .$$

(3). By induction on the length of Γ. For induction step,

$$\cfrac{\cfrac{\top \supset \gamma \supset \top * \gamma \quad \cfrac{\top * \gamma \supset \top}{(\gamma \supset \top * \gamma) \supset \gamma \supset \top}\ \text{(pref.)}}{\top \supset \gamma \supset \top}\ \text{(cut)} \quad \cfrac{\top \supset \Gamma \supset \top}{(\gamma \supset \top) \supset \gamma \supset \Gamma \supset \top}\ \text{(pref.)}}{\top \supset \gamma \supset \Gamma \supset \top}\ \text{(cut)}\quad .$$

(4).

$$\dfrac{\alpha \supset \alpha \vee \beta}{(\Gamma \supset \alpha) \supset \Gamma \supset \alpha \vee \beta}\ (\text{pref.}^*).$$

(5).

$$\dfrac{\dfrac{\alpha \wedge \beta \supset \alpha}{(\alpha \supset \Delta \supset \theta) \supset \alpha \wedge \beta \supset \Delta \supset \theta}\ (\text{suff.})}{(\Gamma \supset \alpha \supset \Delta \supset \theta) \supset \Gamma \supset \alpha \wedge \beta \supset \Delta \supset \theta}\ (\text{pref.}^*).$$

(6). By induction on the length of Γ. For induction step, letting $\Gamma' := \gamma, \Gamma$ and $\theta := (\Gamma \supset \alpha) \wedge (\Gamma \supset \beta)$, we see

$$\dfrac{(\Gamma' \supset \alpha) \wedge (\Gamma' \supset \beta) \supset \gamma \supset \theta \quad \dfrac{\theta \supset \Gamma \supset \alpha \wedge \beta}{(\gamma \supset \theta) \supset \Gamma' \supset \alpha \wedge \beta}\ (\text{pref.})}{(\Gamma' \supset \alpha) \wedge (\Gamma' \supset \beta) \supset \Gamma' \supset \alpha \wedge \beta}\ (\text{cut}).$$

(7). Similar to (6).

(8). By induction on the length of Γ. For induction step,

$$\dfrac{(\alpha \supset \beta) \supset (\Gamma \supset \alpha) \supset \Gamma \supset \beta \quad ((\Gamma \supset \alpha) \supset \Gamma \supset \beta) \supset (\gamma \supset \Gamma \supset \alpha) \supset \gamma \supset \Gamma \supset \beta}{(\alpha \supset \beta) \supset (\gamma \supset \Gamma \supset \alpha) \supset \gamma \supset \Gamma \supset \beta}\ (\text{cut}).$$

(9). By induction on the length of Γ. For induction step,

$$\dfrac{(\alpha \supset \beta) \supset (\Gamma \supset \alpha) \supset \Gamma \supset \beta \quad \dfrac{\gamma \supset \Gamma \supset \alpha}{((\Gamma \supset \alpha) \supset \Gamma \supset \beta) \supset \gamma \supset \Gamma \supset \beta}\ (\text{suff.})}{(\alpha \supset \beta) \supset \gamma \supset \Gamma \supset \beta}\ (\text{cut}).$$

(10).

$$\dfrac{\dfrac{\dfrac{\Gamma \supset \alpha}{1 \supset \Gamma \supset \alpha}\ (\text{necessitation})}{(\alpha \supset \beta) \supset 1 \supset \Gamma \supset \beta}\ (\text{suff.}^+) \quad (1 \supset \Gamma \supset \beta) \supset \Gamma \supset \beta}{(\alpha \supset \beta) \supset \Gamma \supset \beta}\ (\text{cut}).$$

□

Proposition 6 (The Lambek calculus, linear logic and BCK logic). *The Lambek calculus FL is a normal substructural logic with the axioms $B1$, $B3$, $B6$ and the axioms:*

$B9.\ p \supset \bot \supset q,$ $\qquad B10.\ p \supset 1 \supset p,$

$B11.\ (p \supset q \supset r) \supset p * q \supset r;$

intuitionistic linear logic ILL is a normal substructural logic with the axioms for FL and the axiom:

$$B12.\ (p \supset q \supset r) \supset q \supset p \supset r;$$

BCK logic is a normal substructural logic with the axioms for ILL and the axiom:

$$B13.\ p \supset q \supset p.$$

Proof. We will show that a sequent $\Gamma \to \theta$ is provable in the Gentzen-type sequent calculus in [7] if and only if $\Gamma \supset \theta \in FL$. It is straightforward to see that the axioms and the rules in Definition 1, and the axioms $B1$, $B3$, $B6$, $B9$, $B10$, and $B11$ are derivable in the sequent calculus. Conversely, for initial sequents of the forms $\Gamma \to \top$ and $\Gamma, \bot, \Delta \to \theta$, we see

$$\dfrac{\top \supset \Gamma \supset \top \qquad \dfrac{1 \supset \top \quad 1}{\top}\,(\text{m.p.})}{\Gamma \supset \top}\,(\text{m.p.}),$$

