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Shrink Indecomposable Fractals

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Abstract: Iterated Function Systems (IFSs) are among the best-known methods for constructing fractals. The sequence of pictures $E_0, E_1, E_2, \ldots$ generated by an IFS $\{X; f_1, f_2, \ldots, f_t\}$ converges to a unique limit $\mathcal{E}$, which is independent of the choice of starting set E_0, but completely determined by the choice of the maps f_i.
Random context picture grammars (rcpgs) are a method of syntactic picture generation. The terminals are subsets of the Euclidean plane and the replacement of variables involves the building of functions that will eventually be applied to terminals. Context is used to enable or inhibit production rules.
We show that every IFS can be simulated by an rcpg that uses inhibiting context only. Since rcpgs use context to control the sequence in which functions are applied, they can generate a wider range of fractals or, more generally, pictures than IFSs. We give an example of such a fractal. Then we show that under certain conditions the sequence of pictures generated by an rcpg converges to a unique limit.
Category: F.4.2.

1 Introduction

A method of syntactic picture generation, using random context picture grammars (rcpgs), was described and studied elsewhere [Ewert and Van der Walt 97], [Ewert and Van der Walt 98], [Ewert and Van der Walt 99], [Ewert and Van der Walt 99b]. In this paper we generalize the notion of an rcpg somewhat, retaining the name. This concept can be considered a generalization both of 2-dimensional collage grammars [Drewes, Kreowski, and Lapoire 97] and of Iterated Function Systems (IFSs).

An IFS $\{X; f_1, f_2, \ldots, f_t\}$ is an iterative method for constructing fractals from the finite set of contractive maps $f_1, f_2, \ldots, f_t$ defined on the complete metric space X. The sequence of pictures $E_0, E_1, E_2, \ldots$ generated by an IFS converges to a unique limit $\mathcal{E}$, which is independent of the choice of starting set E_0, but completely determined by the choice of the maps f_i. The method was developed principally by Barnsley and co-workers, who obtained impressively life-like images both of nature scenes and the human face [Barnsley 88], [Barnsley and Hurd 93].

First we show that any picture sequence generated by an IFS can also be generated by an rcpg that uses forbidding context only. Secondly, since rcpgs use context to control the sequence in which functions are applied, they can generate a wider range of fractals or, more generally, pictures than IFSs. We give an example of such a fractal. Then we introduce the prefix property for the sequence of pictures generated by an rcpg and show that every picture sequence that can be generated by an IFS has that property. Finally we prove our main result, namely that every sequence of pictures with the prefix property converges to a unique limit.

2 Random Context Picture Grammars

We define a random context picture grammar and illustrate the main concepts with an example, the iteration sequence of the Sierpiński gasket.

A *random context picture grammar* $G = (V_{\mathrm{N}}, V_{\mathrm{T}}, V_{\mathrm{F}}, P, (S, \epsilon))$ has a finite alphabet V of *labels*, consisting of disjoint subsets V_{N} of *variables*, V_{T} of *terminals* and V_{F} of *function identifiers*. The *productions*, finite in number, are of the form $A \to \{(A_1, \rho_1), (A_2, \rho_2), \ldots, (A_t, \rho_t)\}\ (\mathcal{P}; \mathcal{F})$, where $A \in V_{\mathrm{N}}$, $A_1, \ldots, A_t \in V_{\mathrm{N}} \cup V_{\mathrm{T}}$, $\rho_1, \ldots, \rho_t \in V_{\mathrm{F}}^*$ and $\mathcal{P}, \mathcal{F} \subseteq V_{\mathrm{N}}$. Finally, there is an *initial configuration* (S, ϵ), where $S \in V_{\mathrm{N}}$ and ϵ denotes the empty string.

A *pictorial form* Π is a finite set $\{(B_1, \varphi_1), (B_2, \varphi_2), \ldots, (B_s, \varphi_s)\}$, where $B_1, \ldots, B_s \in V_{\mathrm{N}} \cup V_{\mathrm{T}}$ and $\varphi_1, \ldots, \varphi_s \in V_{\mathrm{F}}^*$. We denote the set $\{B_1, \ldots, B_s\}$ by $l(\Pi)$.

For an rcpg G and pictorial forms Π and Γ we write $\Pi \Longrightarrow_{\mathrm{G}} \Gamma$ if there is a production $A \to \{(A_1, \rho_1), (A_2, \rho_2), \ldots, (A_t, \rho_t)\}\ (\mathcal{P}; \mathcal{F})$ in G, Π contains an element (A, φ), $l(\Pi \setminus \{(A, \varphi)\}) \supseteq \mathcal{P}$ and $l(\Pi \setminus \{(A, \varphi)\}) \cap \mathcal{F} = \emptyset$, and $\Gamma = (\Pi \setminus \{(A, \varphi)\}) \cup \{(A_1, \varphi\rho_1), (A_2, \varphi\rho_2), \ldots, (A_t, \varphi\rho_t)\}$. As usual, $\Longrightarrow_{\mathrm{G}}^*$ denotes the reflexive transitive closure of $\Longrightarrow_{\mathrm{G}}$.

A *picture* is a pictorial form Π with $l(\Pi) \subseteq V_{\mathrm{T}}$. The *gallery* $\mathcal{G}(G)$ *generated by an rcpg* G is the set of pictures Π such that $\{(S, \epsilon)\} \Longrightarrow_{\mathrm{G}}^* \Pi$.

The gallery of an rcpg G is *rendered* by specifying functions $\Psi_{\mathrm{G}} : V_{\mathrm{T}} \to \wp(\mathbf{R}^2)$ and $\Upsilon_{\mathrm{G}} : V_{\mathrm{F}} \to \mathrm{F}(\mathbf{R}^2)$, where $\mathrm{F}(\mathbf{R}^2) = \{g \mid g : \mathbf{R}^2 \to \mathbf{R}^2\}$. This yields a representation of a picture $\Pi = \{(B_1, \varphi_1), (B_2, \varphi_2), \ldots, (B_s, \varphi_s)\}$ in $\mathbf{R}^2$ by $r(\Pi) = \bigcup_{i=1}^{s} \Upsilon_{\mathrm{G}}(\varphi_i)(\Psi_{\mathrm{G}}(B_i))$, where Υ_{G} has been extended to V_{F}^* in the obvious manner, $\Upsilon_{\mathrm{G}}(\epsilon)$ representing the identity function.

