Journal of Universal Computer Science, vol. 5, no. 8 (1999), 471-481
submitted: 14/8/99, accepted: 20/8/99, appeared: 28/8/99 © Springer Pub. Co.

Laha Distribution: Computer Generation and Applications to Life Time Modelling

Ileana Popescu
(University of Bucharest,
Faculty of Mathematics, Department of Computer Science,
14, Academiei St., RO-70109 Bucharest, Romania
Email: ileanap@fx.ro)

Monica Dumitrescu
(University of Bucharest,
Faculty of Mathematics, Department of Probability and Statistics,
14, Academiei St., RO-70109 Bucharest, Romania
Email: mdumi@pro.math.unibuc.ro)

Abstract: Laha distribution has been introduced in 1958 as an example of a non-normal distribution where the quotient follows the Cauchy law. In this paper we present two procedures for the computer generation of this distribution and we discuss its applications to life time modelling.

Key Words: Computer generation, composition procedure, ratio of uniforms method, life time.

Category: G.3.

1 Introduction

The evolvement of actual systems is influenced by random factors, hence their effect must be included in the simulation model. One of the most important issues in numerical simulation and Monte Carlo methods is the computer generation of random samples for some random variables or stochastic processes. The construction of effective algorithms for generation of random variables is a "must" in simulation models. Such algorithms are available for most of the statistical distributions.

Laha distribution has been introduced in 1958 [4], as an example of a non-normal distribution where the quotient follows the Cauchy law. Others interesting properties of Laha distribution can be identified. Therefore the interest for its generation is fully justified.

In this paper we present algorithms for generation of Laha distribution and we discuss its applications to life time modelling. In Section 2 of this paper we present properties of the original Laha distribution, which we denote by $L(0,1)$; an extension to a family of distributions, $L(\mu,\sigma^2)$, $\mu \in R$, $\sigma^2 > 0$ is considered too. In Section 3, the computer generation of the $L(0,1)$ distribution, (denoted by L) and of its version L^+, concentrated on $[0,\infty)$, are discussed. We present two generation methods: the composition procedure (using a mixing between a uniform distribution on $[0,\alpha]$ and a Student distribution $t(r)$ on (α,∞), $1 \le r \le 4$), and the ratio of uniforms method. Basic algorithms and simulation results are presented. In Section 4 we emphasize a property of Laha distribution, which allows us to consider an L^+ distributed random variable as the life time of a manufacturing system, with a non-monotonous, positive failure intensity, characterized by the existence of a critical failure interval (or a running in period).

2 Definitions and properties

Definitions 1) *A random variable X has a Laha distribution $L(0,1)$ if its distribution is absolutely continuous with respect to the Lebesgue measure, and its probability density is*

$$f(x) = \frac{\sqrt{2}}{\pi(1+x^4)},\ x \in R. \tag{1}$$

2) *A random variable X has a Laha distribution $L(\mu, \sigma^2)$, with parameters $\mu \in R, \sigma^2 > 0$, if its probability density is*

$$f(x;\mu,\sigma^2) = \frac{(\sigma^2)^{3/2}\sqrt{2}}{\pi\left[\sigma^4 + (x-\mu)^4\right]},\ x \in R.$$

The following property has been established by Laha [3]; it shows a similarity to the normal distribution $N(0,1)$.

Proposition 1 *Let X_1, X_2 be two independent, identical distributed random variables, with the distribution $L(0,1)$. Then, the ratio X_1/X_2 has a Cauchy distribution.*

There are several other resemblances and several differences between Laha and normal distributions, easy to see in the following result, which is obtained by direct calculation.

Proposition 2 *Let X be a random variable, with a $L(\mu,\sigma^2)$ distribution. Then $E(X) = \mu$, $Var(X) = \sigma^2$, but the moments $E(X^p)$ do not exist for $p \geq 3$. The characteristic function of $L(\mu,\sigma^2)$ distribution is*

$$\varphi(t;\mu,\sigma^2) = \exp\{it\mu - \frac{\sigma\sqrt{2}}{2} \mid t \mid\} \left[\cos\left(\frac{\sigma\sqrt{2}}{2}t\right) + \sin\left(\frac{\sigma\sqrt{2}}{2} \mid t \mid\right)\right],\ t \in R.$$

We notice that $\varphi(t;\mu,\sigma^2)$ is equal to the product of the characteristic function of a Cauchy distribution of parameters $\left(\mu, \frac{\sigma\sqrt{2}}{2}\right)$ and the real function $g(t;\sigma^2) = \cos\left(\frac{\sigma\sqrt{2}}{2}t\right) + \sin\left(\frac{\sigma\sqrt{2}}{2} \mid t \mid\right)$, $t \in R$.

