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Some Algebraic Structures with Iteration Lemmata

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Abstract: This paper deals with solutions of algebraic, linear, and rational systems of equations over an ω-complete semiring, and their iteration lemmata. These are guaranteed if the underlying structure has an associative multiplicative operation, and its elements have a norm. A number of such structures like words, vectors, traces, trees, graphs, are presented.

Keywords : ω-complete semirings, systems of equations, algebraic, linear, rational languages, norm, iteration lemmata, associative structures.

1 Introduction

The iteration lemmata for regular, linear and context-free languages are well known. They are based on the catenation operation (with unit element λ) on the free monoid V^* over some alphabet V, and the norm $|w|$ (length of words).

In this paper other binary operations $\circ$ on the power set on underlying monoids M are introduced, as well as other general norms μ. The operations have to be associative with zero element $\emptyset$ and unit element $\{\lambda\}$, and distributive with $\cup$ such that the resulting structure is an ω-complete semiring. The norm μ has to be monotone with respect to $\circ$ and $\cup$, with some minimal norms for $\emptyset$ and $\{\lambda\}$, and be defined for all finite sets.

If rational, linear and algebraic languages are defined as fixed points of corresponding systems of equations on ω-complete semirings, it can be shown that iteration lemmata similar to the classical ones hold for such languages.

2 Definitions

Let M be a monoid with binary operation $\circ$ and unit element λ. Extend $\circ$ to an associative binary operation $\circ : \mathcal{P}(M) \times \mathcal{P}(M) \to \mathcal{P}(M)$, distributive with $\cup$ ($A \circ (B \cup C) = (A \circ B) \cup (A \circ C)$ and $(A \cup B) \circ C = (A \circ B) \cup (B \circ C)$), with unit element $\{\lambda\}$ ($\{\lambda\} \circ A = A \circ \{\lambda\} = A$), and zero element $\emptyset$ ($\emptyset \circ A = A \circ \emptyset = \emptyset$).

Then $\mathcal{S} = (\mathcal{P}(M), \cup, \circ, \emptyset, \{\lambda\})$ is an ω-complete semiring, i.e. if $A_i \subseteq A_{i+1}$ for $0 \leq i$ then $B \circ \bigcup_{i \geq 0} A_i = \bigcup_{i \geq 0}(B \circ A_i)$ and $(\bigcup_{i \geq 0} A_i) \circ B = \bigcup_{i \geq 0}(A_i \circ B)$.

Define also $A^{\circ(0)} = \{\lambda\}$, $A^{\circ(1)} = A, A^{\circ(k+1)} = A \circ A^{\circ(k)}$, $A^{\circ} = \bigcup_{k \geq 0} A^{\circ(k)}$.

Let $\mu : \mathcal{P}(M) \to I\!N$ be a (partial) function (norm) defined for all finite sets, with the following properties :

$\mu(\emptyset) = 0$, $\mu(\{\lambda\}) \leq 1$, $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$,

$\mu(A), \mu(B) \leq \mu(A \cup B) \leq max\{\mu(A), \mu(B)\}$,

$\mu(A), \mu(B) \leq \mu(A \circ B)$ for $A \neq \emptyset, B \neq \emptyset$, $\mu(A \circ B) \leq \mu(A) + \mu(B)$, $\mu(A) = \infty$ for $|A| = \infty$.

Let $\mathcal{X} = \{X_1, \ldots, X_n\}$ be a set of variables such that $X \cap M = \emptyset$.

A *monomial* over $\mathcal{S}$ with variables in X is a finite expression of the form : $A_1 \circ A_2 \circ \ldots \circ A_k$, where $A_i \in \mathcal{X}$ or $A_i \subseteq M, |A_i| < \infty, i = 1, \ldots, k$. (without loss of generality, $A_i = \{\alpha_i\}$ with $\alpha_i \in M$ suffices). A *polynomial* $p(X)$ over $\mathcal{S}$ is a finite union of monomials where $X = (X_1, \cdots, X_n)$.

A *system of equations* over $\mathcal{S}$ is a finite set of equations :

$E = \{X_i = p_i(X) \mid i = 1, \ldots, n\}$, where $p_i(X)$ are polynomials.

The *solution* of the system E is a n-tuple $(L_1, \ldots, L_n)$ of languages over M, where $L_i = p_i(L_1, \ldots, L_n)$ and the n-tuple is minimal with this property, i.e. if $(L'_1, \ldots, L'_n)$ is another n-tuple that satisfies E, then $(L_1, \ldots, L_n) \leq (L'_1, \ldots, L'_n)$ (where the order is defined componentwise with respect to inclusion).

¿From the theory of semirings follows that any system of equations over $\mathcal{S}$ has a unique solution, and this is exactly the least fixed point starting with $(X_1, \cdots, X_n) = (\emptyset, \cdots, \emptyset)$. For the theory of semirings see [4, 6].