$$\dfrac{\dfrac{\top \supset \bot \supset \Delta \supset \theta}{(\Gamma \supset \top) \supset \Gamma \supset \bot \supset \Delta \supset \theta}\,(\text{pref.}^*) \qquad \Gamma \supset \top}{\Gamma \supset \bot \supset \Delta \supset \theta}\,(\text{m.p.}),$$

and for rules (cut), $(1w)$, $(*\to)$, $(\to *)$ and $(\supset\to)$, we see

$$\dfrac{\dfrac{\dfrac{\Gamma \supset \alpha}{(\alpha \supset \Sigma \supset \theta) \supset \Gamma \supset \Sigma \supset \theta}\,(\text{suff.}^*)}{(\Delta \supset \alpha \supset \Sigma \supset \theta) \supset \Delta \supset \Gamma \supset \Sigma \supset \theta}\,(\text{pref.}^*) \qquad \Delta \supset \alpha \supset \Sigma \supset \theta}{\Delta \supset \Gamma \supset \Sigma \supset \theta}\,(\text{m.p.}),$$

$$\dfrac{\dfrac{(\Delta \supset \theta) \supset 1 \supset \Delta \supset \theta}{(\Gamma \supset \Delta \supset \theta) \supset \Gamma \supset 1 \supset \Delta \supset \theta}\,(\text{pref.}^*) \qquad \Gamma \supset \Delta \supset \theta}{\Gamma \supset 1 \supset \Delta \supset \theta}\,(\text{m.p.}),$$

$$\dfrac{\dfrac{(\alpha \supset \beta \supset \Delta \supset \theta) \supset \alpha * \beta \supset \Delta \supset \theta}{(\Gamma \supset \alpha \supset \beta \supset \Delta \supset \theta) \supset \Gamma \supset \alpha * \beta \supset \Delta \supset \theta}\,(\text{pref.}^*) \qquad \Gamma \supset \alpha \supset \beta \supset \Delta \supset \theta}{\Gamma \supset \alpha * \beta \supset \Delta \supset \theta}\,(\text{m.p.}),$$

$$\dfrac{\dfrac{\dfrac{\alpha \supset \beta \supset \alpha * \beta \qquad \dfrac{\Delta \supset \beta}{(\beta \supset \alpha * \beta) \supset \Delta \supset \alpha * \beta}\,(\text{suff.}^*)}{\alpha \supset \Delta \supset \alpha * \beta}\,(\text{cut})}{(\Gamma \supset \alpha) \supset \Gamma \supset \Delta \supset \alpha * \beta}\,(\text{pref.}^*) \qquad \Gamma \supset \alpha}{\Gamma \supset \Delta \supset \alpha * \beta}\,(\text{m.p.}),$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \supset \alpha}{(\alpha \supset \beta) \supset \Gamma \supset \beta}\,(\text{suff.}^*)}{(\beta \supset \Sigma \supset \theta) \supset (\alpha \supset \beta) \supset \Gamma \supset \Sigma \supset \theta}\,(\text{suff.}^*)}{(\Delta \supset \beta \supset \Sigma \supset \theta) \supset \Delta \supset (\alpha \supset \beta) \supset \Gamma \supset \Sigma \supset \theta}\,(\text{pref.}^*) \qquad \Delta \supset \beta \supset \Sigma \supset \theta}{\Delta \supset (\alpha \supset \beta) \supset \Gamma \supset \Sigma \supset \theta}\,(\text{m.p.}).$$

$(\vee\to)$, $(\to\vee 1)$, $(\to\vee 2)$, $(\wedge 1\to)$, $(\wedge 2\to)$, $(\to\wedge)$ are straightforward using Lemma 5.

For ILL and BCK, note that $B12$ and $B13$ correspond to the exchange rule $(e\to)$ and the weakening rule $(w\to)$, respectively. $\square$

Proposition 7 (Logic with strict implication). *Corsi's system is regarded as the*
$\{\top, \bot, \vee, \wedge, \supset\}$*-fragment of the normal substructural logic F with the axioms for B^+, the axiom $B7$, and the axiom:*

$$B14.\ (q \supset p) \wedge (q \supset r) \supset p \supset r.$$

Proof. Note that $(1 \supset \top) \wedge (\top \supset 1) \in F$. Then it is easy to see that the axioms Ax1-Ax10 and the rule MP in [1] are derivable in F. For the rule AF, we see

$$\dfrac{\beta \supset 1 \quad \dfrac{\alpha}{1 \supset \alpha}\ \text{(necessitation)}}{\beta \supset \alpha}\ \text{(cut)} \quad .$$

Conversely, it is straightforward to see that the axioms and the rules for $\top$, $\bot$, $\vee$, $\wedge$, $\supset$ of F are derivable in the system in [1]. $\square$

3 Kripke-type semantics

Definition 8. A *Kripke frame* for normal substructural logics is a structure $\langle M, \cap, \cdot, \varepsilon, \omega \rangle$ satisfying the following conditions:

1. $\langle M, \cap \rangle$ is a meet-semilattice with the greatest element ω,
2. $\cdot$ is a binary operation on M and $\varepsilon \in M$ such that
   (a) $\varepsilon \cdot x = x$, $\omega \cdot x = \omega$,
   (b) $y \leq z$ implies $x \cdot y \leq x \cdot z$ for all $x, y, z \in M$,
   (c) $(x \cap y) \cdot z = (x \cdot z) \cap (y \cdot z)$ for all $x, y, z \in M$.

