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# Rigorous Numerical Studies of the Existence of Periodic Orbits for the Hénon Map

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**Abstract:** In this paper we perform a rigorous study of the Hénon map. We prove with computer assistance the existence of symbolic dynamics for $h^2$ and $h^7$ and the existence of periodic orbits of all periods but 3 and 5.

**Key Words:** chaos, computer assisted proof, interval arithmetic.

## 1 Introduction

In this paper we consider the Hénon map defined by the following equation:

$$h(x, y) = (1 + y - ax^2, bx), \tag{1}$$

where $a = 1.4$ and $b = 0.3$ are the "classical" parameter values. Although the definition of the Hénon map is very simple it displays very complicated dynamics. A typical trajectory of the map is shown in Fig. 1.

In Section 2 we recall the technique of TS-maps and formulate two theorems used in the following sections. In Section 3 we prove the existence of symbolic dynamics for $h^2$ and what follows the existence of periodic points of $h$ for all even periods.

In [Zgliczyński 97b] the dynamics of topological horseshoe was proved for $h^7$. From this follows the existence of symbolic dynamics for $h^7$ and the existence of periodic orbits of $h$ of period $7n$ for all natural $n$. In Section 4 we repeat the proof described in [Zgliczyński 97b] using interval arithmetic. We show that using this tool the number of points for which we must check certain conditions can be significantly reduced. Then checking some more conditions we prove the existence of periodic points with period 8 and all periods greater or equal to 10.

Finally by means of the interval Newton method we prove that within the region $[-5, 5] \times [-5, 5]$ there exists no periodic point with period 3 or 5 and we prove that there exist periodic points with period 9.

During all the computer–assisted proofs we use the procedures for interval computations form BIAS and PROFIL packages [Knüppel 93]. Programs were compiled using gnu C++ compiler (gcc version 2.7.2.1) and run on Sun Ultra 1 computer. The source code of the programs is available at the following www location: `http://fractal.zet.agh.edu.pl/~galias/int.html`. Additionally all the results were checked using the package for interval computations prepared by the author in Turbo-Pascal 7.0 programming environment and run on Pentium 166MHz.

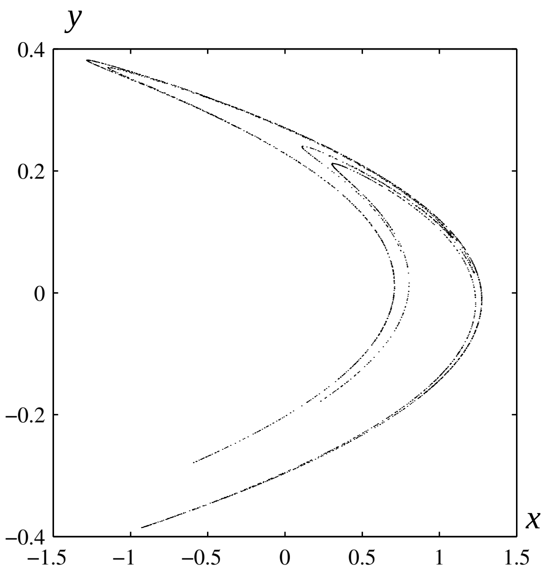


Figure 1: A trajectory of the Hénon map. 3000 points of the trajectory starting from the initial conditions: $x = 0$, $y = 0$ after a short transient (100 iterations) are plotted.

## 2 TS–Maps

One of the tools we use in our study is the technique of TS-maps (topological shifts) introduced in [Zgliczyński 97a, Zgliczyński 97b]. This technique can be used to prove the existence of an infinite number of periodic orbits for a given system. It combines existence results based on the fixed point index theory and computer–assisted computations, necessary to verify assumptions of the existence theorem.

Here we consider a special case of TS–maps defined on two sets $N_0$ and $N_1$. For the general case see [Zgliczyński 97b]. Let the sets $N_0$, $N_1$, $E_0$, $E_1$, $E_2$ be as depicted in Fig. 2. The important property of this sets is that $E_0$ lies on the left hand side of the sets $N_0$ and $N_1$, the set $E_1$ lies between $N_0$ and $N_1$ and $E_2$ lies on the right hand side of $N_0$ and $N_1$. Certain deformations of these sets are also possible (see [Zgliczyński 97b]). Let $W = N_0 \cup N_1 \cup E_0 \cup E_1 \cup E_2$. By $intW$ we denote the interior of $W$. We will say that the image of $N_i$ *covers horizontally* the set $N_j$ if the image of one of the vertical edges of $N_i$ lies on the right hand side of $N_j$ while the image of the second vertical edge lies on the left hand side of $N_j$. For example image of $N_0$ covers horizontally $N_1$ if $f(L(N_0)) \subset E_1$ and $f(R(N_0)) \subset E_2$ or $f(L(N_0)) \subset E_2$ and $f(R(N_0)) \subset E_1$, where $L(N_0)$ and $R(N_0)$ denote respectively the left and right vertical edges of $N_0$.

