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Kan Extensions of Institutions

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Abstract: Institutions were introduced by Goguen and Burstall [GB84, GB85, GB86, GB92] to formally capture the notion of logical system. Interpreting institutions as functors, and morphisms and representations of institutions as natural transformations, we give elegant proofs for the completeness of the categories of institutions with morphisms and representations, respectively, show that the duality between morphisms and representations of institutions comes from an adjointness between categories of functors, and prove the cocompleteness of the categories of institutions over small signatures with morphisms and representations, respectively.
Category: F.3, F.4

1 Introduction

There are different logical systems successfully used in theoretical computer science, such as first and higher order logic, equational logic, Horn clause logic, temporal logics, modal logics, infinitary logics, and many others. As a consequence of the fact that many general results of these logics are not dependent on the particular ingredients of their underlying logic, abstracting Tarski's classic semantic definition of truth [Tar44], Goguen and Burstall [GB84, GB85, GB86, GB92] developed the notion of *institution* to formalize the informal notion of "logical system". The main requirement is the existence of a satisfaction relation between models and sentences which is consistent under change of notation.

Much interest has been shown in the study of institutions since they first appeared in 1986. Institutions have been given for lambda calculus, higher order logic with polymorphic types, second order and modal logics. Mosses [Mos89] shows that (his) unified algebras form an institution, Goguen [Gog91] shows that (his) hidden-sorted equational logic is an institution, Mossakowski [Mos96] gives hierarchies of institutions for total, partial and order-sorted logics, Roşu [Ros94] gives an institution for order-sorted equational logic. Diaconescu, Goguen and Stefaneas [DGS93] and Roşu [Ros99] use institutions to study modularization. Diaconescu [Dia98] introduces extra theory morphisms for institutions to give logical semantics for multiparadigm languages like CafeOBJ [DF98]. Fiadeiro and Sernadas [FS88] introduces the notion of π-institution based on deduction rather than satisfaction, and Pawlowski [Paw96] introduces the notion of context institution to deal with variable contexts and substitutions. Cerioli and Meseguer [CM97], Cerioli [Cer93], Tarlecki [Tar96a], Mossakowski [Mos96] study relationships and translations between institutions. Much interesting work using institutions has been done by Tarlecki [Tar84, Tar86a, Tar86b, Tar86c, Tar87, Tar96a] and by Sannella and Tarlecki [ST86, ST87, ST88].

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As suggested by Goguen and Burstall [GB92], an institution can be regarded as a functor from its category of signatures to some special category. This more categorical view allows us to show that some known results on institutions are instances of results in category theory, and also to obtain new results.

Two main apparently distinct maps between institutions are being considered in the literature: institution morphisms due to Goguen and Burstall [GB92] and institution representations due to Tarlecki [Tar87, Tar96a]. We show that the categories of institutions with morphisms ($\mathcal{INS}$) and of institutions with representations ($\mathcal{INS}^{repr}$), respectively, are special flattened indexed categories, thus their completeness following immediately.

Arrais and Fiadeiro [MJ96] showed that given an adjunction between two categories of signatures, an institution morphism gives birth to an institution representation and the vice-versa. We show that this duality is actually a natural consequence of the fact that the adjointness between the categories of signatures can be contravariantly lifted to functor categories.

Given a functor between two categories of signatures, any institution over the source signature category can be extended to an institution over the target signature category along that functor in two canonical ways given by the left and the right Kan extensions, respectively. As a consequence, the categories of institutions over small signatures with morphisms and representations, respectively, are cocomplete.

2 Category Theory

We assume the reader familiar with many categorical concepts. We use semicolon for morphisms composition and it is written in diagrammatic order, that is, if $f\colon A \to B$ and $g\colon B \to C$ are two morphisms then $f;g\colon A \to C$ is their composition. We also use ";" for vertical composition of natural transformations and "⨟" for horizontal composition of natural transformations.

It is known that $\mathbf{Cat}$ (and implicitly $\mathbf{Cat}^{op}$) and $\mathbf{Set}$ are both complete and cocomplete. The reader is assumed familiar with limits and colimits in $\mathbf{Set}$ and $\mathbf{Cat}$.

2.1 Indexed Categories

Let $\mathbf{Ind}$ be any category, called "of indexes".