Definition 9. A *valuation* $\models$ on a Kripke frame $\langle M, \cap, \cdot, \varepsilon, \omega \rangle$ is a mapping which assigns a filter of M (i.e. a nonempty subset X of M such that $x, y \in X$ if and only if $x \cap y \in X$) to each propositional variables. In the sequel, we will write $x \models p$ for $x \in \models (p)$. Each valuation $\models$ can be extended to a mapping from the set of all formulas to the power set of M by

1. $x \models 1$ if and only if $x \geq \varepsilon$,
2. $x \models \top$ for all x,
3. $x \models \bot$ if and only if $x = \omega$,
4. $x \models \alpha \supset \beta$ if and only if $x \cdot y \leq z$ and $y \models \alpha$ imply $z \models \beta$ for all $y, z \in M$,
5. $x \models \alpha \vee \beta$ if and only if $y \models \alpha$ or $y \models \beta$, and $z \models \alpha$ or $z \models \beta$ for some $y, z \in M$ with $y \cap z \leq x$,

6. $x \models \alpha \wedge \beta$ if and only if $x \models \alpha$ and $x \models \beta$,

7. $x \models \alpha * \beta$ if and only if $y \models \alpha$ and $z \models \beta$ for some $y, z \in M$ with $y \cdot z \leq x$.

Proposition 10. *Let $\models$ be a valuation on a Kripke frame $\langle M, \cap, \cdot, \varepsilon, \omega \rangle$. Then*

$$\{x \in M | x \models \alpha\}$$

is a filter for any formula α.

Proof. By induction on the complexity of α. Here we show only the case $\alpha \equiv \sigma \supset \tau$. Suppose that $\omega \cdot y \leq z$ and $y \models \sigma$. Then since $\omega = \omega \cdot y \leq z$ and $\omega \models \tau$ by the induction hypothesis, we see $z \models \tau$, and hence $\omega \models \sigma \supset \tau$. Next suppose that $x \models \sigma \supset \tau$ and $x \leq y$, and let $u, v \in M$ be such that $y \cdot u \leq v$ and $u \models \sigma$. Then since $x \cdot u \leq y \cdot u \leq v$, we see $v \models \tau$, and hence $y \models \sigma \supset \tau$. Finally suppose that $x \models \sigma \supset \tau$ and $y \models \sigma \supset \tau$, and let $u, v \in M$ be such that $(x \cap y) \cdot u \leq v$ and $u \models \sigma$. Then since $x \cdot u \models \tau$ and $y \cdot u \models \tau$, $(x \cdot u) \cap (y \cdot u) \models \tau$ by the induction hypothesis. Thus $(x \cdot u) \cap (y \cdot u) = (x \cap y) \cdot u \models \tau$, and so $v \models \tau$. Therefore $x \cap y \models \sigma \supset \tau$. □

Definition 11. A *Kripke model* is a structure $\langle M, \cap, \cdot, \varepsilon, \omega, \models \rangle$ such that

1. $\langle M, \cap, \cdot, \varepsilon, \omega \rangle$ is a Kripke frame,

2. $\models$ is a valuation on $\langle M, \cap, \cdot, \varepsilon, \omega \rangle$.

A formula α is *true* in a Kripke model $\langle M, \cap, \cdot, \varepsilon, \omega, \models \rangle$ if

$$\varepsilon \models \alpha,$$

and *valid* in a Kripke frame $\langle M, \cap, \cdot, \varepsilon, \omega \rangle$ if it is true for any valuation $\models$ on the Kripke frame.

Proposition 12. *Let $\mathcal{C}$ be a class of Kripke frames. Then*

$$L(\mathcal{C}) := \{\alpha | \alpha \textit{ is valid in all frames of } \mathcal{C}\}$$

is a normal substructural logic.

Proof. Here we only show that $L(\mathcal{C})$ is closed under the $\vee$-elimination rule. Let $\alpha \supset \gamma, \beta \supset \gamma \in L(\mathcal{C})$, let $\langle M, \cap, \cdot, \varepsilon, \omega \rangle \in \mathcal{C}$, and let $\models$ be a valuation on $\langle M, \cap, \cdot, \varepsilon, \omega \rangle$. Suppose that $\varepsilon \cdot y \leq z$ and $y \models \alpha \vee \beta$. Then $y \leq z$ and there exist $u, v \in M$ such that $u \cap v \leq y$, $u \models \alpha$ or $u \models \beta$, and $v \models \alpha$ or $v \models \beta$. Since $\varepsilon \models \alpha \supset \gamma$ and $\varepsilon \models \beta \supset \gamma$, we have $u \models \gamma$ and $v \models \gamma$. Thus $u \cap v \models \gamma$ by Proposition 10 and $u \cap v \leq y \leq z$, and so $z \models \gamma$. Therefore $\varepsilon \models \alpha \vee \beta \supset \gamma$. □