Let $f$ be a continuous map defined on $N_0 \cup N_1$. In our analysis we consider two special cases of TS-maps. They are described in detail in [Galias 97]. The first case involves maps with topological horseshoe embedded (compare Fig. 2a).

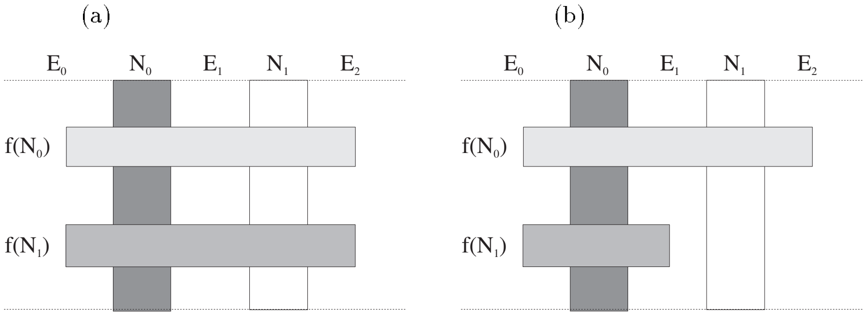


Figure 2: Images of sets $N_0$ and $N_1$ for the horseshoe map (a) and the deformed horseshoe map (b). For the horseshoe map the images of vertical edges of $N_0$ lie one in $E_0$ and the second in $E_2$ and similarly for $N_1$. For the deformed horseshoe the only difference is that the image of one of the vertical edges of $N_1$ is enclosed in $E_1$ instead of $E_2$

***Theorem 1.*** *If $f(N_0), f(N_1) \subset intW$, the image of $N_0$ covers horizontally the sets $N_0$ and $N_1$ (vertical edges of $N_0$ are mapped by $f$ in such a way that the image of one of the edges is enclosed in $E_0$, while the second one is enclosed in $E_2$), and the image of $N_1$ covers horizontally $N_0$ and $N_1$ then for any finite sequence $a_0, a_1, \ldots, a_{n-1} \in \{0,1\}^n$ there exists a point $x$ satisfying*

$$f^i(x) \in N_{a_i} \quad \textit{for} \quad i = 0, \ldots, n-1 \quad \textit{and} \quad f^n(x) = x.$$

In this case one can also prove that the full shift on two symbols with the transition matrix [Robinson 95]

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{2}$$

is embedded in the map $f$. Non-zero element $a_{ij}$ of the transition matrix means that the image of $N_i$ covers horizontally $N_j$ (we can find a point $x \in N_i$ such that $f(x) \in N_j$).

The next theorem is important for maps with the deformed horseshoe embedded (compare Fig. 2b). From the set of $n$-element sequences with elements from the set $\{0,1\}$ let us choose sequences, which do not contain the subsequence $(1,1)$:

$$T_n = \{ (a_0, \ldots, a_{n-1}) \in \{0,1\}^n \colon (a_j, a_{(j+1) \bmod n}) \neq (1,1) \text{ for } 0 \leq j < n \}. \tag{3}$$

***Theorem 2.*** *If $f(N_0), f(N_1) \subset intW$, image of $N_0$ covers horizontally the sets $N_0$ and $N_1$, image of $N_1$ covers horizontally the sets $N_0$, then for any finite sequence $a = (a_0, a_1, \ldots, a_{n-1}) \in T_n$ there exists a point $x$ satisfying*

$$f^i(x) \in N_{a_i} \quad \textit{for} \quad i = 0, \ldots, n-1 \quad \textit{and} \quad f^n(x) = x. \tag{4}$$

In this case the subshift on two symbols with the transition matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \tag{5}$$

is embedded in $f$.

If $f$ is one-to-one one has to check only the conditions concerning the edges of sets $N_i$. Instead of proving that $f(N_i) \subset intW$ it is sufficient to prove that $f(bd(N_i)) \subset intW$, where $bd(N_i)$ denotes the border of the set $N_i$. This is a conclusion from Jordan's theorem (compare [Galias 97]).