A system of equations is called *linear* if all monomials are of the form $A \circ X \circ B$ or A, and *rational* if they are of the form $X \circ A$ or A, with $A \subseteq M$ and $B \subseteq M$. Corresponding families of languages (solutions of such systems of equations) are denoted by $\underline{ALG(\circ)}$, $\underline{LIN(\circ)}$, and $\underline{RAT(\circ)}$. In case $\circ$ is commutative then $\underline{ALG(\circ)} = \underline{LIN(\circ)} = \underline{RAT(\circ)}$.

3 Iteration Lemmata

The following theorems can be proven in a way analogous to the classical iteration lemmata. Proofs can be found in [7].

Theorem 3.1 : Let $L \in \underline{RAT(\circ)}$ with $L \subseteq M$. Then there exist $n(L) > 0$ such that, for any $w \in L$ with $\mu(\{w\}) > n(L)$, there exist $x_1, x_2, x_3 \in M$ such that :

(i) $w \in \{x_1\} \circ \{x_2\} \circ \{x_3\}$.

(ii) $0 < \mu(\{x_1\} \circ \{x_2\}) \leq n(L)$.

(iii) $\{x_1\} \circ \{x_2\}^\circ \circ \{x_3\} \subseteq L$. □

Theorem 3.2 : Let $L \in \underline{LIN(\circ)}$ with $L \subseteq M$. Then there exist $n(L) > 0$ such that, for any $w \in L$ with $\mu(\{w\}) > n(L)$, there exist $x_1, x_2, x_3, x_4, x_5 \in M$ such that :

(i) $w \in \{x_1\} \circ \{x_2\} \circ \{x_3\} \circ \{x_4\} \circ \{x_5\}$.

(ii) $\mu(\{x_1\} \circ \{x_2\} \circ \{x_4\} \circ \{x_5\}) \leq n(L)$.

(iii) $0 < \mu(\{x_2\} \circ \{x_4\})$
(iv) $\forall k \geq 0 : \{x_1\} \circ \{x_2\}^{\circ(k)} \circ \{x_3\} \circ \{x_4\}^{\circ(k)} \circ \{x_5\} \subseteq L$. □

Theorem 3.3 : Let $L \in \underline{ALG(\circ)}$ with $L \subseteq M$. Then there exist $n(L) > 0$ such that, for any $w \in L$ with $\overline{\mu(\{w\})} > n(L)$, there exist $x_1, x_2, x_3, x_4, x_5 \in M$ such that :
(i) $w \in \{x_1\} \circ \{x_2\} \circ \{x_3\} \circ \{x_4\} \circ \{x_5\}$.
(ii) $\mu(\{x_2\} \circ \{x_3\} \circ \{x_4\}) \leq n(L)$.
(iii) $0 < \mu(\{x_2\} \circ \{x_4\})$
(iv) $\forall k \geq 0 : \{x_1\} \circ \{x_2\}^{\circ(k)} \circ \{x_3\} \circ \{x_4\}^{\circ(k)} \circ \{x_5\} \subseteq L$.

□

To prove these theorems the systems of equations are first converted into equivalent systems of equations (with additional variables) where all monomials are in normal form ($X \circ Y$ or α for algebraic, $\alpha \circ X$ or $X \circ \alpha$ or α for linear, and $X \circ \alpha$ or α for rational systems).

Any $w \in L$ can be generated as $w \in \{\beta_1\} \circ \cdots \circ \{\beta_k\}$ where the $\beta_j \in M$ are the leaves of a binary derivation tree with respect to $\circ$, and the children of each node correspond to monomials. Note that μ is monotone with respect to $\cup$ and $\circ$, but bounded by the sum.

4 Associative Structures

Words

Example 4.1 : Let $\circ = \cdot$, the usual catenation (being associative with unit element λ on $M = V^*$), and μ be defined by $\mu(w) = |w|$, extended to sets :
$\mu(\emptyset) = \mu(\{\lambda\}) = 0$, $w \in V^* \Rightarrow \mu(\{w\}) = |w|$, $\mu(A \circ B) = \mu(A) + \mu(B)$,
$\mu(A \cup B) = max\{\mu(A), \mu(B)\}$, $\mu(A^\circ) = \infty$.
Then $(\mathcal{P}(V^*), \cdot, \cup, \{\lambda\}, \emptyset)$ is an ω-complete semiring.

□

Example 4.2 : Let $\circ = ⧢$, the shuffle operation (being associative and commutative on $M = V^*$ with unit element λ), and μ be defined as in Example 4.1.
Then $(\mathcal{P}(V^*), ⧢, \cup, \{\lambda\}, \emptyset)$ is an ω-complete semiring.