If every production in G has $\mathcal{P} = \emptyset$, we call G a *random forbidding context picture grammar* (rFcpg).

Note: For the sake of convenience we write a production $A \to \{(A_1, \epsilon)\}\ (\mathcal{P}; \mathcal{F})$ as $A \to A_1\ (\mathcal{P}; \mathcal{F})$.

We illustrate these concepts with an example.

Example 1. We generate the typical iteration sequence of the Sierpiński gasket with the rFcpg $G_{\mathrm{gasket}} = (\{S, T, U, F\}, \{b\}, \{g_{\mathrm{lb}}, g_{\mathrm{rb}}, g_{\mathrm{t}}\}, P, (S, \epsilon))$, where P is the set:

$$S \to \{(T, g_{\mathrm{lb}}), (T, g_{\mathrm{rb}}), (T, g_{\mathrm{t}})\}\ (\{\}; \{U\}) \tag{1}$$

$$\begin{aligned} T \to\ & U\ (\{\}; \{S, F\})\ \mid & (2)\\ & F\ (\{\}; \{S, U, F\})\ \mid & (3)\\ & b\ (\{F\}; \{\}) & (4)\\ U \to\ & S\ (\{\}; \{T\}) & (5) \end{aligned}$$

$$F \to b\ (\{\}; \{T\}) \tag{6}$$

We give the derivation of a picture Π in $\mathcal{G}(G_{\text{gasket}})$ in detail.

$$
\begin{aligned}
& \{(S,\epsilon)\} \\
\Longrightarrow_{\mathrm{G}} & \{(T,g_{\mathrm{lb}}),(T,g_{\mathrm{rb}}),(T,g_{\mathrm{t}})\} && \text{(rule 1)} \\
\Longrightarrow^*_{\mathrm{G}} & \{(U,g_{\mathrm{lb}}),(U,g_{\mathrm{rb}}),(U,g_{\mathrm{t}})\} && \text{(thrice rule 1)} \\
\Longrightarrow^*_{\mathrm{G}} & \{(S,g_{\mathrm{lb}}),(S,g_{\mathrm{rb}}),(S,g_{\mathrm{t}})\} && \text{(thrice rule 5)} \\
\Longrightarrow^*_{\mathrm{G}} & \{(T,g_{\mathrm{lb}}g_{\mathrm{lb}}),(T,g_{\mathrm{lb}}g_{\mathrm{rb}}),(T,g_{\mathrm{lb}}g_{\mathrm{t}})\}\cup \\
& \{(T,g_{\mathrm{rb}}g_{\mathrm{lb}}),(T,g_{\mathrm{rb}}g_{\mathrm{rb}}),(T,g_{\mathrm{rb}}g_{\mathrm{t}})\}\cup \\
& \{(T,g_{\mathrm{t}}g_{\mathrm{lb}}),(T,g_{\mathrm{t}}g_{\mathrm{rb}}),(T,g_{\mathrm{t}}g_{\mathrm{t}})\} && \text{(thrice rule 1)} \\
\Longrightarrow_{\mathrm{G}} & \{(T,g_{\mathrm{lb}}g_{\mathrm{lb}}),(T,g_{\mathrm{lb}}g_{\mathrm{rb}}),(T,g_{\mathrm{lb}}g_{\mathrm{t}})\}\cup \\
& \{(T,g_{\mathrm{rb}}g_{\mathrm{lb}}),(F,g_{\mathrm{rb}}g_{\mathrm{rb}}),(T,g_{\mathrm{rb}}g_{\mathrm{t}})\}\cup \\
& \{(T,g_{\mathrm{t}}g_{\mathrm{lb}}),(T,g_{\mathrm{t}}g_{\mathrm{rb}}),(T,g_{\mathrm{t}}g_{\mathrm{t}})\} && \text{(rule 3)} \\
\Longrightarrow^*_{\mathrm{G}} & \{(b,g_{\mathrm{lb}}g_{\mathrm{lb}}),(b,g_{\mathrm{lb}}g_{\mathrm{rb}}),(b,g_{\mathrm{lb}}g_{\mathrm{t}})\}\cup \\
& \{(b,g_{\mathrm{rb}}g_{\mathrm{lb}}),(F,g_{\mathrm{rb}}g_{\mathrm{rb}}),(b,g_{\mathrm{rb}}g_{\mathrm{t}})\}\cup \\
& \{(b,g_{\mathrm{t}}g_{\mathrm{lb}}),(b,g_{\mathrm{t}}g_{\mathrm{rb}}),(b,g_{\mathrm{t}}g_{\mathrm{t}})\} && \text{(repeated application of rule 4)} \\
\Longrightarrow_{\mathrm{G}} & \{(b,g_{\mathrm{lb}}g_{\mathrm{lb}}),(b,g_{\mathrm{lb}}g_{\mathrm{rb}}),(b,g_{\mathrm{lb}}g_{\mathrm{t}})\}\cup \\
& \{(b,g_{\mathrm{rb}}g_{\mathrm{lb}}),(b,g_{\mathrm{rb}}g_{\mathrm{rb}}),(b,g_{\mathrm{rb}}g_{\mathrm{t}})\}\cup \\
& \{(b,g_{\mathrm{t}}g_{\mathrm{lb}}),(b,g_{\mathrm{t}}g_{\mathrm{rb}}),(b,g_{\mathrm{t}}g_{\mathrm{t}})\} && \text{(rule 6)}
\end{aligned}
$$

Let $\Upsilon_{\mathrm{G}}(g_{\mathrm{lb}}) = (x,y) \to \left(\frac{x}{2}, \frac{y}{2}\right)$, $\Upsilon_{\mathrm{G}}(g_{\mathrm{rb}}) = (x,y) \to \left(\frac{x}{2}+\frac{1}{2}, \frac{y}{2}\right)$ and $\Upsilon_{\mathrm{G}}(g_{\mathrm{t}}) = (x,y) \to \left(\frac{x}{2}+\frac{1}{4}, \frac{y}{2}+\frac{\sqrt{3}}{4}\right)$.