3 Computer generation

For computer generation of a random variable with a $L(0,1)$ distribution, the **inverse method** is inappropriate, because the distribution function is rather complicated,

$$F_L(x)=\frac{1}{4\pi}\ln\frac{x^2+1+\sqrt{2}x}{x^2+1-\sqrt{2}x}+\frac{1}{2\pi}\left(\arctan\left(\sqrt{2}x+1\right)+\arctan\left(\sqrt{2}x-1\right)\right)+\frac{1}{2},$$

and the equation $F_L(X)=U$ is difficult to solve.

We notice that the L distribution is symmetrical, hence it will be enough to generate **Laha distribution concentrated on** $[0,\infty)$, denoted L^+, which has the probability density

$$f_{L^+}(x)=\frac{2\sqrt{2}}{\pi}\cdot\frac{1}{1+x^4},\ x\geq 0. \tag{2}$$

Remark 1 *If X is a random variable with L^+ distribution, then $E(X)=\frac{\sqrt{2}}{2}$ and $Var(X)=\frac{1}{2}$.*

Remark 2 *If X is a random variable with L^+ distribution, then a variable Y with L distribution is obtained by means of the random sign:*
(i) Generate X, a random variable with L^+ distribution
(ii) Generate V, a random variable uniformly distributed on $(0,1)$
(iii) If $V-0.5\leq 0$, then consider $Y=-X$, else consider $Y=X$.

3.1 Computer generation of L^+ by the composition procedure

The composition procedure is used to generate random variables for which the density function may be written as a mixing of several probability densities $g_i(x), 1\leq i\leq s$. That is,

$$f(x)=\sum_{i=1}^{s}p_i g_i(x),\ 0<p_i<1 \text{ for } 1\leq i\leq s,\ \sum p_i=1.$$

If we denote by N the discrete random variable with probability distribution $P(N=i)=p_i$, $1\leq i\leq s$, and by Y_N a random variable with density function $g_N(x)$, then $X=Y_i$ with probability p_i. When the random variables Y_N, $1\leq N\leq s$ are easy to generate, the simulation of X is obtained through a discrete mixing procedure. For the simulation of a L^+ distribution we use Bucher's method [2] for the generation of $L^+_{|[0,\alpha]}$ (considering a variable uniformly distributed on $[0,\alpha]$), and a rejection method for the generation of $L^+_{|(\alpha,\infty)}$ (considering a variable following a Student distribution).

Proposition 3 *Let us suppose that, for a specified probability density $v(x)$, the following hypotheses are true:*

(1) *The equation $f_{L^+}(x)=v(x)$ has a unique solution $x=\alpha$, and for $x>\alpha$ we have $v(x)>f_{L^+}(x)$.*

(2) *There exists a decomposition*

$$f_{L^+}(x) = p_1 f_1(x) + p_2 f_2(x),$$

with

$$p_1 = p_1(\alpha) = \int_0^\alpha f_{L^+}(x)\,dx = F_{L^+}(\alpha),\ p_2 = p_2(\alpha) = 1 - F_{L^+}(\alpha),$$

$$f_1(x) = \begin{cases} \frac{1}{p_1} f_{L^+}(x), & 0 \le x \le \alpha \\ 0, & x > \alpha \end{cases}$$

$$f_2(x) = \begin{cases} 0, & 0 \le x \le \alpha \\ \frac{1}{p_2} f_{L^+}(x), & x > \alpha \end{cases}$$

Then, a variable X, with density $f_{L^+}(x)$ is obtained by generating with probability p_1 an X with density $f_1(x)$, and with probability p_2 an X with density $f_2(x)$.