## 3 Symbolic Dynamics for $h^2$ — Deformed Horseshoe

In this section we show using the technique described previously that the subshift on two symbols with the transition matrix (5) (the deformed topological horseshoe) is embedded in $h^2$.

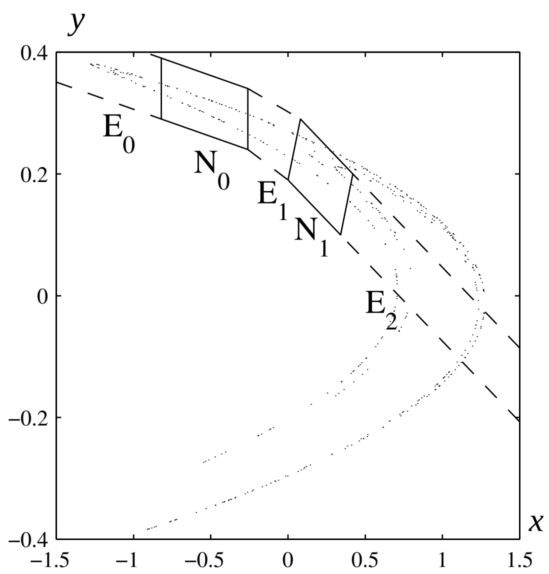


Figure 3: The definition of the sets $N_0$ and $N_1$ for the proof of symbolic dynamics for $h^2$

Let us define the sets $N_i$ as follows: $N_0$ is a quadrangle $\overline{A_1A_2A_3A_4}$ and $N_1$ is the quadrangle $\overline{A_5A_6A_7A_8}$, where $A_1 = (-0.82, 0.29)$, $A_2 = (-0.82, 0.39)$, $A_3 = (-0.26, 0.34)$, $A_4 = (-0.26, 0.24)$, $A_5 = (0, 0.19)$, $A_6 = (0.08, 0.29)$, $A_7 = (0.42, 0.2)$ and $A_8 = (0.34, 0.1)$ (compare Fig. 3). We also define sets $E_0$, $E_1$ and $E_2$ lying respectively to the left, between and to the right of the sets $N_0$ and $N_1$. The set $E_0$ is a half–stripe lying on the left hand side of $N_0$ defined by straight lines $A_2A_3$, $A_4A_1$ and $A_1A_2$. $E_1$ is the quadrangle $\overline{A_4A_3A_6A_5}$. $E_2$ is a half–stripe lying on the right hand side of $N_1$, defined similarly as $E_0$. Let $W = N_1 \cup N_2 \cup E_0 \cup E_1 \cup E_2$.

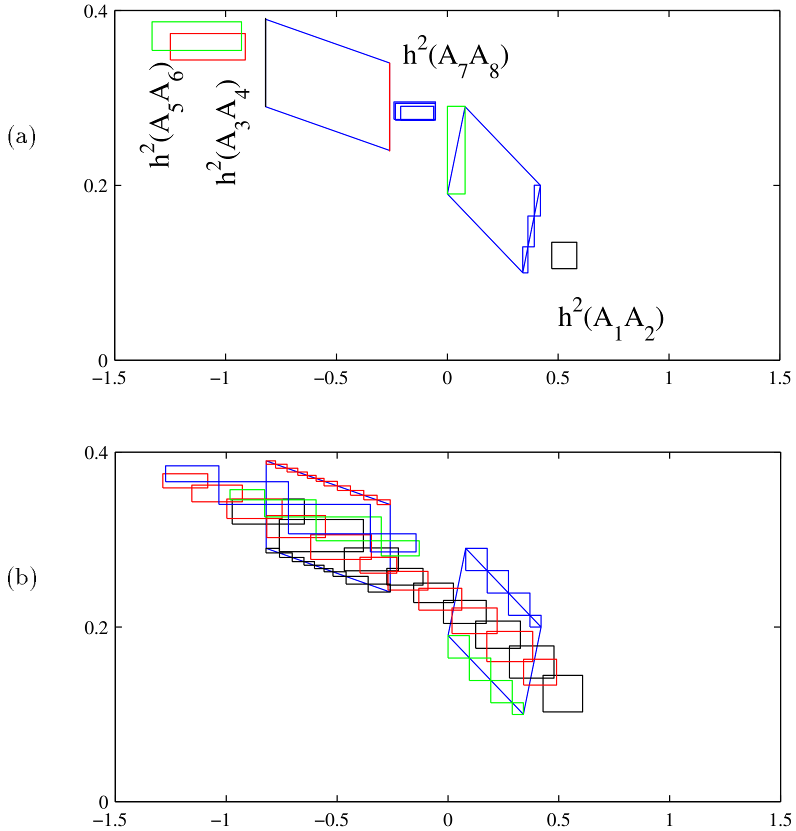


Figure 4: (a) the covering of the vertical edges of $N_0$ and $N_1$ with rectangles (notice that the rectangles covering edges of $N_0$ are very thin as these edges are parallel to the $y$ axis) and its image under $h^2$, (b) the covering of horizontal edges of $N_0$ and $N_1$ and its image under $h^2$ obtained in computer assisted proof