Then $r(\Pi) = \bigcup_{i=1}^{9} \Upsilon_{\mathrm{G}}(\varphi_i)(\Psi_{\mathrm{G}}(b))$, where $\Upsilon_{\mathrm{G}}(\varphi_1) = (x,y) \to \left(\frac{1}{2}\times\frac{x}{2}, \frac{1}{2}\times\frac{y}{2}\right)$, $\Upsilon_{\mathrm{G}}(\varphi_2) = (x,y) \to \left(\frac{1}{2}\left(\frac{x}{2}+\frac{1}{2}\right), \frac{1}{2}\times\frac{y}{2}\right)$, $\Upsilon_{\mathrm{G}}(\varphi_3) = (x,y) \to \left(\frac{1}{2}\left(\frac{x}{2}+\frac{1}{4}\right), \frac{1}{2}\left(\frac{y}{2}+\frac{\sqrt{3}}{4}\right)\right)$, ...

Let $\Psi_{\mathrm{G}}(b)$ be the dark triangle with vertices $\left\{(0,0),(1,0),\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)\right\}$. Then $r(\Pi)$ represents the picture in [Fig. 1]. Alternatively, let $\Psi_{\mathrm{G}}(b)$ be the dark square determined by the vertices $\{(0,0),(1,0),(1,1)\}$. Then $r(\Pi)$ represents [Fig. 2].

3 Iterated Function Systems

An *Iterated Function System* $\{X; f_1, f_2, \ldots, f_t\}$ or $\{X, f_{1-t}\}$ is a pair consisting of a complete metric space X together with a finite set of contractive maps $f_i : X \to X$, $1 \le i \le t$. [Hoggar 92] contains an extensive treatment of IFSs.

Let $\mathcal{H}(X)$ be the set of all nonempty compact subsets of X. For $E \in \mathcal{H}(X)$, let $F(E) = f_1(E) \cup f_2(E) \cup \ldots \cup f_t(E)$. By repeated application of F to E, we obtain a sequence in $\mathcal{H}(X)$, $E_0 = E, E_1 = F(E_0), E_2 = F(E_1), \ldots$. We show that every such sequence can be generated by an rFcpg.

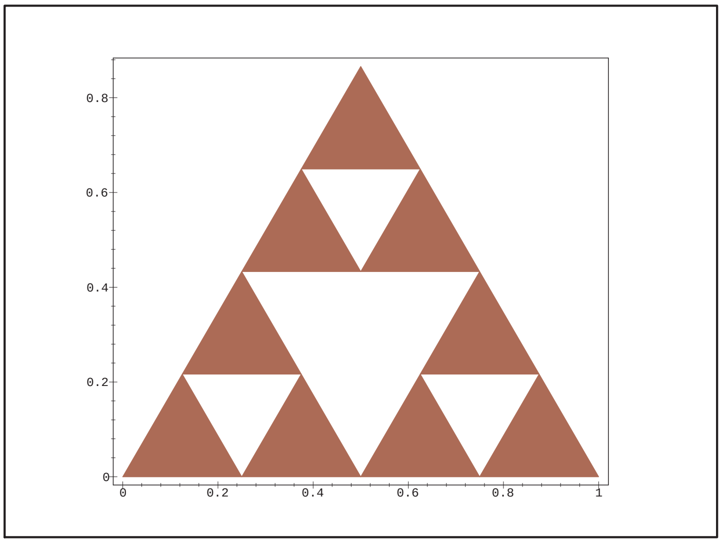

Figure 1: $\Psi_{\mathrm{G}}(\{b\})$ is a dark triangle

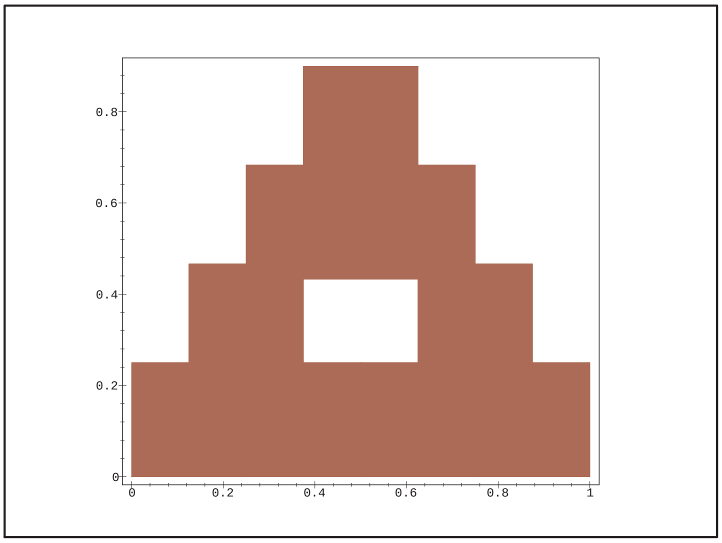

Figure 2: $\Psi_{\mathrm{G}}(\{b\})$ is a dark square

Lemma 1. *Let $\{X, f_{1-t}\}$ be an IFS. Then there is an rFcpg G such that for every integer $l \geq 1$, G generates the set $\{(a, \varphi_1^l), (a, \varphi_2^l), \ldots, (a, \varphi_{t^l}^l)\}$, where the φ_i^l are all t^l possible sequences of length l of the f_j.*