(a) **Bucher's method** [2] for $n = 1$ is used to generate random variables with probability density $f_1(x)$. We consider the factorization

$$f_1(x) = K g_1(x) h_1(x),\ x \in [0, \alpha],$$

where $K > 0$ is a constant, $g_1(x)$ is the density function of an easy to generate random variable on $[0, \alpha]$, and $h_1(x) : [0, \alpha] \to [0, 1]$ is an integrable function. For the L^+ distribution, $K = 2\sqrt{2}\alpha/\pi$, and

$$g_1(x) = \frac{1}{\alpha},\ h_1(x) = \frac{1}{1 + x^4},\ x \in [0, \alpha].$$

(b) The **rejection method** is used to generate random variables with probability density $f_2(x)$. Hence, we need a constant $\beta = \beta(\alpha)$ such that

$$f_2(x) \le \beta v'(x) \text{ for } x \in (\alpha, \infty),$$

where $v'(x)$ is a probability density on (α, ∞).

For the L^+ distribution, we take into consideration the class of Student distributions $t^+(r)$, concentrated on $[0, \infty]$:

$$v(x) \in \left\{ f_{t^+(r)}(x), r = 1, 2, 3, 4 \right\}, \tag{3}$$

with

$$f_{t^+(r)}(x) = \frac{2\Gamma\left(\frac{r+1}{2}\right)}{\sqrt{r\pi}\,\Gamma\left(\frac{r}{2}\right)} \left(1 + \frac{x^2}{r}\right)^{-\frac{r+1}{2}},\ x \ge 0.$$

It follows that

$$v'(x) = \frac{1}{\int_\alpha^\infty f_{t^+(r)}(x)\,dx} f_{t^+(r)}(x) = \frac{1}{1 - F_{t^+(r)}(\alpha)} f_{t^+(r)}(x),\ x \in (\alpha, \infty),$$

and

$$\beta = \beta(\alpha; r) = \frac{\sqrt{2r}\Gamma\left(\frac{r}{2}\right)\left(1 - F_{t^+(r)}(\alpha)\right)}{\sqrt{\pi}\Gamma\left(\frac{r+1}{2}\right)p_2(\alpha)} \max_{x\in(\alpha,\infty)} \frac{\left(1+\frac{x^2}{r}\right)^{\frac{r+1}{2}}}{1+x^4}.$$

The probability of rejection in this procedure is given in the following proposition.

Proposition 4 *Let U and Y be two independent random variables, such that U is uniformly distributed on $[0,1]$ and Y is uniformly distributed on $[0,\alpha]$. Then, denoting*

$$p_{rej}(\alpha) = P\left(U > \frac{1}{1+Y^4}\right),$$

we have

$$p_{rej}(\alpha) = 1 - \frac{1}{2\alpha\sqrt{2}}\left[\frac{1}{2}\ln\frac{\alpha^2+1+\alpha\sqrt{2}}{\alpha^2+1-\alpha\sqrt{2}} + \arctan\frac{\alpha^2-1}{\alpha\sqrt{2}} + \frac{\pi}{2}\right].$$

Proof. The random vector (U,Y) has the probability density

$$\rho(u,y) = \begin{cases} \frac{1}{\alpha}, & (u,y)\in[0,1]\times[0,\alpha] \\ 0, & \text{otherwise.}\end{cases}$$

Put $D = \left\{0 \le u \le 1, 0 \le y \le \alpha, u > \frac{1}{1+y^4}\right\}$. Then

$$p_{rej}(\alpha) = \int_D \rho(u,y)\,dudy = \frac{1}{\alpha}\int_0^\alpha\left(\int_{\frac{1}{1+y^4}}^1 du\right)dy.$$

The result is obtained by direct calculation of this integral. □

Remark 3 *Our choice of the family* (3) *is based on two facts:*

- The equations

$$\frac{2\sqrt{2}}{\pi}\cdot\frac{1}{1+x^4} = \frac{2\Gamma\left(\frac{r+1}{2}\right)}{\sqrt{r\pi}\Gamma\left(\frac{r}{2}\right)}\left(1+\frac{x^2}{r}\right)^{-\frac{r+1}{2}}, \; r \ge 1 \tag{4}$$

have unique solutions for $r = 1,2,3,4$ and they have two solutions for $r \ge 5$.

- Plotting the curves $f_{L^+}(x)$ and $f_{t^+(r)}(x)$ for $r = 1,2,3,4$, one can see that the following inequalities is satisfied:

$$f_{L^+}(x) < f_{t^+(r)}(x) \text{ for } x > \alpha.$$

Numerical solutions of equations (4) have been obtained using Maple Library.

r :	1	2	3	4
α :	1.2897	1.1428	1.0807	1.046

Using Proposition 3 and Remark 3, the generation of a L^+ distribution by means of the composition procedure is implemented through the algorithm CRPL.