Lemma 13. *Let $\mathcal{F} := \langle M, \cap, \cdot, \varepsilon, \omega \rangle$ be a Kripke frame, let $x, y \in M$, and let $\models$ be a valuation on $\mathcal{F}$ such that $x \cdot y \models q$ and $\models (p) = \uparrow y := \{z \in M | y \leq z\}$. Then $x \models p \supset q$.*

Proof. Let $u, v \in M$ be such that $x \cdot u \leq v$ and $u \models p$. Then $y \leq u$, and hence $x \cdot y \leq x \cdot u \leq v$. Therefore $v \models q$, and so $x \models p \supset q$. □

Proposition 14. *Let $\mathcal{F} := \langle M, \cap, \cdot, \varepsilon, \omega \rangle$ be a Kripke frame. Then*

1. *$B1$ is valid in $\mathcal{F}$ if and only if $(x \cdot y) \cap (x \cdot z) \leq x \cdot (y \cap z)$ for all $x, y, z \in M$,*
2. *$B2$ is valid in $\mathcal{F}$ if and only if for all $x, y, z \in M$ with $x \not\leq z$, $y \not\leq z$ and $x \cap y \leq z$, there exist $u, v \in M$ such that $x \leq u$, $y \leq v$ and $u \cap v = z$,*
3. *$B3$ is valid in $\mathcal{F}$ if and only if $x \cdot (y \cdot z) \leq (x \cdot y) \cdot z$ for all $x, y, z \in M$,*
4. *$B4$ is valid in $\mathcal{F}$ if and only if $y \cdot (x \cdot z) \leq (x \cdot y) \cdot z$ for all $x, y, z \in M$,*
5. *$B5$ is valid in $\mathcal{F}$ if and only if $(x \cdot y) \cdot y \leq x \cdot y$ for all $x, y \in M$,*
6. *$B6$ is valid in $\mathcal{F}$ if and only if $x \cdot \varepsilon \leq x$ for all $x \in M$,*
7. *$B7$ is valid in $\mathcal{F}$ if and only if $\varepsilon \leq x$ for all $x \in M$,*
8. *$B8$ is valid in $\mathcal{F}$ if and only if $y \cdot x \leq x \cdot y$ for all $x, y \in M$,*
9. *$B9$ is valid in $\mathcal{F}$ if and only if $\omega \leq x \cdot \omega$ for all $x \in M$,*
10. *$B10$ is valid in $\mathcal{F}$ if and only if $x \leq x \cdot \varepsilon$ for all $x \in M$,*
11. *$B11$ is valid in $\mathcal{F}$ if and only if $(x \cdot y) \cdot z \leq x \cdot (y \cdot z)$ for all $x, y, z \in M$,*
12. *$B12$ is valid in $\mathcal{F}$ if and only if $(x \cdot z) \cdot y \leq (x \cdot y) \cdot z$ for all $x, y, z \in M$,*
13. *$B13$ is valid in $\mathcal{F}$ if and only if $x \leq x \cdot y$ for all $x, y \in M$,*
14. *$B14$ is valid in $\mathcal{F}$ if and only if $x \cdot (x \cdot y) \leq x \cdot y$ for all $x, y \in M$.*

Proof. Here we show (1), (2), (3), (11), and (14).

(1). Suppose that $(x \cdot y) \cap (x \cdot z) \leq x \cdot (y \cap z)$ for all $x, y, z \in M$, and let $\models$ be a valuation on $\mathcal{F}$. Let $x, y \in M$ be such that $x \models (p \supset r) \wedge (q \supset r)$ and $y \models p \vee q$. Then there exist $u, v \in M$ such that $u \cap v \leq y$, $u \models p$ or $u \models q$, and $v \models p$ or $v \models q$, and hence $x \cdot u \models r$ and $x \cdot v \models r$. Thus $(x \cdot u) \cap (x \cdot v) \models r$, and so $x \cdot (u \cap v) \models r$. Hence $x \cdot y \models r$. Therefore $x \models p \vee q \supset r$, and so $\varepsilon \models (p \supset r) \wedge (q \supset r) \supset p \vee q \supset r$. Conversely suppose that $(p \supset r) \wedge (q \supset r) \supset p \vee q \supset r$ is valid in $\mathcal{F}$, and let $\models$ be a valuation on $\mathcal{F}$ such that $\models (p) = \uparrow y$, $\models (q) = \uparrow z$, and $\models (r) = \uparrow (x \cdot y \cap x \cdot z)$. Then $y \cap z \models p \vee q$, $x \models p \supset r$, and $x \models q \supset r$

by Lemma 13, and hence $x \models p \vee q \supset r$. Therefore $x \cdot (y \cap z) \models r$, and so $(x \cdot y) \cap (x \cdot z) \leq x \cdot (y \cap z)$.