Proof. Let $G = (\{S, I, T, U, F\}, \{a\}, \{f_1, f_2, \ldots, f_t\}, P, (S, \epsilon))$, where P is the set:

$$
\begin{aligned}
S &\to \{(I, f_1), (I, f_2), \ldots, (I, f_t)\} \\
I &\to \{(T, f_1), (T, f_2), \ldots, (T, f_t)\} \ (\{\}; \{F, U\}) \ | \\
&\quad\ F \ (\{\}; \{T, U\}) \\
T &\to U \ (\{\}; \{I\}) \\
U &\to I \ (\{\}; \{T\}) \\
F &\to a \ (\{\}; \{I\})
\end{aligned}
$$

□

Example 2. We obtain the iteration sequence of the Sierpiński gasket with the IFS $\{\mathbf{R}^2; g_{\mathrm{lb}}, g_{\mathrm{rb}}, g_{\mathrm{t}}\}$, where $g_{\mathrm{lb}} : (x, y) \to \left(\frac{x}{2}, \frac{y}{2}\right)$, $g_{\mathrm{rb}} : (x, y) \to \left(\frac{x}{2} + \frac{1}{2}, \frac{y}{2}\right)$ and $g_{\mathrm{t}} : (x, y) \to \left(\frac{x}{2} + \frac{1}{4}, \frac{y}{2} + \frac{\sqrt{3}}{4}\right)$.

For any $E \in \mathcal{H}(\mathbf{R}^2)$, $F(E) = g_{\mathrm{lb}}(E) \cup g_{\mathrm{rb}}(E) \cup g_{\mathrm{t}}(E)$. Let $E_0 = E$. Then $E_1 = F(E_0) = g_{\mathrm{lb}}(E_0) \cup g_{\mathrm{rb}}(E_0) \cup g_{\mathrm{t}}(E_0)$, $E_2 = F(E_1) = g_{\mathrm{lb}}g_{\mathrm{lb}}(E_0) \cup g_{\mathrm{lb}}g_{\mathrm{rb}}(E_0) \cup g_{\mathrm{lb}}g_{\mathrm{t}}(E_0) \cup g_{\mathrm{rb}}g_{\mathrm{lb}}(E_0) \cup g_{\mathrm{rb}}g_{\mathrm{rb}}(E_0) \cup g_{\mathrm{rb}}g_{\mathrm{t}}(E_0) \cup g_{\mathrm{t}}g_{\mathrm{lb}}(E_0) \cup g_{\mathrm{t}}g_{\mathrm{rb}}(E_0) \cup g_{\mathrm{t}}g_{\mathrm{t}}(E_0), \ldots$. When we choose E_0 to be a dark triangle, respectively, a dark square, E_2 is represented by [Fig. 1] and [Fig. 2], respectively.

To this IFS corresponds the rFcpg $G = (\{S, I, T, U, F\}, \{a\}, \{g_{\mathrm{lb}}, g_{\mathrm{rb}}, g_{\mathrm{t}}\}, P, (S, \epsilon))$, where P is the set:

$$
\begin{aligned}
S &\to \{(I, g_{\mathrm{lb}}), (I, g_{\mathrm{rb}}), (I, g_{\mathrm{t}})\} \\
I &\to \{(T, g_{\mathrm{lb}}), (T, g_{\mathrm{rb}}), (T, g_{\mathrm{t}})\} \ (\{\}; \{F, U\}) \ | \\
&\quad\ F \ (\{\}; \{T, U\}) \\
T &\to U \ (\{\}; \{I\}) \\
U &\to I \ (\{\}; \{T\}) \\
F &\to a \ (\{\}; \{I\})
\end{aligned}
$$

G generates the pictorial forms $\{(a, g_{\mathrm{lb}}), (a, g_{\mathrm{rb}}), (a, g_{\mathrm{t}})\}$, $\{(a, g_{\mathrm{lb}}g_{\mathrm{lb}}), (a, g_{\mathrm{lb}}g_{\mathrm{rb}}), (a, g_{\mathrm{lb}}g_{\mathrm{t}})\} \cup \{(a, g_{\mathrm{rb}}g_{\mathrm{lb}}), (a, g_{\mathrm{rb}}g_{\mathrm{rb}}), (a, g_{\mathrm{rb}}g_{\mathrm{t}})\} \cup \{(a, g_{\mathrm{t}}g_{\mathrm{lb}}), (a, g_{\mathrm{t}}g_{\mathrm{rb}}), (a, g_{\mathrm{t}}g_{\mathrm{t}})\}, \ldots$.

Since rcpgs use context to control the sequence in which functions are applied, they can generate a wider range of pictures than IFSs. An example of such a picture set is $\mathcal{G}_{\mathrm{trail}}$, which is described below. $\mathcal{G}_{\mathrm{trail}}$ cannot be generated by an rFcpg, as becomes clear when inspecting the proof in [Ewert and Van der Walt 99b], and therefore also not by an IFS.

$\mathcal{G}_{\text{trail}} = \{\Theta_1, \Theta_2, \ldots\}$, where Θ_1, Θ_2 and Θ_3 are shown in [Fig. 3], [Fig. 4] and [Fig. 5], respectively. For the sake of clarity, an enlargement of the lower lefthand ninth of Θ_3 is given in [Fig. 6].

For $i = 2, 3, \ldots$, Θ_i is obtained by dividing each dark square in Θ_{i-1} into four and placing a copy of Θ_1, modified so that it has exactly $i+2$ dark squares, all on the diagonal, into each quarter.

The modification of Θ_1 is effected in its middle dark square only and proceeds in detail as follows: The square is divided into four and the newly-created lower lefthand quarter coloured dark. The newly-created upper righthand quarter is again divided into four and its lower lefthand quarter coloured dark. This successive quartering of the upper righthand square is repeated until a total of $i-1$ dark squares have been created, then the upper righthand square is also coloured dark. The new dark squares thus get successively smaller, except for the last two, which are of equal size.