ALGORITHM CRPL

input h_1, f_1, f_2, v'
calculate p_1, α, β
generate U uniformly distributed on $(0,1)$
if $U > p_1$ then
 repeat
  generate U_1, uniformly distributed on $(0,1)$
  generate X, with the density $v'(x)$
 until $U_1 \leq f_2(X) / \beta v'(X)$
else
 generate U_2, uniformly distributed on $[0,1]$
 repeat
  generate X, uniformly distributed on $[0,\alpha]$
 until $U_2 \leq h_1(X)$
output X.

Remark 4 *(1) Numerical average numbers of iterations for the generation of a random variable X, with the L^+ distribution, by means of the composition procedure are presented in the following table:*

r :	1	2	3	4
$1/p_{rej}$:	4.0987	5.3385	6.1218	6.6605
β :	3.4364	2.2676	1.9302	1.7759
$p_1 \frac{1}{p_{rej}} + (1-p_1)\beta$:	4.0178	4.8354	5.3423	5.6851

(2) In spite of the fact that β decreases as r increases, both the average number of trials and the difficulty of generation of a random variable with probability density $v'(x)$ increase. Hence, we recommend the use of the Cauchy distribution concentrated on $[0,\infty)$ (that is, $t^+(1)$).

(3) The Cauchy distributed random variable may be obtained as the ratio of two independent normally distributed random variables. Two simpler procedures are the following: (**a**) *using the inverse method, based on the relation $X = \tan(\pi U)$, where U is uniformly distributed, and (b) using a rejection procedure associated with the ratio of uniforms method, according to the following algorithm:*

repeat
 generate i.i.d.r.v. U and V, uniformly distributed on $[-1,1]$
until $U^2 + V^2 \leq 1$
$X \longleftarrow V/U$.

3.2 Computer generation of $L(0,1)$ by the ratio of uniforms method

The ratio of uniforms method is based on the following result, established by Kinderman and Monahan [2].

Proposition 5 *Let* $A = \left\{(u,v) \mid 0 \le u \le \sqrt{f\left(\frac{u}{v}\right)}\right\}$, *where* $f \ge 0$ *is an integrable function. If* (U,V) *is a random vector uniformly distributed over* A, *then* V/U *the probability density* $\frac{1}{2 \cdot area(A)} f$.

Our goal is to generate X, with $L(0,1)$ distribution. It suffices to enclose the area A by a rectangle, in which we know how to generate uniform vectors, and only to apply the rejection principle. We notice several properties of the set A :

- It is a subset of $[0,\infty) \times R$.

- It is symmetric with respect to the u axis if f is symmetric with respect to 0.

- It vanishes in the negative v quadrant when f is the density of a non-negative random variable.

- The boundary of A can be found parametrically by $\{(u(x), v(x)) \mid x \in R\}$, where

$$u(x) = \sqrt{f(x)}, v(x) = x\sqrt{f(x)}.$$

Thus, A can be enclosed in a rectangle if and only if both $f(x)$ and $x^2 f(x)$ are bounded.

The enclosing rectangle will be called $B = [0,b) \times [a_-, a_+]$, and the value

$$\frac{b(a_+ - a_-)}{area(A)} = \frac{2b(a_+ - a_-)}{\int\limits_{-\infty}^{\infty} f(x)\,dx}$$

will be called the rejection constant.

The generation algorithm is the following:

ALGORITHM RUML

input f
compute b, a_+, a_-
repeat
  generate U, uniformly distributed on $[0,b]$
  generate V, uniformly distributed on $[a_-, a_+]$
  $X \longleftarrow V/U$
until $U^2 \le f(X)$
output X.

The vector (U,V) is uniformly distributed in A. Thus, the algorithm is valid, *i.e.* X has the density proportional to the function f. The acceptance condition $U^2 \le f(X)$ cannot be simplified by using logarithmic transformations, because U is explicitly needed in the definition of X. We notice that the rejection constant is equal to the expected number of iterations.