(2). Suppose that for all $x, y, z \in M$ with $x \not\leq z$, $y \not\leq z$ and $x \cap y \leq z$, there exist $u, v \in M$ such that $x \leq u$, $y \leq v$ and $u \cap v = z$, and let $\models$ be a valuation on $\mathcal{F}$. Let $z, w \in M$ be such that $\varepsilon \cdot z \leq w$ and $z \models p \wedge (q \vee r)$. Then $z \models p$ and there exist $x, y \in M$ such that $x \cap y \leq z$, $x \models q$ or $x \models r$, and $y \models q$ or $y \models r$. If $x \leq z$ or $y \leq z$, then $z \models p \wedge q$ or $z \models p \wedge r$, and hence $w \models (p \wedge q) \vee (p \wedge r)$. Assume that $x \not\leq z$ and $y \not\leq z$. Then there exist $u, v \in M$ such that $x \leq u$, $y \leq v$ and $u \cap v = z$, and hence $u \cap v \leq z \leq w$, $u \models p \wedge q$ or $u \models p \wedge r$, and $v \models p \wedge q$ or $v \models p \wedge r$. Thus $w \models (p \wedge q) \vee (p \wedge r)$. Therefore $\varepsilon \models p \wedge (q \vee r) \supset (p \wedge q) \vee (p \wedge r)$. Conversely suppose that $p \wedge (q \vee r) \supset (p \wedge q) \vee (p \wedge r)$ is valid in $\mathcal{F}$. Let $x, y, z \in M$ be such that $x \not\leq z$, $y \not\leq z$ and $x \cap y \leq z$, and let $\models$ be a valuation on $\mathcal{F}$ such that $\models (p) = \uparrow z$, $\models (q) = \uparrow x$ and $\models (r) = \uparrow y$. Then $z \models p \wedge (q \vee r)$, and hence $z \models (p \wedge q) \vee (p \wedge q)$. Thus there exist $u, v \in M$ such that $u \cap v \leq z$, $u \models p \wedge q$ or $u \models p \wedge r$, and $v \models p \wedge q$ or $v \models p \wedge r$, and so $z \leq u \cap v$, $x \leq u$ or $y \leq u$, and $x \leq v$ or $y \leq v$. Therefore $u \cap v = z$, and either $x \leq u$ and $y \leq v$, or $x \leq v$ and $y \leq u$.

(3). Suppose that $x \cdot (y \cdot z) \leq (x \cdot y) \cdot z$ for all $x, y, z \in M$, and let $\models$ be a valuation on $\mathcal{F}$. Let $x, y, z, u, v, w \in M$ be such that $\varepsilon \cdot x \leq u$, $x \models q \supset r$, $u \cdot y \leq v$, $y \models p \supset q$, $v \cdot z \leq w$, and $z \models p$. Then $(x \cdot y) \cdot z \leq (u \cdot y) \cdot z \leq v \cdot z \leq w$. Since $x \cdot (y \cdot z) \leq (x \cdot y) \cdot y$ and $x \cdot (y \cdot z) \models r$, we have $w \models r$. Thus $v \models p \supset r$, and so $u \models (p \supset q) \supset p \supset r$. Therefore $\varepsilon \models (q \supset r) \supset (p \supset q) \supset p \supset r$. Conversely suppose that $(q \supset r) \supset (p \supset q) \supset p \supset r$ is valid in $\mathcal{F}$, and let $x, y, z \in M$, and let $\models$ be a valuation on $\mathcal{F}$ such that $\models (p) = \uparrow z$, $\models (q) = \uparrow y \cdot z$, and $\models (r) = \uparrow x \cdot (y \cdot z)$. Then $y \models p \supset q$ and $x \models q \supset r$ by Lemma 13, and hece $x \models (p \supset q) \supset p \supset r$. Therefore $(x \cdot y) \cdot z \models r$, and so $x \cdot (y \cdot z) \leq (x \cdot y) \cdot z$.

(11). Suppose that $(x \cdot y) \cdot z \leq x \cdot (y \cdot z)$ for all $x, y, z \in M$, and let $\models$ be a valuation on $\mathcal{F}$. Let $x, y, u, v \in M$ be such that $\varepsilon \cdot x \leq u$, $x \models p \supset q \supset r$, $u \cdot y \leq v$ and $y \models p * q$. Then there exist $u', v' \in M$ such that $u' \cdot v' \leq y$, $u' \models p$ and $v' \models q$, and hence $(x \cdot u') \cdot v' \leq x \cdot (u' \cdot v') \leq x \cdot y \leq u \cdot y \leq v$ and $(x \cdot u') \cdot v' \models r$. Thus $v \models r$, and so $u \models p * q \supset r$. Therefore $\varepsilon \models (p \supset q \supset r) \supset p * q \supset r$. Conversely suppose that $(p \supset q \supset r) \supset p * q \supset r$ is valid in $\mathcal{F}$, and let $\models$ be a valuation such that $\models (p) = \uparrow y$, $\models (q) = \uparrow z$ and $\models (r) = \uparrow (x \cdot y) \cdot z$. Then $x \models p \supset q \supset r$ by Lemma 13, and hence $x \models p * q \supset r$. Therefore $x \cdot (y \cdot z) \models r$, and so $(x \cdot y) \cdot z \leq x \cdot (y \cdot z)$.