Definition 1. An **indexed category** is a functor $C\colon \mathbf{Ind}^{op} \to \mathbf{Cat}$. $\mathbf{Flat}(C)$ is the category having pairs (i,a) as objects, where i is an object in $\mathbf{Ind}$ and a is an object in C_i, and pairs $(\alpha, f)\colon (i,a) \to (i',a')$ as morphisms, where $\alpha \in \mathbf{Ind}(i,i')$ and $f \in C_i(a, C_\alpha(a'))$.

The following two theorems show conditions under which the flattened category of an indexed category is complete or cocomplete [TBG91]:

Theorem 2. *If $C\colon \mathbf{Ind}^{op} \to \mathbf{Cat}$ is an indexed category such that $\mathbf{Ind}$ is complete, C_i is complete for all indices $i \in |\mathbf{Ind}|$, and $C_\alpha\colon C_j \to C_i$ is continuous for all index morphisms $\alpha\colon i \to j$, then $\mathbf{Flat}(C)$ is complete.*

Theorem 3. *If $C\colon \mathbf{Ind}^{op} \to \mathbf{Cat}$ is an indexed category such that* $\mathbf{Ind}$ *is cocomplete, C_i is cocomplete for all indices $i \in |\mathbf{Ind}|$, and $C_\alpha\colon C_j \to C_i$ has a left adjoint for all index morphisms $\alpha\colon i \to j$, then $\mathbf{Flat}(C)$ is cocomplete.*

Given an indexed category $C\colon \mathbf{Ind}^{op} \to \mathbf{Cat}$, one can easily build another indexed category $C^{op}\colon \mathbf{Ind}^{op} \to \mathbf{Cat}$ such that C^{op}_i is $(C_i)^{op}$ and $C^{op}_\alpha\colon C^{op}_{i'} \to C^{op}_i$ is $(C_\alpha)^{op}$ for every $\alpha \in \mathbf{Ind}(i, i')$. The following corollaries are immediate from Theorems 2 and 3, respectively:

Corollary 4. *If $C\colon \mathbf{Ind}^{op} \to \mathbf{Cat}$ is an indexed category such that* $\mathbf{Ind}$ *is complete, C_i is cocomplete for all indices $i \in |\mathbf{Ind}|$, and $C_\alpha\colon C_j \to C_i$ is cocontinuous for all index morphisms $\alpha\colon i \to j$, then $\mathbf{Flat}(C^{op})$ is complete.*

Corollary 5. *If $C\colon \mathbf{Ind}^{op} \to \mathbf{Cat}$ is an indexed category such that* $\mathbf{Ind}$ *is cocomplete, C_i is complete for all indices $i \in |\mathbf{Ind}|$, and $C_\alpha\colon C_j \to C_i$ has a right adjoint for all index morphisms $\alpha\colon i \to j$, then $\mathbf{Flat}(C^{op})$ is cocomplete.*

2.2 Functor Categories and Kan Extensions

Let $\mathbf{T}$ be a category. For any category $\mathbf{S}$, let $\mathbf{T}^{\mathbf{S}}$ be the category of functors from $\mathbf{S}$ to $\mathbf{T}$ and natural transformations, and for any functor $\Phi\colon \mathbf{S} \to \mathbf{S}'$, let $\mathbf{T}^{\Phi}\colon \mathbf{T}^{\mathbf{S}'} \to \mathbf{T}^{\mathbf{S}}$ be the functor defined as $\mathbf{T}^{\Phi}(I') = \Phi; I'$ for functors $I'\colon \mathbf{S}' \to \mathbf{T}$ and $\mathbf{T}^{\Phi}(\sigma) = 1_\Phi \,⨟\, \sigma$ for natural transformations $\sigma\colon I'_1 \Rightarrow I'_2$. Also, let $\mathbf{T}^{-}\colon \mathbf{Cat}^{op} \to \mathbf{Cat}$ be the functor that takes a category $\mathbf{S}$ to $\mathbf{T}^{\mathbf{S}}$ and a functor $\Phi\colon \mathbf{S} \to \mathbf{S}'$ to $\mathbf{T}^{\Phi}$. Then obviously $\mathbf{T}^{-}\colon \mathbf{Cat}^{op} \to \mathbf{Cat}$ is an indexed category, and