$\mathcal{G}_{\text{trail}}$ is generated by the rcpg $G = (\{S\} \cup \{A, L, R, T, A_{\text{t}}\} \cup \{M\} \cup \{X_{\text{e}}, X_{\text{t}}, E_{\text{x}}\} \cup \{A_{\text{e}}, Z_{\text{x}}\} \cup \{X\} \cup \{L_{\text{x}}\} \cup \{Y_{\text{e}}, Y_{\text{t}}, B\} \cup \{B_{\text{e}}\} \cup \{E_{\text{y}}, Y, Z_{\text{y}}\} \cup \{L_{\text{y}}\}, \{g_{14}, g_{24}, g_{34}, g_{44}, g_{19}, g_{29}, g_{39}, g_{49}, g_{59}, g_{69}, g_{79}, g_{89}, g_{99}\}, \{b, w\}, P, (S, \epsilon))$, where P is the set:

$$
\begin{aligned}
S &\to \{(L, g_{19}), (w, g_{29}), (w, g_{39}), (w, g_{49}), (A, g_{59}), (w, g_{69}), (w, g_{79})\} \cup \\
&\quad\ \{(w, g_{89}), (R, g_{99})\}
\end{aligned}
$$

$$
\begin{aligned}
A &\to A_{\text{t}}\ (\{\}; \{B, B_{\text{e}}, A_{\text{e}}, A_{\text{t}}\}) \\
A &\to b\ (\{A_{\text{t}}\}; \{\}) \\
L &\to b\ (\{A_{\text{t}}\}; \{\}) \\
R &\to b\ (\{A_{\text{t}}\}; \{\}) \\
T &\to b\ (\{A_{\text{t}}\}; \{\}) \\
A_{\text{t}} &\to b\ (\{\}; \{A, L, R, T\})
\end{aligned}
$$

$$
\begin{aligned}
A &\to \{(A, g_{14}), (M, g_{24}), (M, g_{34}), (A_{\text{e}}, g_{44})\}\ (\{\}; \{B, B_{\text{e}}, A_{\text{e}}, A_{\text{t}}\}) \\
L &\to \{(M, g_{14}), (M, g_{24}), (M, g_{34}), (M, g_{44})\}\ (\{A_{\text{e}}\}; \{\}) \\
R &\to \{(M, g_{14}), (M, g_{24}), (M, g_{34}), (M, g_{44})\}\ (\{A_{\text{e}}\}; \{L_{\text{x}}\}) \\
T &\to \{(M, g_{14}), (M, g_{24}), (M, g_{34}), (M, g_{44})\}\ (\{A_{\text{e}}\}; \{\})
\end{aligned}
$$

$$
\begin{aligned}
M &\to (\{(L_{\text{x}}, g_{19}), (w, g_{29}), (w, g_{39}), (w, g_{49}), (X_{\text{e}}, g_{59}), (w, g_{69}), (w, g_{79})\} \cup \\
&\quad\ \{(w, g_{89}), (R, g_{99})\})\ (\{A_{\text{e}}\}; \{L, R, T, L_{\text{x}}\})
\end{aligned}
$$

$$
\begin{aligned}
X_{\text{e}} &\to \{(X_{\text{t}}, g_{14}), (w, g_{24}), (w, g_{34}), (X_{\text{e}}, g_{44})\}\ (\{A\}; \{X_{\text{t}}, E_{\text{x}}\}) \\
X_{\text{t}} &\to X\ (\{E_{\text{x}}\}; \{\}) \\
A &\to E_{\text{x}}\ (\{X_{\text{t}}\}; \{E_{\text{x}}\}) \\
E_{\text{x}} &\to Z_{\text{x}}\ (\{\}; \{X_{\text{t}}\})
\end{aligned}
$$

$$
\begin{aligned}
A_{\text{e}} &\to M\ (\{\}; \{A, E_{\text{x}}\}) \\
Z_{\text{x}} &\to \{(M, g_{14}), (M, g_{24}), (M, g_{34}), (M, g_{44})\}\ (\{Z_{\text{x}}\}; \{A_{\text{e}}\}) \\
Z_{\text{x}} &\to M\ (\{\}; \{Z_{\text{x}}, A_{\text{e}}\})
\end{aligned}
$$