(14). Suppose that $x \cdot (x \cdot y) \leq x \cdot y$ for all $x, y \in M$, and let $\models$ be a valuation on $\mathcal{F}$. Let $x, y, u, v \in M$ be such that $\varepsilon \cdot x \leq u$, $x \models (p \supset q) \wedge (q \supset r)$, $u \cdot y \leq v$ and $y \models p$. Then $x \cdot (x \cdot y) \models r$ and $x \cdot (x \cdot y) \leq x \cdot y \leq u \cdot y \leq v$, and hence $v \models r$. Therefore $u \models p \supset r$, and so $\varepsilon \models (p \supset q) \wedge (q \supset r) \supset p \supset r$. Conversely suppose that $(p \supset q) \wedge (q \supset r) \supset p \supset r$ is valid in $\mathcal{F}$, let $x, y \in M$, and let $\models$ be a valuation such that $\models (p) = \uparrow y$, $\models (q) = \uparrow x \cdot y$ and $\models (r) = \uparrow x \cdot (x \cdot y)$. Then $x \models p \supset q$ and $x \models q \supset r$ by Lemma 13. Thus $x \models p \supset r$, and so $x \cdot y \models r$.

Therefore $x \cdot (x \cdot y) \leq x \cdot y$. □

4 A canonical model construction

Definition 15. Let L be a normal substructural logic. An *L-pretheory* x is a subset of the set Φ of all formulas such that

1. $\top \in x$,
2. if $\alpha \in x$ and $\alpha \supset \beta \in L$, then $\beta \in x$,
3. if $\alpha, \beta \in x$, then $\alpha \wedge \beta \in x$.

Lemma 16. *Let L be a normal substructural logic. Then*

1. *if x and y are L-pretheories, then so is $x \cap y$,*
2. *if x and y are L-pretheories, then so is $x \cdot y := \{\beta | \exists \alpha \in y(\alpha \supset \beta \in x)\}$,*
3. *$L \cdot \{\alpha\}$ is an L-pretheory,*
4. *if x is an L-pretheory, then $L \cdot x = x$,*
5. *if x, y and z are L-pretheories, then $(x \cap y) \cdot z = (x \cdot z) \cap (y \cdot z)$.*

Proof. (1). Straightforward.

(2). Let x and y be L-pretheories. Then since $\top \supset \top \supset \top \in L$ by Lemma 5 (3), $\top \supset \top \in x$, and hence $\top \in x \cdot y$. Suppose that $\beta \in x \cdot y$ and $\beta \supset \gamma \in L$. Then there exists $\alpha \in y$ such that $\alpha \supset \beta \in x$. Since L is closed under the prefixing rule, $(\alpha \supset \beta) \supset \alpha \supset \gamma \in L$, and hence $\alpha \supset \gamma \in x$. Therefore $\gamma \in x \cdot y$. Suppose that $\beta, \gamma \in x \cdot y$. Then there exist $\alpha, \alpha' \in y$ such that $\alpha \supset \beta, \alpha' \supset \gamma \in x$, and hence $\alpha \wedge \alpha' \in y$ and $(\alpha \wedge \alpha' \supset \beta) \wedge (\alpha \wedge \alpha' \supset \gamma) \in x$ by Lemma 5 (5). Since $(\alpha \wedge \alpha' \supset \beta) \wedge (\alpha \wedge \alpha' \supset \gamma) \supset \alpha \wedge \alpha' \supset \beta \wedge \gamma \in L$, $\alpha \wedge \alpha' \supset \beta \wedge \gamma \in x$, and therefore $\beta \wedge \gamma \in x \cdot y$.

(3). Since $\alpha \supset \top \in L$, $\top \in L \cdot \{\alpha\}$. Suppose that $\beta \in L \cdot \{\alpha\}$ and $\beta \supset \gamma \in L$. Then $\alpha \supset \beta \in L$, and hence $\alpha \supset \gamma \in L$. Therefore $\gamma \in L \cdot \{\alpha\}$. Suppose that $\beta, \gamma \in L \cdot \{\alpha\}$. Then $(\alpha \supset \beta) \wedge (\alpha \supset \gamma) \in L$, and hence $\alpha \supset \beta \wedge \gamma \in L$. Thus $\beta \wedge \gamma \in L \cdot \{\alpha\}$.

(4). Straightforward.

(5). Let x, y and z be L-pretheories. Then trivially, $(x \cap y) \cdot z \subseteq (x \cdot z) \cap (y \cdot z)$. Suppose that $\gamma \in (x \cdot z) \cap (y \cdot z)$. Then there exist $\alpha, \beta \in z$ such that $\alpha \supset \gamma \in x$ and $\beta \supset \gamma \in y$, and hence $\alpha \wedge \beta \in z$ and $\alpha \wedge \beta \supset \gamma \in x \cap y$ by Lemma 5 (5). Therefore $\gamma \in (x \cap y) \cdot z$.