Proposition 6. *If $\mathbf{T}$ is complete (cocomplete) then $\mathbf{T}^{\mathbf{S}}$ is complete (cocomplete) for any category $\mathbf{S}$ and $\mathbf{T}^{\Phi}$ is continuous (cocontinuous) for any functor $\Phi\colon \mathbf{S} \to \mathbf{S}'$.*
Hint: *The limits (colimits) in $\mathbf{T}^{\mathbf{S}}$ are built "pointwise" (see [Lan71], pg. 112).*

Definition 7. Given functors $K\colon \mathbf{S}_1 \to \mathbf{S}_2$ and $I_1\colon \mathbf{S}_1 \to \mathbf{T}$, a **right Kan extension of I_1 along K** is a pair containing a functor $I_2\colon \mathbf{S}_2 \to T$ and a natural transformation $\mu\colon K; I_2 \Rightarrow I_1$ which is universal from $\mathbf{T}^K$ to I_1, that is, for every $I'_2\colon \mathbf{S}_2 \to T$ and $\mu'\colon K; I'_2 \Rightarrow I_1$ there is a unique natural transformation $\sigma\colon I'_2 \Rightarrow I_2$ such that $\mu' = (1_K \,⨟\, \sigma); \mu$. Dually, a **left Kan extension of I_1 along K** is a functor $I_2\colon \mathbf{S}_2 \to T$ and a natural transformation $\rho\colon I_1 \Rightarrow K; I_2$ which is universal from I_1 to $\mathbf{T}^K$, that is, for every $I'_2\colon \mathbf{S}_2 \to T$ and $\rho'\colon I_1 \Rightarrow K; I'_2$ there is a unique natural transformation $\sigma\colon I_2 \Rightarrow I'_2$ such that $\rho' = \rho; (1_K \,⨟\, \sigma)$.

The following major result (see [Lan71]) plays an important role in our paper:

Proposition 8. *Given a small category $\mathbf{S}_1$,*

- *If $\mathbf{T}$ is complete then any functor $I_1\colon \mathbf{S}_1 \to \mathbf{T}$ has a right Kan extension along any $K\colon \mathbf{S}_1 \to \mathbf{S}_2$, and $\mathbf{T}^K$ has a right adjoint, and*
- *If $\mathbf{T}$ is cocomplete then any functor $I_1\colon \mathbf{S}_1 \to \mathbf{T}$ has a left Kan extension along any $K\colon \mathbf{S}_1 \to \mathbf{S}_2$, and $\mathbf{T}^K$ has a left adjoint.*

The following theorem gives other conditions (see Corollary 11) under which Kan extensions exist.

Theorem 9. $\mathbf{T}^{-}$ *contravariantly lifts adjoints to functor categories.*
Hint: *If* $\langle \Phi, \Psi, \eta, \epsilon \rangle \colon \mathbf{S} \to \mathbf{S}'$ *is an adjointness then so is* $\langle \mathbf{T}^{\Psi}, \mathbf{T}^{\Phi}, \mathbf{T}^{\eta}, \mathbf{T}^{\epsilon} \rangle$, *from* $\mathbf{T}^{\mathbf{S}} \to \mathbf{T}^{\mathbf{S}'}$, *where* $(\mathbf{T}^{\eta})_I = \eta \fatsemi 1_I$ *and* $(\mathbf{T}^{\epsilon})_{I'} = \epsilon \fatsemi 1_{I'}$ *for every functors* $I \colon \mathbf{S} \to \mathbf{T}$ *and* $I' \colon \mathbf{S}' \to \mathbf{T}$.

Then within the same notations,

Corollary 10. $Nat(\Psi; I, I') \simeq Nat(I, \Phi; I')$ *and this bijection is natural in* I *and* I'.

More precisely, a natural transformation $\mu \colon \Psi; I \Rightarrow I'$ is taken to $(\eta \fatsemi 1_I); (1_{\Phi} \fatsemi \mu)$ and conversely, a natural transformation $\rho \colon I \Rightarrow \Phi; I'$ is taken to $(1_{\Psi} \fatsemi \rho); (\epsilon \fatsemi 1_{I'})$.

Corollary 11. *Given any functor* $I_1 \colon \mathbf{S}_1 \to \mathbf{T}$, *then*

- I_1 *has a right Kan extension along any* $K \colon \mathbf{S}_1 \to \mathbf{S}_2$ *which has a left adjoint, and* $\mathbf{T}^K$ *has a right adjoint,*
- I_1 *has a left Kan extension along any* $K \colon \mathbf{S}_1 \to \mathbf{S}_2$ *which has a right adjoint, and* $\mathbf{T}^K$ *has a left adjoint.*

2.3 Twisted Relations

Twisted relations are introduced in [GB92]. We show that they give a complete and cocomplete category.