□

Vectors

Example 4.3 : Consider the set $I\!N^k$ of positive k dimensional vectors. For $x = (x_1, \cdots, x_k)$ and $y = (y_1, \cdots, y_k) \in I\!N^k$ define $x \circ y = x + y$ and the norm

$\mu(x) = max\{x_1, \cdots, x_k\}$. $\circ$ is an commutative and associative operaion on $I\!N^k$, and can be extended to $\mathcal{P}(I\!N^k)$.

Then $(\mathcal{P}(I\!N^k), +, \cup, \{0\}, \emptyset)$ is a commutative ω-complete semiring. □

Matrices

Example 4.4 : Consider the set $\mathcal{M}_k(I\!N)$ all $k \times k$-matrices with coefficients from $I\!N$ and matrix (operator) norm $||M|| \geq 1.$, Let the associative operation be defined by the normal matrix multiplication $M_1 \circ M_2 = M_1 \cdot M_2$. Let I be the unit matrix.

Again, $(\mathcal{P}\}(\mathcal{M}_k(I\!N)), \cdot, \cup, I, \emptyset)$ is an ω-complete semiring.

The norm is defined by $\mu(M) = log_2(2 \cdot ||M||) = 1 + log_2(||M||)$ (in this case as a positive real number). Then

$max\{\mu(M_1), \mu(M_2)\} \leq \mu(M_1 \cdot M_2) = 1 + log_2(||M_1 \cdot M_2||)$
$= 1 + log_2(||M_1|| \cdot ||M_2||) < 2 + log_2(||M_1||) + log_2(||M_2||) = \mu(M_1) + \mu(M_2)$.

□

In the next two examples two closely related associative structures on words are presented to show that iteration lemmata also may hold for a subset of the structure, actually without only one zero element. This is done by introducing 'garbage' symbols such that the condition on the norm is fulfilled. They also serve as a method for other algebraic structures with similar properties.

Words

Example 4.5 : Let Σ be an alphabet, $\# \notin \Sigma$, and consider $\mathcal{A} = \Sigma^* \cup \{\#\}^+$.

Define an operation $\odot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ in the following way : for $x, y \in \Sigma$ and $w, w_1, w_2 \in \Sigma^*$

$w_1 x \odot x w_2 = w_1 x w_2$, $w_1 x \odot y w_2 = \#^{|w_1|+|w_2|+1}$ if $x \neq y$,
$w \odot \#^k = \#^k \odot w = \#^{|w|+k-1}$ ($|w| > 0$, $k > 0$) ,
$\#^m \odot \#^n = \#^{m+n-1}$ ($m, n > 0$)
$\lambda \odot w = w \odot \lambda = w$, $\lambda \odot \#^k = \#^k \odot \lambda = \#^k$

Then $\mathcal{M} = (\mathcal{A}, \odot, \lambda)$ is a monoid since $\odot$ is an associative operation on $\mathcal{A}$.

Extend $\odot$ to $\mathcal{P}(\mathcal{A})$ by

$$A \odot B = \bigcup_{a \in A, b \in B} a \odot b$$

Then $\mathcal{S} = (\mathcal{P}(\mathcal{A}), \cup, \odot, \emptyset, \{\lambda\})$ is an ω-complete semiring.

Therefore $\odot$-rational, $\odot$-linear and $\odot$-algebraic languages over $\mathcal{A}$ can be defined as minimal solutions of corresponding systems of equations.

Note that any such language L is a disjoint union $L_1 \uplus L_2$ with $L_1 \subseteq \Sigma^*$ and $L_2 \subseteq \{\#\}^+$.

Defining the *norm* of any $a \in \mathcal{A}$ by
$\mu(w) = |w|$ for $w \in \Sigma^*$ and $\mu(\#^k) = k$ ($k > 0$)
one gets $\mu(a) + \mu(b) \leq \mu(a \odot b) + 1 \leq \mu(a) + \mu(b) + 1$, and extended to sets
$max(\mu(A), \mu(B)) \leq \mu(A \odot B) \leq \mu(A) + \mu(B)$.
Therefore the iteration lemmata for $\odot$-rational, $\odot$-linear, and $\odot$-algebraic languages hold. This is especially true for elements from $L \cap \Sigma^*$.

□

Example 4.6 : Let Σ be an alphabet, $\Omega \notin \Sigma$, and consider $\mathcal{B} = \Sigma^* \cup \{\Omega\}$.
Define an operation $\otimes : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ in the following way : for $w, w_1, w_2 \in \Sigma^*$
$w_1 x \otimes x w_2 = w_1 x w_2$, $w_1 x \otimes y w_2 = \Omega$ if $x \neq y$
$w \otimes \Omega = \Omega \otimes w = \lambda \otimes \Omega = \Omega \otimes \lambda = \Omega \otimes \Omega = \Omega$.
$\lambda \otimes w = w \otimes \lambda = w$
$\otimes$ is an associative operation, and $\mathcal{M}_\Omega = (\mathcal{B}, \otimes, \lambda)$ is a monoid.