Proposition 17. *Let L be a normal substructural logic, and let M_L be the set of all L-pretheories. Then $\mathcal{F}_L := \langle M_L, \cap, \cdot, L, \Phi \rangle$ is a Kripke frame.*

Proof. Straightforward. □

Proposition 18. *Let L be a normal substructural logic. Then*

1. *if $B1 \in L$, then $(x \cdot y) \cap (x \cdot z) \subseteq x \cdot (y \cap z)$ for all $x, y, z \in M_L$,*
2. *if $B2 \in L$, then for all $x, y, z \in M_L$ with $x \not\subseteq z$, $y \not\subseteq z$ and $x \cap y \subseteq z$, there exist $u, v \in M_L$ such that $x \subseteq u$, $y \subseteq v$ and $u \cap v = z$,*
3. *if $B3 \in L$, then $x \cdot (y \cdot z) \subseteq (x \cdot y) \cdot z$ for all $x, y, z \in M_L$,*
4. *if $B4 \in L$, then $y \cdot (x \cdot z) \subseteq (x \cdot y) \cdot z$ for all $x, y, z \in M_L$,*
5. *if $B5 \in L$, then $(x \cdot y) \cdot y \subseteq x \cdot y$ for all $x, y \in M_L$,*
6. *if $B6 \in L$, then $x \cdot L \subseteq x$ for all $x \in M_L$,*
7. *if $B7 \in L$, then $L \subseteq x$ for all $x \in M_L$,*
8. *if $B8 \in L$, then $y \cdot x \subseteq x \cdot y$ for all $x, y \in M_L$,*
9. *if $B9 \in L$, then $\Phi \subseteq x \cdot \Phi$ for all $x \in M_L$,*
10. *if $B10 \in L$, then $x \subseteq x \cdot L$ for all $x \in M_L$,*
11. *if $B11 \in L$, then $(x \cdot y) \cdot z \subseteq x \cdot (y \cdot z)$ for all $x, y, z \in M_L$,*
12. *if $B12 \in L$, then $(x \cdot z) \cdot y \subseteq (x \cdot y) \cdot z$ for all $x, y, z \in M_L$,*
13. *if $B13 \in L$, then $x \subseteq x \cdot y$ for all $x, y \in M_L$,*
14. *if $B14 \in L$, then $x \cdot (x \cdot y) \subseteq x \cdot y$ for all $x, y \in M_L$.*

Proof. Here we show (1), (2), (5), and (11).

(1). Let $x, y, z \in M_L$, and let $\gamma \in (x \cdot y) \cap (x \cdot z)$. Then there exist $\alpha \in y$ and $\beta \in z$ such that $\alpha \supset \gamma, \beta \supset \gamma \in x$, and hence $\alpha \vee \beta \supset \gamma \in x$ by $B1$. Therefore, since $\alpha \vee \beta \in y \cap z$, we have $\gamma \in x \cdot (y \cap z)$.

(2). Let $x, y, z \in M_L$ be such that $x \not\subseteq z$, $y \not\subseteq z$ and $x \cap y \subseteq z$, and define u and v by

$$u := \{\theta \in \Phi | \exists \alpha \in x \exists \gamma \in z (\alpha \wedge \gamma \supset \theta \in L)\},$$
$$v := \{\theta \in \Phi | \exists \beta \in y \exists \gamma \in z (\beta \wedge \gamma \supset \theta \in L)\}.$$

Then it is straightforward to see that u and v are L-pretheories with $x \subseteq u$, $y \subseteq v$ and $z \subseteq u \cap v$. Suppose that $\theta \in u \cap v$. Then there exist $\alpha \in x$, $\beta \in y$ and $\gamma, \gamma' \in z$ such that $\alpha \wedge \gamma \supset \theta, \beta \wedge \gamma' \supset \theta \in L$, and hence $\alpha \wedge \gamma \wedge \gamma' \supset \theta, \beta \wedge \gamma \wedge \gamma' \supset \theta \in L$. Thus $(\alpha \wedge \gamma \wedge \gamma') \vee (\beta \wedge \gamma \wedge \gamma') \supset \theta \in L$, and so $(\alpha \vee \beta) \wedge \gamma \wedge \gamma' \supset \theta \in L$ by $B2$. Therefore, since $\alpha \vee \beta \in x \cap y \subseteq z$, we have $\theta \in z$.

(5). Let $x, y, z \in M_L$, and let $\gamma \in (x \cdot y) \cdot y$. Then there exist $\alpha, \beta \in y$ such that $\alpha \supset \beta \supset \gamma \in x$. Since $(\alpha \supset \beta \supset \gamma) \supset \alpha \wedge \beta \supset \alpha \wedge \beta \supset \gamma \in L$ by Lemma 5 (5), we see $\alpha \wedge \beta \supset \alpha \wedge \beta \supset \gamma \in x$, and hence $\alpha \wedge \beta \supset \gamma \in x$ by $B5$. Therefore, since $\alpha \wedge \beta \in y$, we have $\gamma \in x \cdot y$.

(11). Let $x, y, x \in M_L$, and let $\gamma \in (x \cdot y) \cdot z$. Then there exist $\alpha \in y$ and $\beta \in z$ such that $\alpha \supset \beta \supset \gamma \in x$, and hence $\alpha * \beta \supset \gamma \in x$ by $B11$. Since $\alpha \supset \beta \supset \alpha * \beta \in L$, we see $\beta \supset \alpha * \beta \in y$, and hence $\alpha * \beta \in y \cdot z$. Therefore $\gamma \in x \cdot (y \cdot z)$. □

Lemma 19. *Let L be a normal substructural logic, and let $x \in M_L$. Then*

1. *$\alpha \in x$ if and only if $L \cdot \{\alpha\} \subseteq x$,*
2. *$L \cdot \{\alpha\} \cap L \cdot \{\beta\} \subseteq L \cdot \{\alpha \vee \beta\}$,*
3. *$(L \cdot \{\alpha\}) \cdot (L \cdot \{\beta\}) \subseteq L \cdot \{\alpha * \beta\}$.*

Proof. (1). If $\beta \in L \cdot \{\alpha\}$, then $\alpha \supset \beta \in L$, and hence $\beta \in x$. Since $\alpha \in L \cdot \{\alpha\} \subseteq x$, the converse is trivial.