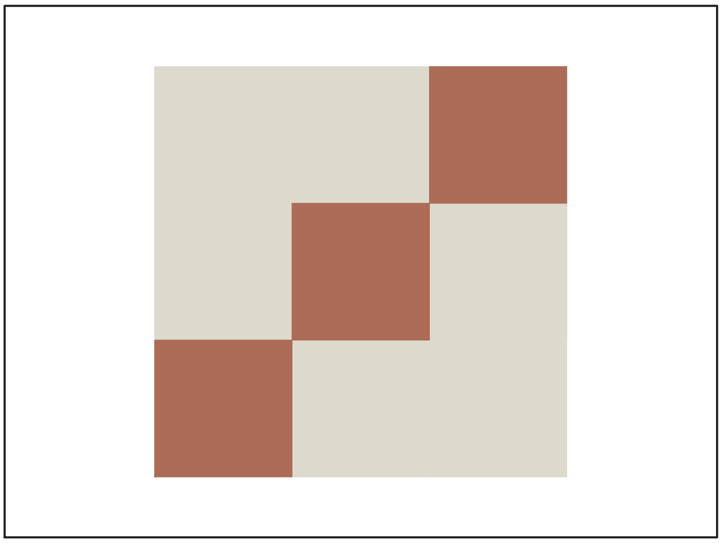

Figure 3: Θ_1 of $\mathcal{G}_{\text{trail}}$

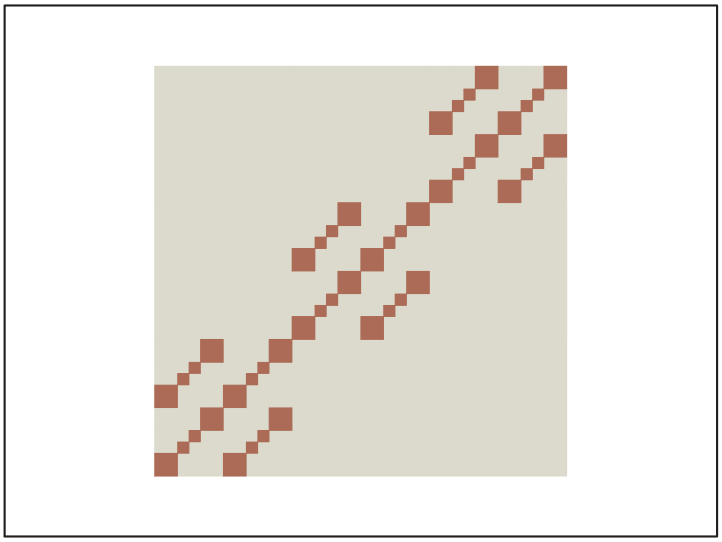

Figure 4: Θ_2 of $\mathcal{G}_{\text{trail}}$

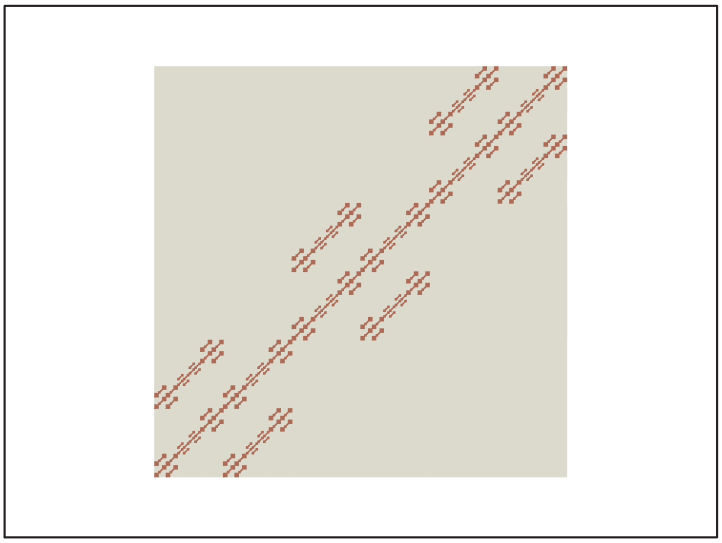

Figure 5: Θ_3 of $\mathcal{G}_{\mathrm{trail}}$

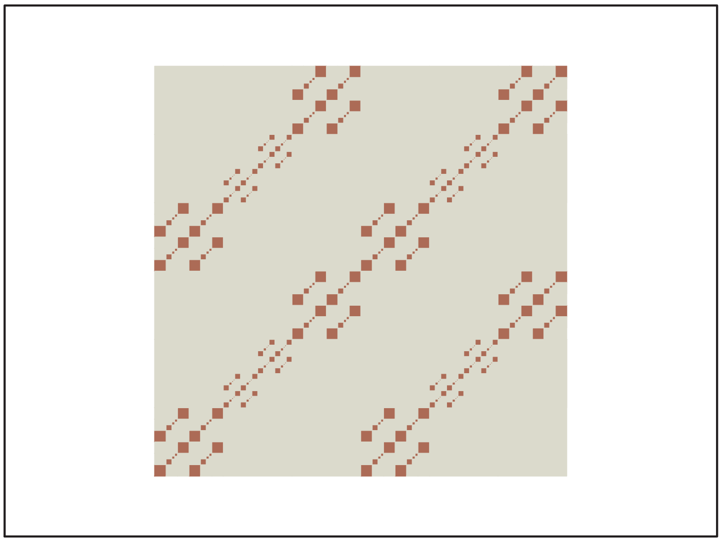

Figure 6: Bottom lefthand ninth of Θ_3 of $\mathcal{G}_{\mathrm{trail}}$ enlarged

$$
\begin{aligned}
X &\to B\ (\{\};\{A_\mathrm{e}\})\\
X_\mathrm{e} &\to B_\mathrm{e}\ (\{\};\{A_\mathrm{e}\})\\
L_\mathrm{x} &\to L\ (\{\};\{A_\mathrm{e}, Z_\mathrm{x}, X, X_\mathrm{e}\})\\
M &\to (\{(L_\mathrm{y}, g_{19}), (w, g_{29}), (w, g_{39}), (w, g_{49}), (Y_\mathrm{e}, g_{59}), (w, g_{69}), (w, g_{79})\} \cup\\
&\quad\ \{(w, g_{89}), (R, g_{99})\})\ (\{B_\mathrm{e}\};\{L_\mathrm{x}, L_\mathrm{y}\})\\
Y_\mathrm{e} &\to \{(Y_\mathrm{t}, g_{14}), (w, g_{24}), (w, g_{34}), (Y_\mathrm{e}, g_{44})\}\ (\{B\};\{Y_\mathrm{t}, E_\mathrm{y}\})\\
Y_\mathrm{t} &\to Y\ (\{E_\mathrm{y}\};\{\})\\
B &\to E_\mathrm{y}\ (\{Y_\mathrm{t}\};\{E_\mathrm{y}\})\\
E_\mathrm{y} &\to Z_\mathrm{y}\ (\{\};\{Y_\mathrm{t}\})\\
Y &\to T\ (\{\};\{Y_\mathrm{t}, E_\mathrm{y}, B\})\\
Y_\mathrm{e} &\to T\ (\{\};\{Y_\mathrm{t}, E_\mathrm{y}, B\})\\
Z_\mathrm{y} &\to B\ (\{\};\{Y, Y_\mathrm{e}\})\\
L_\mathrm{y} &\to L\ (\{\};\{Z_\mathrm{y}, Y_\mathrm{e}\})\\
B &\to A\ (\{\};\{M, L_\mathrm{y}\})\\
B_\mathrm{e} &\to A\ (\{\};\{M, L_\mathrm{y}\})
\end{aligned}
$$