$\otimes$ can be extended to $\mathcal{P}(\mathcal{B})$ as above.
Then $\mathcal{S}_\Omega = (\mathcal{P}(\mathcal{B}), \cup, \otimes, \emptyset, \{\lambda\})$ is an ω-complete semiring.
Again, $\otimes$-rational, $\otimes$-linear and $\otimes$-algebraic languages can be defined. Note that such a language L_Ω either contains Ω or not.

Now define a relation $\approx$ on $\mathcal{A}$ by
$a \approx b$ if either $a, b \in \Sigma^*$ and $a = b$, or $a = \#^m, b = \#^n$ ($m, n > 0$).
$\approx$ is an equivalence relation on $\mathcal{A}$. Thus $\mathcal{M}/\approx$ and $\mathcal{S}/\approx$ can be defined.
Note that $\mathcal{M}_\Omega \simeq \mathcal{M}/\approx$ and $\mathcal{S}_\Omega \simeq \mathcal{S}/\approx$, where $\simeq$ means *isomorphic*.

Now define a mapping $h : \mathcal{S} \to \mathcal{S}_\Omega$ by $h(w) = w$ for $w \in \Sigma^*$ and $h(a) = \Omega$ for $a \in \{\#\}^+$.
Thus, $h^{-1}(\Omega) = \{\#\}^+$ and $h^{-1}(w) = w$ for $w \in \Sigma^*$.

If L is some $\odot$-language, defined by a $\odot$-system of equations, define the corresponding $\otimes$-system of equations, replacing every $\#^k$ by Ω (only in the constants of the system of equations) and every $\odot$ by $\otimes$, yielding a $\otimes$-language L_Ω. Then $h(L) = L_\Omega$ and $L \cap \Sigma^* = L_\Omega \cap \Sigma^*$.

On the other hand, if L_Ω is some $\otimes$-language, define the corresponding $\odot$-system of equations, replacing every Ω by $\#$ (only in the constants of the equation system) and every $\otimes$ by $\odot$, yielding a $\odot$-language L. Then $h(L) = L_\Omega$ and $L \cap \Sigma^* = L_\Omega \cap \Sigma^*$.

Since the Σ part of any $\otimes$-language L_Ω is identical to the Σ part of the corresponding $\odot$-language L, the iteration lemmata for $\odot$-languages also are valid for the Σ part of L_Ω.

Note that the iteration lemmata for $\otimes$-languages cannot be proved by the same method as for $\odot$-languages since there is no (almost) monotone and bounded norm on $\mathcal{S}_\Omega$.

$\square$

Traces

Details on traces can be found in [2, 3].

Let V be an alphabet and $C \subseteq V \times V$ a symmetric, not necessarily reflexive, relation, called an independence relation. Define $uabv \sim ubav$ if $(a,b) \in I$ and consider its reflexive and transitive closure $\sim^*$. Then $\sim^*$ is an equivalence relation, and the set $\mathcal{T} = V^*/\sim^*$, also written V^*/C, being a monoid, is called the trace monoid of V with respect to C.

For $t_1 = [u], t_2 = [v] \in \mathcal{T}$ the binary operation on traces is defined by $t_1 \circ t_2 = [uv]$, and the neutral element is $[\lambda]$. For $t = [w] \in \mathcal{T}$ a norm can be defined by $\mu(t) = |w|$.

A (Mazurkiewicz) trace can be uniquely factorized into left (right) Foata normal form [2, 5] : $t = [x_1] \circ \cdots \circ [x_k]$ with $\forall a, b \in [x_i] : (a,b) \in C$ ($1 \leq i \leq k$) and $\forall b \in [x_{i+1}]\ \exists a \in [x_i] : (a,b) \notin C$ (left) or $\forall a \in [x_i]\ \exists a \in [x_{i+1}] : (a,b) \notin C$ (right) ($1 \leq i < k$). Here the factors $[x_i]$ can be considered as multisets forming some maximal commutative clique.

The number of such factors, left or right, is identical, and also defines a norm $\mu(t)$ on $\mathcal{T}$ with $\mu([\lambda]) = 0$, which can be extended to set of traces, yielding also the structure $(\mathcal{P}(\mathcal{T}), \circ, \cup, \{[\lambda]\}, \emptyset)$.

Example 4.7 : Let $\#$ be an additional symbol with $(\#, \#) \notin C$, and define $\mathcal{C} = \mathcal{T} \cup \{[\#^k] \mid 1 \leq k\}$.