(2). Suppose that $\gamma \in L \cdot \{\alpha\} \cap L \cdot \{\beta\}$. Then $\alpha \supset \gamma, \beta \supset \gamma \in L$, and hence $\alpha \vee \beta \supset \gamma \in L$ by the $\vee$-elimination rule. Therefore $\gamma \in L \cdot \{\alpha \vee \beta\}$.

(3). Suppose that $\theta \in (L \cdot \{\alpha\}) \cdot (L \cdot \{\beta\})$. Then there exists γ such that $\alpha \supset \gamma \supset \theta, \beta \supset \gamma \in L$, and also

$$\dfrac{\dfrac{\alpha \supset \gamma \supset \theta \quad \dfrac{\beta \supset \gamma}{(\gamma \supset \theta) \supset \beta \supset \theta}\,(\text{suff.})}{\alpha \supset \beta \supset \theta}\,(\text{cut})}{\alpha * \beta \supset \theta}\,(\text{residuation}) .$$

Therefore $\alpha * \beta \supset \theta \in L$, and so $\theta \in L \cdot \{\alpha * \beta\}$.

Theorem 20. *Let L be a normal substructural logic, and let $\models_L$ be a mapping from the set of all propositional variables to the set of all subsets of M_L defined by*

$$\models_L (p) := \{x \in M_L | p \in x\}.$$

Then $\mathcal{M}_L := \langle M_L, \cap, \cdot, L, \Phi, \models_L \rangle$ is a Kripke model such that $\alpha \in L$ if and only if α is true in $\mathcal{M}_L$.

Proof. It is easy to see that $\models_L$ is a valuation on $\mathcal{F}_L$, and hence $\mathcal{M}_L$ is a Kripke model. It remains to show that

$$L \models_L \alpha \Leftrightarrow \alpha \in L.$$

To this end, we will show that

$$x \models_L \alpha \Leftrightarrow \alpha \in x$$

for all $\alpha \in \Phi$ and $x \in M_L$ by induction on the complexity of α.
Basis: Straightforward.
Induction step:
Case 1. $\alpha \equiv \sigma \vee \tau$: Suppose that $x \models_L \sigma \vee \tau$. Then there exist $u, v \in M_L$ such that $u \cap v \subseteq x$, $u \models_L \sigma$ or $u \models_L \tau$, and $v \models_L \sigma$ or $v \models_L \tau$. By the induction hypothesis $\sigma \in u$ or $\tau \in u$, and $\sigma \in v$ or $\tau \in v$. Since $\sigma \supset \sigma \vee \tau, \tau \supset \sigma \vee \tau \in L$, we have $\sigma \vee \tau \in u$ and $\sigma \vee \tau \in v$. Therefore $\sigma \vee \tau \in u \cap v \subseteq x$. Conversely suppose that $\sigma \vee \tau \in x$. Then $L \cdot \{\sigma\} \cap L \cdot \{\tau\} \subseteq x$ by Lemma 19 (1) and (2), and since $L \cdot \{\sigma\} \models_L \sigma$ and $L \cdot \{\tau\} \models_L \tau$ by the induction hypothesis, we have $x \models_L \sigma \vee \tau$.
Case 2. $\alpha \equiv \sigma \wedge \tau$: Straightforward.
Case 3. $\alpha \equiv \sigma * \tau$: Suppose that $x \models_L \sigma * \tau$. Then there exist $u, v \in M_L$ such that $u \cdot v \subseteq x$, $u \models_L \sigma$, and $v \models_L \tau$. By the induction hypothesis $\sigma \in u$ or $\tau \in v$, and since $\sigma \supset \tau \supset \sigma * \tau \in L$, we have $\tau \supset \sigma * \tau \in u$. Therefore $\sigma * \tau \in u \cdot v$, and so $\sigma * \tau \in x$. Conversely suppose that $\sigma * \tau \in x$. Then $(L \cdot \{\sigma\}) \cdot (L \cdot \{\tau\}) \subseteq x$ by Lemma 19 (1) and (3), and hence $x \models_L \sigma * \tau$.
Case 4. $\alpha \equiv \sigma \supset \tau$: Suppose that $x \models_L \sigma \supset \tau$. Then since $L \cdot \{\sigma\} \models_L \sigma$ by the induction hypothesis, $x \cdot (L \cdot \{\sigma\}) \models_L \tau$, and hence $\tau \in x \cdot (L \cdot \{\sigma\})$ by the induction hypothesis. Therefore there exists θ such that $\theta \supset \tau \in x$ and $\sigma \supset \theta \in L$, and hence $(\theta \supset \tau) \supset \sigma \supset \tau \in L$ by the suffixing rule. Thus $\sigma \supset \tau \in x$. The converse is straightforward. □

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