$\mathcal{G}_\mathrm{trail}$ is rendered by defining $\Upsilon_\mathrm{G}(g_{14}) = (x,y) \to \left(\frac{x}{2}, \frac{y}{2}\right)$, $\Upsilon_\mathrm{G}(g_{24}) = (x,y) \to \left(\frac{x}{2}+\frac{1}{2}, \frac{y}{2}\right)$, $\Upsilon_\mathrm{G}(g_{34}) = (x,y) \to \left(\frac{x}{2}, \frac{y}{2}+\frac{1}{2}\right)$, $\Upsilon_\mathrm{G}(g_{44}) = (x,y) \to \left(\frac{x}{2}+\frac{1}{2}, \frac{y}{2}+\frac{1}{2}\right)$, $\Upsilon_\mathrm{G}(g_{19}) = (x,y) \to \left(\frac{x}{3}, \frac{y}{3}\right)$, $\Upsilon_\mathrm{G}(g_{29}) = (x,y) \to \left(\frac{x}{3}+\frac{1}{3}, \frac{y}{3}\right)$, $\Upsilon_\mathrm{G}(g_{39}) = (x,y) \to \left(\frac{x}{3}+\frac{2}{3}, \frac{y}{3}\right)$, $\Upsilon_\mathrm{G}(g_{49}) = (x,y) \to \left(\frac{x}{3}, \frac{y}{3}+\frac{1}{3}\right)$, $\Upsilon_\mathrm{G}(g_{59}) = (x,y) \to \left(\frac{x}{3}+\frac{1}{3}, \frac{y}{3}+\frac{1}{3}\right)$, $\Upsilon_\mathrm{G}(g_{69}) = (x,y) \to \left(\frac{x}{3}+\frac{2}{3}, \frac{y}{3}+\frac{1}{3}\right)$, $\Upsilon_\mathrm{G}(g_{79}) = (x,y) \to \left(\frac{x}{3}, \frac{y}{3}+\frac{2}{3}\right)$, $\Upsilon_\mathrm{G}(g_{89}) = (x,y) \to \left(\frac{x}{3}+\frac{1}{3}, \frac{y}{3}+\frac{2}{3}\right)$ and $\Upsilon_\mathrm{G}(g_{99}) = (x,y) \to \left(\frac{x}{3}+\frac{2}{3}, \frac{y}{3}+\frac{2}{3}\right)$, moreover, $\Psi_\mathrm{G}(b)$ as the dark square determined by the vertices $\{(0,0), (1,0), (1,1)\}$ and $\Psi_\mathrm{G}(w)$ as the light square determined by the vertices $\{(0,0), (1,0), (1,1)\}$.

4 Shrink Indecomposable Fractals

According to Banach's Fixed Point Theorem, also known as the Contraction Mapping Theorem, the map F associated with an IFS $\{X, f_{1-t}\}$ has a unique fixed point $\mathcal{E}$ (i.e., there exists a unique $\mathcal{E} \in \mathcal{H}(X)$ such that $F(\mathcal{E}) = \mathcal{E}$) and the sequence $E_0, E_1, E_2, \ldots$ converges to $\mathcal{E}$. $\mathcal{E}$ is independent of the choice of starting set E_0, but completely determined by the choice of the f_i.

Since $\mathcal{E} = F(\mathcal{E}) = f_1(\mathcal{E}) \cup f_2(\mathcal{E}) \cup \ldots \cup f_t(\mathcal{E})$, we may call a fractal generated by an IFS *shrink decomposable*. We now present a theorem that could be considered a generalization of the contraction mapping theorem and that, similarly to the latter, guarantees the existence and construction of fractals. The range of fractals constructed in this way is wider than in the case of Banach's theorem; we call those that cannot be generated by an IFS *shrink indecomposable*. An example of such a fractal is the limit set Θ of the gallery $\mathcal{G}_\mathrm{trail}$ of [Section 3].

Let therefore X be a complete metric space with metric d. Let $\Phi = \{f_1, \ldots, f_t\}$ be a finite set of contractive maps $f_i : X \to X$, i.e., for all $x, y \in X$, $d(f_i(x), f_i(y)) \leq r_i d(x,y)$ for some r_i, $0 \leq r_i < 1$. Let $r = \max(r_1, \ldots, r_t)$.

As before, let $\mathcal{H}(X)$ be the set of all nonempty compact subsets of X. For $E \in \mathcal{H}(X)$, let $F(E) = \varphi_1(E) \cup \varphi_2(E) \cup \ldots \cup \varphi_p(E)$, where $\varphi_i \in \Phi^+$. We call F a *collage map*; this is a slightly more general usage of the term than commonly found in literature. We call the φ_i the *constituents* of F.

Let b be any point and B any set in X. Then the distance between b and B is given by $d(b, B) = \min_{b' \in B} d(b, b')$. This minimum exists [Hoggar 92]. The *Hausdorff distance* between elements of $\mathcal{H}(X)$ is then defined as

$$d(B', B) = \max\left(\max_{b \in B'} d(b, B), \max_{b \in B} d(b, B')\right) \quad .$$

The Hausdorff distance is a metric on $\mathcal{H}(X)$ [Hoggar 92]. We then have:

Lemma 2. *A collage map on $\mathcal{H}(X)$ is a contractive map on $\mathcal{H}(X)$.*

Let $\varphi_1, \varphi_2 \in \Phi^+$. φ_1 is called a *proper prefix* of φ_2 if $\varphi_2 = \varphi_1 f_{i_1} f_{i_2} \ldots f_{i_k}$ for some $k \geq 1$. A sequence $F_1, F_2, \ldots$ of collage maps is said to have the *prefix property* if, for all $1 \leq m \leq n$, every constituent of F_m is a proper prefix of a constituent of F_n and every constituent of F_n has a constituent of F_m for a proper prefix. For example, any rFcpg that simulates an IFS using the construction of [Lemma 1] generates a sequence of collage maps with the prefix property. Moreover, it is easily seen that the sequence of collage maps representing $\Theta_1, \Theta_2, \ldots$ of $\mathcal{G}_{\text{trail}}$ has the prefix property. However, it is unknown whether it is decidable if an arbitrary rcpg generates sequences of collage maps with the prefix property.