Let the trace $t_1 = [x_1] \circ \cdots \circ [x_k] \circ [y_1]$ be factorized into right Foata normal form, and the trace $t_2 = [y_2] \circ [z_1] \circ \cdots \circ [z_m]$ into left Foata normal form. Then $\odot$ is defined by

$t_1 \odot t_2 = [x_1 \cdots x_k y_1 z_1 \cdots z_m]$ if $[y_1] = [y_2]$, and $t_1 \odot t_2 = [\#^{k+m+1}]$ if $[y_1] \neq [y_2]$,

$[\lambda] \odot t = t \odot [\lambda] = t$, $[\lambda] \odot [\#^k] = [\#^k] \odot [\lambda] = [\#^k]$, $[\#^k] \odot [\#^m] = [\#^{k+m}]$

$\odot$ is also an associative operation, and $\mathcal{M}_C = (\mathcal{C}, \odot, [\lambda])$ is a monoid.

Again, $\odot$ can be extended to $\mathcal{P}(\mathcal{C})$, and $\mathcal{S}_C = (\mathcal{P}(\mathcal{C}), \cup, \odot, \emptyset, \{[\lambda]\})$ is an ω-complete semiring.

Therefore $\odot$-rational, $\odot$-linear and $\odot$-algebraic languages over $\mathcal{C}$ can be defined as minimal solutions of corresponding systems of equations.

Defining now the *norm* of any $t \in \mathcal{C}$ as in Example 3.4 for $t \in \mathcal{T}$, and $\mu([\#^k]) = k$ ($k > 0$) one gets $\mu(s) + \mu(t) \leq \mu(s \odot t) + 1 \leq \mu(s) + \mu(t) + 1$, and extended to sets this yields $max(\mu(A), \mu(B)) \leq \mu(A \odot B) \leq \mu(A) + \mu(B)$.

Therefore the iteration lemmata for $\odot$-rational, $\odot$-linear, and $\odot$-algebraic trace languages hold.

□

Example 4.8 : Let $\Omega \notin \Sigma$, and consider $\mathcal{D} = \mathcal{T} \cup \{\Omega\}$.

Let the trace $t_1 = [x_1] \circ \cdots \circ [x_k] \circ [y_1]$ be factorized into right Foata normal form, and the trace $t_2 = [y_2] \circ [z_1] \circ \cdots \circ [z_m]$ into left Foata normal form.

Define an operation $\otimes : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$ in the following way :
$t_1 \otimes t_2 = [x_1 \cdots x_k y_1 z_1 \cdots z_m]$ if $[y_1] = [y_2]$, and
$t_1 \otimes t_2 = \Omega$ if $[y_1] \neq [y_2]$,
$[\lambda] \otimes t = t \otimes [\lambda] = t$
$[\lambda] \otimes \Omega = \Omega \otimes [\lambda] = \Omega$
$\Omega \otimes \Omega = \Omega$

$\otimes$ is an associative operation, and $\mathcal{M}_{C,\Omega} = (\mathcal{D}, \otimes, \lambda)$ is a monoid.

$\otimes$ can be extended to $\mathcal{P}(\mathcal{D})$ as above.

Then $\mathcal{S}_{C,\Omega} = (\mathcal{P}(\mathcal{D}), \cup, \otimes, \emptyset, \{\lambda\})$ is an ω-complete semiring.

Again, $\otimes$-rational, $\otimes$-linear and $\otimes$-algebraic trace languages can be defined. Note that such a language L_Ω either contains Ω or not.

Now similar relations as between the structures from examples 4.5 and 4.6 can be stated between the structure $\mathcal{S}_C = (\mathcal{P}(\mathcal{C}), \cup, \odot, \emptyset, \{[\lambda]\})$ from example 4.7 and the structure from example 4.8, $\mathcal{S}_{C,\Omega} = (\mathcal{P}(\mathcal{D}), \cup, \otimes, \emptyset, \{\lambda\})$.

Therefore the iteration lemmata also hold for $\otimes$-trace languages without Ω.

□

Trees

Example 4.9 :

Consider labelled trees with labels $x \in C$ where C is a finite set of labels. If T is any tree let $\rho(T)$ be the label of the root, and $\lambda(T)$ the set of labels of the leaves. Define the norm $\mu(T)$ of T to be the depth of T. Let Λ denote the empty tree, and Ω a special element.

Define $T_1 \circ T_2$ to be the tree T obtained from T_1 and T_2 by identifying each leaf of T_1 with label $\rho(T_2)$ with the root of T_2, and labelling it as before, provided $\rho(T_2) \in \lambda(T_1)$ and that $(\lambda(T_1) - \{\rho(T_2)\}) \cap \lambda(T_2) = \emptyset$. Otherwise, define $T_1 \circ T_2 = \Omega$. Furthermore, define $\Lambda \circ T = T \circ \Lambda = T$ and $\Omega \circ T = T \circ \Omega = \Omega \circ \Omega = \Omega$.

Then $\circ$ is an associative operation, and
$max(\mu(T_1), \mu(T_2) \leq \mu(T_1 \circ \mu(T_2) \leq \mu(T_1) + \mu(T_2)$ for all $T_1, T_2 \neq \Omega$.