With the computer assistance we have proved that the image of $N_0$ covers horizontally $N_0$ and $N_1$ and the image of $N_1$ covers horizontally $N_0$. This is formally written in the following lemma.

***Lemma 3.***

1. $h^2(\overline{A_1A_2}) \subset E_2$ *and* $h^2(\overline{A_3A_4}) \subset E_0$,
2. $h^2(\overline{A_5A_6}) \subset E_0$ *and* $h^2(\overline{A_7A_8}) \subset E_1$,
3. $h^2(\overline{A_1A_4})$, $h^2(\overline{A_2A_3})$, $h^2(\overline{A_5A_8})$, $h^2(\overline{A_6A_7}) \subset intW$.

*Proof.* For the proof of 1 and 2 we have covered the vertical edges $\overline{A_1A_2}$, $\overline{A_3A_4}$, $\overline{A_5A_6}$ and $\overline{A_7A_8}$ by 1, 1, 1 and 3 rectangles respectively. Using interval arithmetic we have proved that their images under $h^2$ are enclosed in the appropriate sets

$E_i$. The covering of vertical edges with rectangles (two-dimensional intervals) and their images under $h^2$ computed during the proof are shown in Fig. 4a. One can clearly see that $h^2(\overline{A_1A_2})$ lies on the right hand side of $N_1$, $h^2(\overline{A_3A_4})$ and $h^2(\overline{A_5A_6})$ lie on the left hand side of $N_0$ and $h^2(\overline{A_7A_8})$ lies between $N_0$ and $N_1$.

For the proof of 3 we have covered the horizontal edges $\overline{A_1A_4}$, $\overline{A_2A_3}$, $\overline{A_5A_8}$ and $\overline{A_6A_7}$ by 9, 11, 4 and 4 rectangles respectively. The covering of horizontal edges with rectangles and their images under $h^2$ are shown in Fig. 4b. We have checked that the images are enclosed within the set $intW$.

For the whole proof of the existence of symbolic dynamics for $h^2$ it was sufficient to compute images of 34 rectangles under $h^2$.

From Lemma 3 and Theorem 2 it follows that for every sequence of symbols $a = (a_0, a_1, \ldots, a_{n-1}) \in T_n$ there exists a point $x$ such that

$$h^{2i}(x) \in N_{a_i} \quad \text{for} \quad i = 0, \ldots, n-1 \quad \text{and} \quad h^{2n}(x) = x.$$

In particular for every positive integer $n$ there exists a periodic point of $h^2$ with period $n$. Hence for every even integer $n$ there exists a periodic point of the Hénon map with period $n$. In this way we have also proved that the subshift on two symbols with the transition matrix (5) is embedded in $h^2$.

## 4 Symbolic Dynamics for $h^7$ — Topological Horseshoe

In [Zgliczyński 97b] the author introduced the quadrangles $N_0 = \overline{A_1A_2A_3A_4}$, $N_1 = \overline{A_5A_6A_7A_8}$ shown in Fig. 5 (notice that they are different to the sets defined in the previous section), where $A_1 = (0.460, 0.01)$, $A_2 = (0.595, 0.28)$, $A_3 = (0.691, 0.28)$, $A_4 = (0.556, 0.01)$, $A_5 = (0.588, 0.01)$, $A_6 = (0.723, 0.28)$, $A_7 = (0.755, 0.28)$ and $A_8 = (0.62, 0.01)$. He also defined the set $E_0$ as a part of the plane lying above the straight line $A_1A_4$ and on the left hand side of line $A_1A_2$, $E_1 = \overline{A_4A_3A_6A_5}$ and the set $E_2$ consisting of points lying below line $A_5A_8$ or below line $A_6A_7$ and one the right hand side of line $A_7A_8$. The set $W$ is defined as before as $W = N_0 \cup N_1 \cup E_0 \cup E_1 \cup E_2$. For these sets using the technique of TS-maps he proved the existence of the topological horseshoe. He proved that the full shift on two symbols with the transition matrix (2) is embedded within the map $h^7$. Zgliczyński did not use the interval arithmetic. Instead he computed the $7^{th}$ iteration of the Hénon map at some points and estimated the position of nearby points after seven iterations by means of Lipschitz constant of the Hénon map. The proof required computation of $h^7$ for approximately 60000 points.