We can now formulate a generalization of the Banach Fixed Point Theorem:

Theorem 3. *Let $F_1, F_2, \ldots$ be a sequence of collage maps with the prefix property. Let $E_0 \in \mathcal{H}(X)$. Then the sequence $E_0, E_1 = F_1(E_0), E_2 = F_2(E_0), \ldots$ converges to a limit $\mathcal{E} \in \mathcal{H}(X)$. Moreover, we have the following estimates:*

1. $d(E_n, \mathcal{E}) \leq \frac{r}{1-r} d(E_{n-1}, E_n), n \geq 1$
2. $d(E_n, \mathcal{E}) \leq \frac{r^n}{1-r} d(E_0, E_1), n \geq 0$

Proof. Let $a = \max_{f \in \Phi} d(f(E_0), E_0)$. Let $n > m \geq 1$. The assertion of the theorem follows from

$$d(E_n, E_m) \leq \frac{r^m}{1-r} a \quad ,$$

which we establish using the following known or easily proven facts:

1. For $f \in \Phi$ and $E_i, E_j \in \mathcal{H}(X)$, $d(f(E_i), f(E_j)) \leq r d(E_i, E_j)$.
2. For $\varphi \in \Phi^+$ and $E_i, E_j \in \mathcal{H}(X)$, $d(\varphi(E_i), \varphi(E_j)) \leq r^{|\varphi|} d(E_i, E_j)$.
3. For $\varphi \in \Phi^+$, $d(\varphi(E_0), E_0) \leq \frac{a}{1-r}$.

Proof. Suppose $\varphi = f_{i_1} f_{i_2} \ldots f_{i_s}$, for $f_{i_j} \in \Phi$ and some integer s. Then

$$\begin{aligned}
& d(f_{i_1} f_{i_2} \ldots f_{i_s}(E_0), E_0) \\
& \leq d(f_{i_1}(E_0), E_0) + d(f_{i_1} f_{i_2}(E_0), f_{i_1}(E_0)) \\
& \quad + d(f_{i_1} f_{i_2} f_{i_3}(E_0), f_{i_1} f_{i_2}(E_0)) + \ldots \\
& \quad + d(f_{i_1} f_{i_2} \ldots f_{i_s}(E_0), f_{i_1} f_{i_2} \ldots f_{i_{s-1}}(E_0)) \\
& \leq a(1 + r + r^2 + \ldots + r^s + r^{s+1} + \ldots) \\
& = a \frac{1}{1-r}
\end{aligned}$$

$\square$

4. For $E_i, E_j, E_k, E_l \in \mathcal{H}(X)$, $d(E_i \cup E_j, E_k \cup E_l) \le \max(d(E_i, E_k), d(E_j, E_l))$. [Hoggar 92]

Now suppose

$$F_m = \varphi_1 \cup \ldots \cup \varphi_p$$

and

$$F_n = \varphi_1 (\mu_{11} \cup \ldots \cup \mu_{1q_1}) \cup \ldots \cup \varphi_p (\mu_{p1} \cup \ldots \cup \mu_{pq_p}) \quad .$$

Then

$$\begin{aligned}
&d(E_n, E_m) \\
&= d(\varphi_1 (\mu_{11} \cup \ldots \cup \mu_{1q_1})(E_0) \cup \ldots \cup \varphi_p (\mu_{p1} \cup \ldots \cup \mu_{pq_p})(E_0), \\
&\quad \varphi_1 (E_0) \cup \ldots \cup \varphi_p (E_0)) \\
&\le \max_j \left(d\left(\varphi_j \left(\mu_{j1} \cup \ldots \cup \mu_{jq_j}\right)(E_0), \varphi_j (E_0)\right)\right) \\
&\le r^m \max_j \left(d\left(\left(\mu_{j1} \cup \ldots \cup \mu_{jq_j}\right)(E_0), E_0\right)\right) \\
&\le r^m \max_j \max_k d(\mu_{jk}(E_0), E_0) \\
&\le r^m \frac{a}{1-r} \\
&= a \frac{r^m}{1-r}
\end{aligned}$$

□

5 Conclusion

We showed that any IFS can be simulated by an rFcpg. Moreover, we gave an example of a fractal that can be generated by an rcpg, but not by any IFS. Then we introduced the prefix property for the picture sequence generated by an rcpg and proved that every sequence with this property converges to a unique limit.

References

[Barnsley 88] Barnsley, M. F.: "Fractals Everywhere"; Academic Press, Boston / Massachusetts 1988.

[Barnsley and Hurd 93] Barnsley, M. F., Hurd, L. P.: "Fractal Image Compression"; Peters, Wellesley, Massachusetts 1993.

[Drewes, Kreowski, and Lapoire 97] Drewes, F., Kreowski, H.-J., and Lapoire, D.: "Criteria to Disprove Context-Freeness of Collage Languages"; Proc. FCT'97, Springer, Berlin / Heidelberg / New York 1997, 169—178.

[Ewert and Van der Walt 97] Ewert, S., Van der Walt, A.: "Generating Pictures Using Random Forbidding Context"; International Journal of Pattern Recognition and Artificial Intelligence, 12, 7 1998, 939—950.

[Ewert and Van der Walt 98] Ewert, S., Van der Walt, A.: "Random Context Picture Grammars"; Publicationes Mathematicae, to appear.

[Ewert and Van der Walt 99] Ewert, S., Van der Walt, A.: "Generating Pictures Using Random Permitting Context"; International Journal of Pattern Recognition and Artificial Intelligence, 13, 3 1999, 339—355.

[Ewert and Van der Walt 99b] Ewert, S., Van der Walt, A.: "A Hierarchy Result for Random Forbidding Context Picture Grammars"; International Journal of Pattern Recognition and Artificial Intelligence, to appear.

[Hoggar 92] Hoggar, S. G.: "Mathematics for Computer Graphics"; Cambridge University Press, Cambridge, Great Britain 1992.