Extend $\circ$ to $\mathcal{P}(\mathcal{T})$. Then $(\mathcal{P}(\mathcal{T}), \circ, \cup, \{\Lambda\}, \emptyset)$ is an ω-complete semiring.

□

Example 4.10 :

As in the previous example consider labelled trees.

Define $\{T_1\} \circ \{T_2\}$ to be the set of all trees obtained from T_1 and T_2 by identifying a leaf of T_1 with label $\rho(T_2)$ with the root of T_2, and labelling it as before, provided $\rho(T_2) \in \lambda(T_1)$ and $(\lambda(T_1) - \{\rho(T_2)\}) \cap \lambda(T_2) = \emptyset$. Otherwise let the result be $\{\Omega\}$.

Then $\circ$ is an associative operation and $(\mathcal{P}(\mathcal{T}), \circ, \cup, \{\Lambda\}, \emptyset)$ is an ω-complete semiring.

□

Graphs

The next examples show some associative operations on graphs.

Let C be a finite set of *colours* (or *labels*). A vertex *coloured* (or vertex *labelled*) graph is a structure $G = (V, E, C, \gamma)$ where V is a finite set of vertices, $E \subseteq V \times V - D$ the set of edges with $D = \{(x, x) | x \in V\}$, and $\gamma : V \to C$ a mapping attaching every vertex a colour. Actually, V, E, and γ depend on G. If necessary, this dependence will be indicated.

For $x \in V$ let $I(x) = \{y \in V | (y, x) \in E\}$ and $O = \{y \in V | (x, y) \in E\}$. The *indegree* of x is defined by $d_-(x) = |I(x)|$, and the *outdegree* by $d_+(x) = |O(x)|$. This can be extended to the entire graph by $I = \{x \in V | d_-(x) = 0\}$ (*initial vertices*) and $O = \{x \in V(G) | d_+(x) = 0\}$ (*terminal vertices*). Let $I \cap O = \emptyset$, and assume $I \neq \emptyset$, $O \neq \emptyset$.

Let $\delta(x, y)$ be the (directed) distance between the vertices x and y. Additionally, assume also that $\forall x \in I \ \exists y \in O : \delta(x, y) < \infty$, and define the norm $\mu(G) = max\{\delta(x, y) \mid \delta(x, y) < \infty, x \in I, y \in O\}$ as a norm of G.

Consider also the sets $\gamma(I)$ and $\gamma(O)$, representing the colours of *initial* and *terminal* vertices.

Let $\mathcal{G}$ denote the set of all such graphs.

Example 4.11 : Now consider two such graphs, $G_1 = (V_1, E_1, C, \gamma_1)$ and $G_2 = (V_2, E_2, C, \gamma_2)$. Take a copy of G_2 such that $V_1 \cap V_2 = \emptyset$. An operation $\odot$ will be defined for singletons of such graphs.

Let $c \in C$ be a colour, and consider the sets $\gamma_1^{-1}(c) \cap O_1$, $\gamma_2^{-1}(c) \cap I_2$.

If $\forall c \in C : |\gamma_1^{-1}(c) \cap O_1| = |\gamma_2^{-1}(c) \cap I_2| = k(c)$ then take a permutation π on $\{1, \cdots, k(c)\}$, and identify $x_i \in \gamma_1^{-1}(c) \cap O_1$ with $y_{\pi(i)} \in \gamma_2^{-1}(c) \cap I_2$.

If $\exists c \in C : |\gamma_1^{-1}(c) \cap O_1| = |\gamma_2^{-1}(c) \cap I_2| > 0$

then define the resulting graph $G = (V, E, C, \gamma) \in \{G_1\} \odot \{G_2\}$ by

$V = V_1 \cup V_2 - \bigcup_{c \in C} \gamma_1^{-1}(c) \cap O_1$, $E = E_1 \cup E_2$ (with identification of vertices) . $I = I_1$, $O = O_2$

$\gamma(z) = \gamma_1(z)$ if $z \in V_1 - \bigcup_{c \in C} \gamma_1^{-1}(c) \cap O_1$ and $\gamma(z) = \gamma_2(z)$ if $z \in V_2$.

Only $G = (V, E, C, \gamma)$ defined in such a way are elements of $\{G_1\} \odot \{G_2\}$.