Definition 12. Let **Trel** be the category of "twisted relations", having triples $\langle A, \mathcal{R}, B \rangle$ as objects, where A is a category, B is a set and $\mathcal{R} \subseteq |A| \times B$, and pairs $\langle F, g \rangle \colon \langle A, \mathcal{R}, B \rangle \to \langle A', \mathcal{R}', B' \rangle$ as morphisms, where $F \colon A' \to A$ is a functor and $g \colon B \to B'$ is a function such that the diagram

$$\begin{tikzcd} |A| \arrow[r, dash, "\mathcal{R}"] & B \arrow[d, "g"] \\ |A'| \arrow[u, "F"] \arrow[r, dash, "\mathcal{R}'"'] & B' \end{tikzcd}$$

commutes.

Let $Left \colon \mathbf{Trel} \to \mathbf{Cat}^{op}$ take a triple $\langle A, \mathcal{R}, B \rangle$ to A and a morphism $\langle F, g \rangle$ to F, and let $Right \colon \mathbf{Trel} \to \mathbf{Set}$ take a triple $\langle A, \mathcal{R}, B \rangle$ to B and a morphism $\langle F, g \rangle$ to g. It is straightforward that $Left$ and $Right$ are functors.

Proposition 13. **Trel** *is both complete and cocomplete.*

Proof. Let $\mathbf{J}$ be a small category and $D\colon \mathbf{J} \to \mathbf{Trel}$ be a functor.

First, let $(A, \{F_j\}_{j\in|\mathbf{J}|})$ and $(B, \{g_j\}_{j\in|\mathbf{J}|})$ be limits of $D;Left$ and $D;Right$, respectively (they exist because $\mathbf{Cat}^{op}$ and $\mathbf{Set}$ are complete), and let $\mathcal{R} \subseteq |A| \times B$ be the relation defined as $(a,b) \in \mathcal{R}$ iff $(a_j, g_j(b)) \in \mathcal{R}_j$ for each $j \in |\mathbf{J}|$ and each $a_j \in |A_j|$ with $F_j(a_j) = a$. Notice that for every $a \in |A|$ there are some $j \in |\mathbf{J}|$ and $a_j \in |A_j|$ such that $F_j(a_j) = a$ (because of the way in which colimits are built in $\mathbf{Cat}$). It is easy now to check that $(\langle A, \mathcal{R}, B\rangle, \{\langle F_j, g_j\rangle\}_{j\in|\mathbf{J}|})$ is a limit of D. Therefore, $\mathbf{Trel}$ is complete.

Let $(\{F_j\}_{j\in|\mathbf{J}|}, A)$ and $(\{g_j\}_{j\in|\mathbf{J}|}, B)$ be colimits of $D;Left$ and $D;Right$, respectively (they exist because $\mathbf{Cat}^{op}$ and $\mathbf{Set}$ are cocomplete), and let $\mathcal{R} \subseteq |A| \times B$ be the relation defined as $(a,b) \in \mathcal{R}$ iff $(F_j(a_j), b_j) \in \mathcal{R}_j$ for each $j \in |\mathbf{J}|$ and each $b_j \in B_j$ with $g_j(b_j) = b$. As before, notice that for every $b \in B$ there are some $j \in |\mathbf{J}|$ and $b_j \in B_j$ such that $g_j(b_j) = b$, and that $(\{\langle F_j, g_j\rangle\}_{j\in|\mathbf{J}|}, \langle A, \mathcal{R}, B\rangle)$ is a colimit of D. Thus $\mathbf{Trel}$ is cocomplete.

3 Institutions

Institutions were introduced by Goguen and Burstall [GB92] to formally capture the notion of logical system.