For the generation of the $L(0,1)$ distribution we consider $f(x)$ equal to the $L(0,1)$ density, $b = \sup \sqrt{f(x)} = \sqrt{\sqrt{2}/\pi}$, $a_+ = \sup x\sqrt{f(x)} = \sqrt{1/\pi\sqrt{2}}$, and $a_- = \inf x\sqrt{f(x)} = -\sqrt{1/\pi\sqrt{2}}$.

Remark 5 *Laha distribution has several applications to simulation, such as:*
(a) The generation of a random variable with a Cauchy distribution as a ratio of two independent, $L(0,1)$ distributed variables.
(b) The generation of truncated, normally distributed on $[-\alpha, \alpha]$ random variables, by means of a rejection method. Such a procedure is based on the fact that there exists a positive value α, such that

$$f_{N(0,1)}(x) \leq f_{L(0,1)}(x) \;\; \textit{for } x \in [-\alpha, \alpha].$$

(c) The generation of Student distributed random variables, $t(r), r = 1,2,3,4$, truncated on $[-\alpha, \alpha]$, by means of a similar rejection method.

4 Applications to life time modelling

The most frequently used model for the life time of a manufacturing system is exponential distributed random variable. For such systems there is no ageing process, and after each renewal the system starts "as good as new". But most of the manufacturing systems are affected by ageing. Here we propose positive Laha distribution as a model for the life time of a system which has a running in period of time. That is, there exists an interval where the system is most vulnerable because its ageing increases rapidly. Afterwards, the ageing is still present, but it diminishes in time.

4.1 The life time

The fundamental mathematical items involved into maintenance of systems are the following:
- the life time of the system, which is a positive random variable X, with probability density $f(x), x \geq 0$;
- the survival function

$$S(x) = 1 - F_X(x) = \int_x^\infty f(t)\,dt, x \geq 0;$$

- the failure intensity (or the mortality rate)

$$h(x) = \frac{f(x)}{S(x)}, \quad x \geq 0.$$

If $h(x) > 0$ for $x \geq 0$, the system ageing is positive, hence it deteriorates in time. If $h(x) = ct.$ for $x \geq 0$, there is no ageing process. Finally, if $h(x) < 0$ for $x \geq 0$, the system ageing is negative, hence the performances improve in time.

The probability distribution most frequently used in modelling the life time is the **Exponential** $Exp(\lambda)$ distribution:

$$f(x;\lambda) = \lambda e^{-\lambda x}, x \geq 0, \lambda > 0.$$

In this case there is no ageing process, as

$$h(x) = \frac{\lambda e^{-\lambda x}}{1-(1-e^{-\lambda x})} = \lambda.$$

But most of the manufacturing processes are affected by ageing. **Kodlin** [3] has introduced a model for the life time X, which we denote by $K(\alpha,\beta)$, given by the probability density

$$f(x;\alpha,\beta) = (\alpha+\beta x)\cdot \exp\left(-\alpha x - \beta\frac{x^2}{2}\right), x \geq 0, \alpha > 0, \beta \geq 0.$$

For $\beta = 0$, Kodlin's distribution coincides with $Exp(\alpha)$, and there is no ageing process. For $\beta > 0$, the failure intensity is

$$h(x) = \alpha + \beta x,$$

hence the manufacturing process is affected by a increasing positive ageing, hence its capacities **linearly** deteriorate in time.

Proposition 6 *The failure intensity $h_{L^+}(x)$ associated with a positive Laha distribution L^+ has the following properties:*

$$h_{L^+}(x) > 0 \textit{ for } x \geq 0,$$

and, there exists a positive value x_0, $(x_0 \approx 1.0279)$, such that $h_{L^+}(x_0) = \max_{x\geq 0} h_{L^+}(x)$, $h_{L^+}(x)$ increases for $0 \leq x < x_0$ and $h_{L^+}(x)$ decreases for $x > x_0$.