If $\forall c \in C : |\gamma_1^{-1}(c) \cap O_1| = |\gamma_2^{-1}(c) \cap I_2| = 0$
then define the resulting graph $G = (V, E, C, \gamma)$ by
$V = V_1 \cup V_2$, $E = E_1 \cup E_2$, $I = I_1 \cup I_2$, $O = O_1 \cup O_2$
$\gamma(z) = \gamma_1(z)$ if $z \in V_1$ and $\gamma(z) = \gamma_2(z)$ if $z \in V_2$.
$\{G_1\} \odot \{G_2\} = \{G\}$

In all other cases let $\{G_1\} \odot \{G_2\} = \{X^{\mu(G_1)+\mu(G_2)}\}$,
where $X = (\{0,1\}, \{(0,1)\}, \{\#\}, \gamma)$ with $\# \notin C$ and $\gamma(0) = \gamma(1) = \#$, $\{G\} \odot \{X^k\} = \{X^k\} \odot \{G\} = \{X^{\mu(G)+k}\}$, $\{X^k\} \odot \{X^m\} = \{X^{k+m}\}$, $0 < k, m$, $X^1 = X$.

Thus graphs are connected only if the numbers of vertices in O_1 with the same colour coincide with those in I_2. Especially, if all such numbers are 0 then the graphs are put in parallel.

Let furthermore Λ be the *empty graph*, the neutral element, with the following properties :
$\{\Lambda\} \odot \{G\} = \{G\} \odot \{\Lambda\} = \{G\}$, $\{\Lambda\} \odot \{\Lambda\} = \{\Lambda\}$, $\{\Lambda\} \odot \{\Omega\} = \{\Omega\} \odot \{\Lambda\} = \{\Omega\}$.

Extend $\odot$ to sets of graphs by

$$A \odot B = \bigcup_{G \in A, H \in B} \{G\} \odot \{H\} \ ,$$

Then $(\{G_1\} \odot \{G_2\}) \odot \{G_3\} = \{G_1\} \odot (\{G_2\} \odot \{G_3\})$, and also for sets $(A \odot B) \odot D = A \odot (B \odot D)$, i.e. $\odot$ is an associative operation.

Let $\mathcal{G}_\# = \mathcal{G} \cup \{X^k \mid k > 0\}$. Then the structure $\mathcal{S} = (\mathcal{P}(\mathcal{G}_\#), \cup, \odot, \emptyset, \{\Lambda\})$ is an ω-complete semiring. Therefore $\odot$-rational, $\odot$-linear, and $\odot$-algebraic languages can be defined.

The norm satisfies $max(\mu(G_1), \mu(G_2)) \leq \mu(G) \leq \mu(G_1) + \mu(G_2)$ for all $G \in \{G_1\} \odot \{G_2\}$, and extended to sets of graphs, fulfills all conditions. Therefore the iteration lemmata hold also for $\odot$-languages.

□

Example 4.12 : Now consider two such graphs, $G_1 = (V_1, E_1, C, \gamma_1)$ and $G_2 = (V_1, E_2, C, \gamma_2)$. Assume $V_1 \cap V_2 = \emptyset$. An operation $\otimes$ will be defined for singletons of such graphs.

Let $c \in C$ be a colour, and consider the sets $\gamma_1^{-1}(c) \cap O_1$, $\gamma_2^{-1}(c) \cap I_2$.

If $\forall c \in C : |\gamma_1^{-1}(c) \cap O_1| = |\gamma_2^{-1}(c) \cap I_2| = k(c)$ then take a permutation π on $\{1, \cdots, k(c)\}$, and identify $x_i \in \gamma_1^{-1}(c) \cap O_1$ with $y_{\pi(i)} \in \gamma_2^{-1}(c) \cap I_2$.

If $\exists c \in C : |\gamma_1^{-1}(c) \cap O_1| = |\gamma_2^{-1}(c) \cap I_2| > 0$
then define the resulting graph $G = (V, E, C, \gamma) \in \{G_1\} \otimes \{G_2\}$ by

$V = V_1 \cup V_2 - \bigcup_{c \in C} \gamma_1^{-1}(c) \cap O_1$, $E = E_1 \cup E_2$ (with identification of vertices) . $I = I_1$, $O = O_2$

$\gamma(z) = \gamma_1(z)$ if $z \in V_1 - \bigcup_{c \in C} \gamma_1^{-1}(c) \cap O_1$ and $\gamma(z) = \gamma_2(z)$ if $z \in V_2$.

Only $G = (V, E, C, \gamma)$ defined in such a way are elements of $\{G_1\} \otimes \{G_2\}$.

If $\forall c \in C : |\gamma_1^{-1}(c) \cap O_1| = |\gamma_2^{-1}(c) \cap I_2| = 0$
then define the resulting graph $G = (V, E, C, \gamma)$ by
$V = V_1 \cup V_2$, $E = E_1 \cup E_2$, $I = I_1 \cup I_2$, $O = O_1 \cup O_2$
$\gamma(z) = \gamma_1(z)$ if $z \in V_1$ and $\gamma(z) = \gamma_2(z)$ if $z \in V_2$.
$\{G_1\} \otimes \{G_2\} = \{G\}$

In all other cases define $\{G_1\} \otimes \{G_2\} = \{\Omega\}$ where Ω is a zero element with the following properties by definition :
$\{G\} \otimes \{\Omega\} = \{\Omega\} \otimes \{G\} = \{\Omega\} \otimes \{\Omega\} = \{\Omega\}$.