Definition 14. An **institution** $(\mathbf{Sign}, \mathbf{Mod}, \mathbf{Sen}, \models)$ consists of a category $\mathbf{Sign}$ whose objects are called **signatures**, a functor $\mathbf{Mod}\colon \mathbf{Sign} \to \mathbf{Cat}^{op}$ giving for each signature Σ a category of **Σ-models**, a functor $\mathbf{Sen}\colon \mathbf{Sign} \to \mathbf{Set}$ giving for each signature a set of **Σ-sentences**, and a Σ-indexed relation $\models\; = \{\models_\Sigma \mid \Sigma \in \mathbf{Sign}\}$, where $\models_\Sigma \subseteq |\mathbf{Mod}(\Sigma)| \times \mathbf{Sen}(\Sigma)$, such that for each signature morphism $\varphi\colon \Sigma \to \Sigma'$, the following diagram commutes,

$$\begin{array}{ccccc}
\Sigma & & |\mathbf{Mod}(\Sigma)| & \overset{\models_\Sigma}{\text{———}} & \mathbf{Sen}(\Sigma) \\
\Big\downarrow{\scriptstyle\varphi} & & {\scriptstyle\mathbf{Mod}(\varphi)}\Big\uparrow & & \Big\downarrow{\scriptstyle\mathbf{Sen}(\varphi)} \\
\Sigma' & & |\mathbf{Mod}(\Sigma')| & \underset{\models_{\Sigma'}}{\text{———}} & \mathbf{Sen}(\Sigma')
\end{array}$$

that is, the following *Satisfaction Condition*

$$m' \models_{\Sigma'} \mathbf{Sen}(\varphi)(a) \text{ iff } \mathbf{Mod}(\varphi)(m') \models_\Sigma a$$

holds for each $m' \in |\mathbf{Mod}(\Sigma')|$ and each $e \in \mathbf{Sen}(\Sigma)$.

Proposition 15. *There is a bijection*[2] *between institutions over the signature* $\mathbf{Sign}$ *and functors* $\mathbf{Sign} \to \mathbf{Trel}$.

Proof. Given an institution $(\mathbf{Sign}, \mathbf{Mod}, \mathbf{Sen}, \models)$, then one can build a functor $\mathcal{I}\colon \mathbf{Sign} \to \mathbf{Trel}$ that takes every signature $\Sigma \in |\mathbf{Sign}|$ to the triple $\langle \mathbf{Mod}(\Sigma), \models_\Sigma, \mathbf{Sen}(\Sigma)\rangle$, and every morphism of signatures $\varphi\colon \Sigma \to \Sigma'$ to the "twisted" morphism $\langle \mathbf{Mod}(\varphi), \mathbf{Sen}(\varphi)\rangle$.

Conversely, given a functor $\mathcal{I}\colon \mathbf{Sign} \to \mathbf{Trel}$, one can build an institution $(\mathbf{Sign}, \mathbf{Mod}, \mathbf{Sen}, \models)$ such that for every $\Sigma \in |\mathbf{Sign}|$, $\mathbf{Mod}(\Sigma) = A_\Sigma$,

[2] Actually we mean one-to-one correspondence between classes.

$\mathbf{Sen}(\Sigma) = B_\Sigma$, and $\models_\Sigma = \mathcal{R}_\Sigma$, where $\mathcal{I}(\Sigma) = \langle A_\Sigma, \mathcal{R}_\Sigma, B_\Sigma \rangle$, and such that for every morphism of signatures $\varphi\colon\ \Sigma \to \Sigma'$, $\mathbf{Mod}(\varphi) = F_\varphi$ and $\mathbf{Sen}(\varphi) = g_\varphi$, where $\mathcal{I}(\varphi) = \langle F_\varphi, g_\varphi \rangle$.

Because of this bijection, we can interchangeably use any of the tuple or functor notation when referring to institutions.

Some other more categorical views of institutions are explored in [GB92], where the target category is a comma category. Depending on the functors involved in the comma category, one can obtain institutions as in our approach, institutions with no morphisms between models, or even institutions with morphisms between sentences. The morphisms between sentences are thought of as "proofs", as advocated by Lambek and (Phil) Scott [LS86].

3.1 Institution Morphisms

Institution morphisms were introduced together with the institutions in [GB92].