Proof. The survival function is

$$S_{L^+}(x) = 1 - F_{L^+}(x),$$

hence the failure intensity is

$$h_{L^=}(x) = \frac{\frac{2\sqrt{2}}{\pi}\cdot\frac{1}{1+x^4}}{1-\left[\frac{1}{2\pi}\ln\left(\frac{x^2+1+\sqrt{2}x}{x^2+1-\sqrt{2}x}\right)+\frac{1}{\pi}\left(\arctan(\sqrt{2}x+1)+\arctan(\sqrt{2}x-1)\right)\right]}$$

Notice that $h_{L^+}(0) = .90032$, $\lim_{x\to\infty} h_{L^+}(x) = 0$. The derivative of $h_{L^+}(x)$ is $h'_{L^+}(x) = g_1(x)/g_2(x)$, with

$$g_1(x) = -8\pi\sqrt{2}x^3(1-F_{L^+}(x))\left(x^2+1-\sqrt{2}x\right)^2\left(x^2+1+\sqrt{2}x\right)^2 + 8\left(1+x^4\right)^2,$$

$$g_2(x) = \pi^2\left(1+x^4\right)^2\left(x^2+1-\sqrt{2}x\right)^2\left(x^2+1+\sqrt{2}x\right)^2(1-F_{L^+}(x))^2 > 0.$$

By numerical calculation, we obtain the existence of the point $x_0 \approx 1.0279$, such that $h'_{L^+}(x_0) = 0$, $h'_{L^+}(x) > 0$ for $0 \leq x < x_0$, and $h'_{L^+}(x_0) < 0$ for $x > x_0$. □

Comment: According to this property, the positive Laha distribution can be used in modelling the life time of a manufacturing system, with a **non-monotonous, positive failure intensity**: During the first "unit" of time, $[0, x_0]$, the ageing of the system increases and it is "most vulnerable" at the moment x_0, while afterwards, ageing is still present, but it diminishes in time. Such manufacturing processes can be characterized by the existence of a "**critical failure interval**" (or a "**running in**" interval).

4.2 A Markov renewal model

Now, let us consider a Markov renewal process as the model for a manufacturing process with a non-monotonous failure intensity. Let E be a countable set, called the set of the states of the system, and let $\{Y_n, n \in N\}$ be a sequence of random variables taking values in E. Let $\{X_n, n \in N\}$ be a sequence of *i.i.d.* random variables, with positive Laha distribution. Consider the sequence $T_0 = 0, T_{n+1} = T_n + X_{n+1}$, $n \in N$. The stochastic process $\{Y_n, T_n, n \in N\}$ is a Markov renewal process iff

$$P(Y_{n+1} = j, T_{n+1} - T_n \leq x \mid Y_0, ..., Y_n, T_0, ..., T_n) =$$
$$P(Y_{n+1} = j, T_{n+1} - T_n \leq x \mid Y_n).$$

We suppose that, for any $i, j \in E, x \geq 0$, the probabilities

$$P(Y_{n+1} = j, T_{n+1} - T_n \leq x \mid Y_n = i) = Q(i, j, x)$$

are independent on n. Hence, the family $\{Q(i, j, x) \mid i, j \in E, x \geq 0\}$ is a semi-Markov kernel over E. Let us denote by $P(i, j) = \lim_{x\to\infty} Q(i, j, x)$ the associated transition probabilities of the Markov chain $\{Y_n, n \in N\}$.

Using theorem 1.13 [1], we obtain:

Proposition 7 *For the positive Laha* (L^+) *distributed life times* $\{X_n, n \in N\}$, *we have*

$$Q(i, j, x) = P(i, j) \cdot$$
$$\left[\frac{1}{2\pi}\ln\left(\frac{x^2+1+\sqrt{2}x}{x^2+1-\sqrt{2}x}\right) + \frac{1}{\pi}\left(\arctan\left(\sqrt{2}x+1\right) + \arctan\left(\sqrt{2}x-1\right)\right)\right]$$

and the minimal semi-Markov process

$$Z_t = \begin{cases} Y_n, & if\ T_n \leq t < T_{n+1} \\ \Delta, & if\ t \geq \sup_n T_n \end{cases}$$

is not Markovian, where Δ *is a point not in* E, *and* $t \geq 0$.

For a deeper discussion on this model, we must specify the semi-Markov kernel over E, but this exceeds the goal of this paper.

References

[1] Çinlar, E. (1975) , *Introduction to Stochastic Processes*, Prentice-Hall, NJ.
[2] Devroye, L. (1986) , *Non-uniform Random Variate Generation*, Springer Verlag, NY.
[3] Kodlin, D. (1967) , *A new response time distribution*, Biometrics, 23, 227 – 239.
[4] Laha, R.G. (1958) , *An example of a non-normal distribution where the quotient follows the Cauchy law*, Proceedings of the National Academy of Sciences, 44, 222 – 223.