Let furthermore Λ be the *empty graph*, the neutral element, with the following properties :
$\{\Lambda\} \otimes \{G\} = \{G\} \otimes \{\Lambda\} = \{G\}, \{\Lambda\} \otimes \{\Lambda\} = \{\Lambda\}, \{\Lambda\} \otimes \{\Omega\} = \{\Omega\} \otimes \{\Lambda\} = \{\Omega\}$.

Then $(\{G_1\} \otimes \{G_2\}) \otimes \{G_3\} = \{G_1\} \otimes (\{G_2\} \otimes \{G_3\})$, i.e. the operation $\otimes$ is associative.

Extend $\otimes$ to sets of graphs by

$$A \otimes B = \bigcup_{G \in A, H \in B} \{G\} \otimes \{H\} \ ,$$

Then $(\{G_1\} \otimes \{G_2\}) \otimes \{G_3\} = \{G_1\} \otimes (\{G_2\} \otimes \{G_3\})$, and also for sets $(A \otimes B) \otimes D = A \otimes (B \otimes D)$, i.e. $\otimes$ is an associative operation.

Let $\mathcal{G}_\Omega = \mathcal{G} \cup \{\Omega\}$. Then the structure $\mathcal{S}_\Omega = (\mathcal{P}(\mathcal{G}_\Omega), \cup, \otimes, \emptyset, \{\Lambda\})$ is an ω-complete semiring. Therefore $\otimes$-rational, $\otimes$-linear, and $\otimes$-algebraic languages can be defined.

Now similar relations as between the structures from examples 4.5 and 4.6 can be stated between the structure $\mathcal{S} = (\mathcal{P}(\mathcal{G}_\#), \cup, \odot, \emptyset, \{\Lambda\})$ from example 4.11 and the structure from example 4.12, $\mathcal{S}_\Omega = (\mathcal{P}(\mathcal{G}_\Omega), \cup, \otimes, \emptyset, \{\Lambda\})$.

Therefore the iteration lemmata also hold for $\otimes$ languages without Ω.

□

Example 4.13 : Consider any $N_1 \subseteq (\bigcup_{c \in C} \gamma_1^{-1}(c)) \cap O_1$ and
$M_2 \subseteq (\bigcup_{c \in C} \gamma_2^{-1}(c)) \cap I_2$ with
$\forall c \in C : |\gamma_1^{-1}(c)) \cap N_1| = |\gamma_2^{-1}(c)) \cap M_2|$.

If $\gamma_1^{-1}(c)) \cap N_1 = \{x_1, \cdots, x_k\}$, $\gamma_2^{-1}(c)) \cap M_2 = \{y_1, \cdots, y_k\}$, and π_c is any permutation on $\{1, \cdots, k\}$ then identify x_i with $y_{\pi_c(i)}$ (for all colours). Let π denote the total permutation $\Pi_{c \in C} \pi_c$, i.e. the one to one mapping $N_1 \leftrightarrow M_2$. Thus $M_2 = \pi(N_1)$ and $|N_1| = |M_2|$.

Let $I = I_1 \cup (I_2 - M_2)$ and $O = (O_1 - N_1) \cup O_2$, $V = V_1 \cup (V_2 - M_2)$, $E = E_1 \cup E_2$, $\gamma(z) = \gamma_1(z)$ if $z \in V_1$, and $\gamma(z) = \gamma_2(z)$ if $z \in (V_2 - M_2)$. Define $G = (V, E, C, \gamma)$. Put any G defined in such a way into $\{G_1\} \circ \{G_2\}$.

Extend $\circ$ to sets of graphs by

$$A \circ B = \bigcup_{G \in A, H \in B} \{G\} \circ \{H\} \ .$$

$\circ$ is an associative operation.

If Λ is defined as the neutral element, i.e. $\{\Lambda\} \circ \{G\} = \{G\} \circ \{\Lambda\} = \{G\}$, then the structure $\mathcal{S} = (\mathcal{P}(\mathcal{G}), \cup, \circ, \emptyset, \{\Lambda\})$ is an ω-complete semiring, and $\circ$-rational, $\circ$-linear, $\circ$-algebraic languages can be defined.

As for the norm trivially holds : $max(\mu(G_1), \mu(G_2)) \leq \mu(G) \leq \mu(G_1) + \mu(G_2)$ for any graph $G \in \{G_1\} \circ \{G_2\}$, the iteration lemmata are valid for such $\circ$ languages.

□

5 Conclusion

It has been shown that there exists a variety of associative structures for which iteration lemmata are valid. Especially, certain operations on graphs are important for concurrent systems [8]. In another paper process algebras related to some graph structures have been considered [1].

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