Definition 16. Given two institutions $\mathcal{I} = (\mathbf{Sign}, \mathbf{Mod}, \mathbf{Sen}, \models)$ and $\mathcal{I}' = (\mathbf{Sign}', \mathbf{Mod}', \mathbf{Sen}', \models')$, an **institution morphism** from $\mathcal{I}'$ to $\mathcal{I}$ consists of a functor $\Psi :\ \mathbf{Sign}' \to \mathbf{Sign}$, a natural transformation $\beta :\ \mathbf{Mod}' \Rightarrow \Psi;\mathbf{Mod}$, and a natural transformation $\alpha\colon\ \Psi;\mathbf{Sen} \Rightarrow \mathbf{Sen}'$, such that the following *Satisfaction Condition* holds for each $\Sigma' \in |\mathbf{Sign}'|$, $m' \in |\mathbf{Mod}'(\Sigma')|$, and $e \in \mathbf{Sen}(\Psi(\Sigma'))$:

$$m' \models_{\Sigma'} \alpha_{\Sigma'}(e) \ \text{ iff } \ \beta_{\Sigma'}(m') \models'_{\Psi(\Sigma')} e$$

Let $\mathcal{INS}$ denote the category of institutions and institution morphisms.

Intuitively, a morphism from $\mathcal{I}'$ to $\mathcal{I}$ is a projection of the logic $\mathcal{I}'$ into the logic $\mathcal{I}$.

Theorem 17. *$\mathcal{INS}$ is isomorphic to* $\mathbf{Flat}((\mathbf{Trel}^{-})^{op})$.

Proof. By Proposition 15, any institution $(\mathbf{Sign}, \mathbf{Mod}, \mathbf{Sen}, \models)$ is taken bijectively to a pair $(\mathbf{Sign}, \mathcal{I}\colon\ \mathbf{Sign} \to \mathbf{Trel})$, that is, an object of $\mathbf{Flat}((\mathbf{Trel}^{-})^{op})$. If $\langle \Psi, \beta, \alpha \rangle$ is a morphism from $(\mathbf{Sign}', \mathbf{Mod}', \mathbf{Sen}', \models')$ to $(\mathbf{Sign}, \mathbf{Mod}, \mathbf{Sen}, \models)$, then let $\langle \Psi, \mu \rangle$ be a morphism in $\mathbf{Flat}((\mathbf{Trel}^{-})^{op})$, from $(\mathbf{Sign}', \mathcal{I}')$ to $(\mathbf{Sign}, \mathcal{I})$

$$\begin{tikzcd}
\mathbf{Sign} \arrow[dr, "\mathcal{I}"] & \\
\arrow[r, phantom, "\Downarrow\mu"] & \mathbf{Trel} \\
\mathbf{Sign}' \arrow[uu, "\Psi"] \arrow[ur, "\mathcal{I}'"'] &
\end{tikzcd}$$

where $\mu\colon\ \Psi;\mathcal{I} \Rightarrow \mathcal{I}'$ is the natural transformation defined as $\mu_{\Sigma'} = \langle \beta_{\Sigma'}, \alpha_{\Sigma'} \rangle$. It is easy (but meticulous) to show that μ is a natural transformation and that this map indeed gives an isomorphism between $\mathcal{INS}$ and $\mathbf{Flat}((\mathbf{Trel}^{-})^{op})$.

Therefore, we can use morphisms in $\mathbf{Flat}((\mathbf{Trel}^{\text{-}})^{op})$ instead of institution morphisms whenever such choice simplifies the exposition. The following result was proved for the first time by Tarlecki in [Tar86a] (see also [Tar96a, Tar96b]):

Corollary 18. *$\mathcal{INS}$ is complete.*

Proof. By Proposition 13, Proposition 6 and Corollary 4, one obtains that the category $\mathbf{Flat}((\mathbf{Trel}^{\text{-}})^{op})$ is complete, so $\mathcal{INS}$ is complete.

3.2 Institution Representations

A slightly different notion of mapping between institutions, namely institution representation, was introduced by Tarlecki in [Tar87] (see also [Tar96a, Tar96b]). This is a special case of Meseguer's map of institutions [Mes89]:

Definition 19. Given two institutions $\mathcal{I} = (\mathbf{Sign}, \mathbf{Mod}, \mathbf{Sen}, \models)$ and $\mathcal{I}' = (\mathbf{Sign}', \mathbf{Mod}', \mathbf{Sen}', \models')$, an **institution representation** from $\mathcal{I}$ to $\mathcal{I}'$ consists of $\Phi\colon \mathbf{Sign} \to \mathbf{Sign}'$, a natural transformation $\beta\colon \Phi;\mathbf{Mod}' \Rightarrow \mathbf{Mod}$, and a natural transformation $\alpha\colon \mathbf{Sen} \Rightarrow \Phi;\mathbf{Sen}'$, such that the following *Representation Condition* holds for each $\Sigma \in |\mathbf{Sign}|$, $m' \in |\mathbf{Mod}'(\Phi(\Sigma))|$, and $e \in \mathbf{Sen}(\Sigma')$:

$$m' \models_{\Phi(\Sigma)} \alpha_\Sigma(e) \ \text{ iff } \ \beta_\Sigma(m') \models_\Sigma e$$

Let $\mathcal{INS}^{repr}$ denote the category of institutions and institution representations.

Intuitively, a representation from $\mathcal{I}'$ to $\mathcal{I}$ is an encoding of the logic $\mathcal{I}'$ into the logic $\mathcal{I}$.

Theorem 20. *$\mathcal{INS}^{repr}$ is isomorphic to* $\mathbf{Flat}(\mathbf{Trel}^{\text{-}})$.

Proof. Similarly but dually to the proof of Theorem 17, if $\langle \Phi, \beta, \alpha \rangle$ is a representation from $(\mathbf{Sign}, \mathbf{Mod}, \mathbf{Sen}, \models)$ to $(\mathbf{Sign}', \mathbf{Mod}', \mathbf{Sen}', \models')$, then let $\langle \Phi, \rho \rangle$ be a morphism in $\mathbf{Flat}(\mathbf{Trel}^{\text{-}})$, from $(\mathbf{Sign}, \mathcal{I})$ to $(\mathbf{Sign}', \mathcal{I}')$:

$$\begin{tikzcd}
\mathbf{Sign} \arrow[dd, "\Phi"'] \arrow[dr, "\mathcal{I}"] & \\
\arrow[r, phantom, "\Downarrow\rho"] & \mathbf{Trel} \\
\mathbf{Sign}' \arrow[ur, "\mathcal{I}'"'] &
\end{tikzcd}$$

such that $\rho\colon \mathcal{I} \Rightarrow \Phi;\mathcal{I}'$ is the natural transformation defined as $\rho_\Sigma = \langle \beta_\Sigma, \alpha_\Sigma \rangle$. This defines an isomorphism between $\mathcal{INS}^{repr}$ and $\mathbf{Flat}(\mathbf{Trel}^{\text{-}})$. The rest of the proof is left to the reader.

Thus, we can use morphisms in $\mathbf{Flat}(\mathbf{Trel}^{\text{-}})$ instead of the more complicated institution representations whenever such choice simplifies the exposition. Completeness of $\mathcal{INS}^{repr}$ was shown for the first time by Tarlecki in [Tar96b]:

Corollary 21. *$\mathcal{INS}^{repr}$ is complete.*

Proof. By Proposition 13, Proposition 6, Theorem 2 and the fact that **Cat** is complete, $\mathbf{Flat}(\mathbf{Trel}^{\text{-}})$ is complete, so $\mathcal{INS}^{repr}$ is complete.

3.3 Duality Between Morphisms and Representations

Arrais and Fiadeiro [MJ96] observed for the first time that given an adjoint pair of functors between two categories of signatures, there is a duality between the institution morphisms and the institution representations associated. We are pleased to notice that this nice result is a natural consequence of the fact that the functor $\mathbf{Trel}^{-}$ contravariantly lifts adjoint pairs to functor categories (Theorem 9):

Theorem 22. *If $\Phi\colon \mathbf{Sign} \to \mathbf{Sign}'$ is a left adjoint to $\Psi\colon \mathbf{Sign}' \to \mathbf{Sign}$ then for any institutions $I\colon \mathbf{Sign} \to \mathbf{Trel}$ and $I'\colon \mathbf{Sign}' \to \mathbf{Trel}$ there is a bijection between institution morphisms $\langle \Psi, \mu\rangle\colon\ I' \to I$ and institution representations $\langle \Phi, \rho\rangle\colon\ I \to I'$. Moreover, this bijection is natural in I and I'.*

Proof. It follows easily by Corollary 10, as the institution morphisms and the institution representations are ordinary morphisms in $\mathbf{Flat}((\mathbf{Trel}^{-})^{op})$ (Theorem 17) and $\mathbf{Flat}(\mathbf{Trel}^{-})$ (Theorem 20), respectively.

The bijection in Corollary 10 takes a natural transformation $\mu\colon\ \Psi; I \Rightarrow I'$ to $(\eta \,\text{⨾}\, 1_I); (1_\Psi \,\text{⨾}\, \mu)$, and its inverse takes a natural transformation $\rho\colon\ I \Rightarrow \Psi; I'$ to $(1_\Psi \,\text{⨾}\, \rho); (\epsilon \,\text{⨾}\, 1_{I'})$, where η and ϵ are the unit and the counit of the adjunction, respectively. Translating that in a more institutional language, by the construction of isomorphisms in Theorems 17 and 20, one gets that:

1. institution morphism $\langle \Psi, \alpha', \beta'\rangle\colon\ I' \Rightarrow I$ yields an institution representation $\langle \Phi, \alpha, \beta\rangle\colon\ I \to I'$, where $\alpha_\Sigma\colon\ \mathbf{Sen}(\eta_\Sigma); \alpha'_{\Psi(\Sigma)}$ and $\beta_\Sigma = \beta'_{\Psi(\Sigma)}; \mathbf{Mod}(\eta_\Sigma)$ for all $\Sigma \in |\mathbf{Sign}|$, and
2. institution representation $\langle \Phi, \alpha, \beta\rangle\colon\ I \Rightarrow I'$ yields an institution morphism $\langle \Psi, \alpha', \beta'\rangle\colon\ I' \to I$, where for all $\Sigma' \in |\mathbf{Sign}'|$, $\alpha'_{\Sigma'} = \alpha_{\Psi(\Sigma')}; \mathbf{Sen}'(\epsilon_{\Sigma'})$ and $\beta'_{\Sigma'} = \mathbf{Mod}(\epsilon_{\Sigma'}); \beta_{\Psi(\Sigma')}$.

This is exactly the construction described in [MJ96].

4 Kan Extensions of Institutions

Provided a signature translation, a whole institution can be translated in two distinct but canonical ways, given by the two Kan extensions associated.

Proposition 23. *Given a small category* $\mathbf{Sign}$ *and a functor* $K\colon \mathbf{Sign} \to \mathbf{Sign}'$*, any institution* $I\colon \mathbf{Sign} \to \mathbf{Trel}$ *has both a right and a left Kan extension along* K*, and the functor* $\mathbf{Trel}^K$ *has both a right and a left adjoint.*

Proof. It follows from Proposition 8, noticing that **Trel** is both complete and cocomplete (Proposition 13).

Definition 24. Let **SCat** be the category of small categories, $\mathcal{SINS}$ be the category of institutions over small signature categories and institution morphisms, and $\mathcal{SINS}^{repr}$ be the category of institutions over small signature categories and institution representations.

It is easy to see that **SCat** is both complete and cocomplete, and that Theorems 17 and 20 can be adapted to categories of small signatures, thus getting that $\mathcal{SINS}$ and $\mathcal{SINS}^{repr}$ are both complete. It is not known if $\mathcal{INS}$ and/or $\mathcal{INS}^{repr}$ are/is cocomplete. The following result seems to be new:

Theorem 25. *$\mathcal{SINS}$ and $\mathcal{SINS}^{repr}$ are cocomplete.*

Proof. By Theorems 17 and 20, it suffices to show that $\mathbf{Flat}((\mathbf{Trel}^{\mathbf{-}})^{op})$ and $\mathbf{Flat}(\mathbf{Trel}^{\mathbf{-}})$ are cocomplete. This follows from Corollary 5 and Theorem 3, noticing that **SCat** is cocomplete, that $\mathbf{Trel}^{\mathbf{Sign}}$ is both cocomplete and complete (Proposition 6 for $\mathbf{T} = \mathbf{Trel}$, using Proposition 13) for all signature categories **Sign**, and that $\mathbf{Trel}^{K}$ has a left adjoint and a right adjoint (Proposition 23).

5 Conclusion

This paper can be regarded as an application of category theory to the theory of institutions. Organizing institutions as functors between certain categories, we both gave elegant categorical proofs to known facts (such as the completeness of the category of institutions, the duality between morphisms and representations of institutions) and obtained new conjectured results (the cocompleteness of the category of institutions). But from the author's point of view, the most fascinating result is Proposition 23, which philosophically says that provided a logic and a translation of its syntax to another syntax, then the whole logic can be translated to a logic over the new syntax in two cannonical ways.

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