Using the same sets $N_i$ and $E_i$ we have repeated the proof. In order to prove the existence of symbolic dynamics associated with the full shift we have to prove that the images of $N_0$ and $N_1$ under $h^7$ cover horizontally the set $N_0 \cup N_1$.

***Lemma 4.*** *The image of $N_0$ under $h^7$ covers horizontally $N_0$ and $N_1$, i.e.,*

$$h^7(A_1A_2) \subset E_2 \text{ and } h^7(A_3A_4) \subset E_0,$$

$$h^7(\overline{A_1A_4}), h^7(\overline{A_2A_3}) \subset intW.$$

*The image of $N_1$ under $h^7$ covers horizontally $N_0$ and $N_1$, i.e.,*

$$h^7(A_5A_6) \subset E_0 \text{ and } h^7(A_7A_8) \subset E_2,$$

$$h^7(\overline{A_5A_8}), h^7(\overline{A_6A_7}) \subset intW.$$

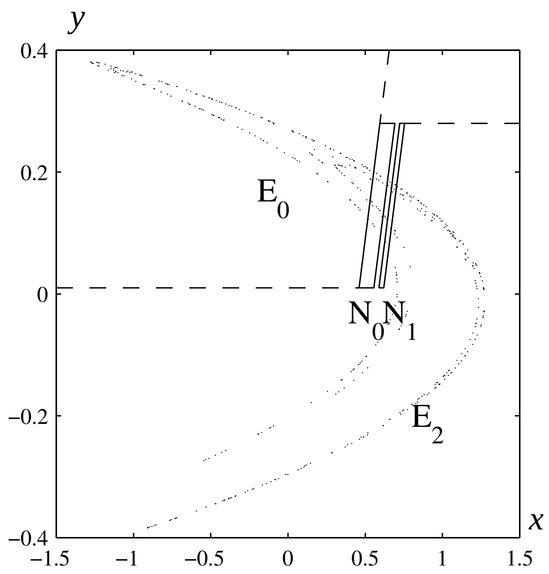


Figure 5: The definition of the sets $N_0$ and $N_1$ for the proof of symbolic dynamics for $h^7$

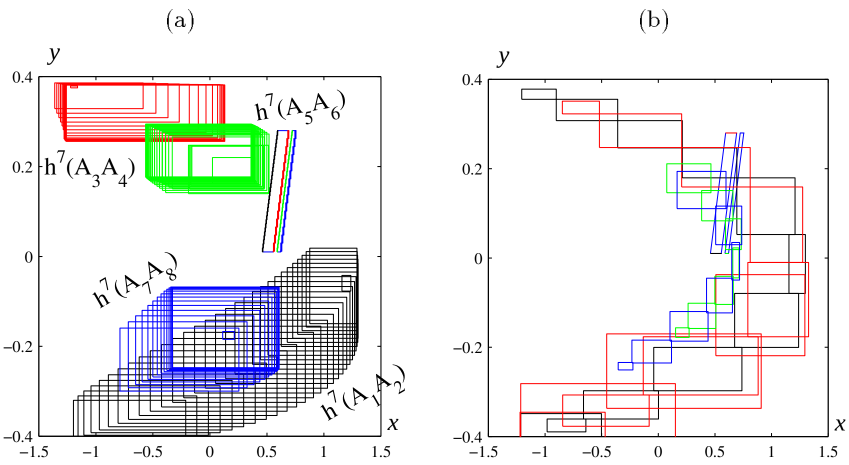


Figure 6: (a) the covering of the vertical edges of $N_0$ and $N_1$ with rectangles and their images under $h^7$, (b) the covering of horizontal edges of $N_0$ and $N_1$ and their images under $h^7$

*Proof.* The proof was carried out using interval arithmetic. The covering of vertical edges with rectangles and their images under $h^7$ are shown in Fig. 6a. Similar covering of horizontal edges and its image are shown in Fig. 6b. We have checked that they are enclosed in appropriate sets. For the proof of the existence of topological horseshoe it was sufficient to compute the images of 131 rectangles under $h^7$.

Notice that the number of rectangles for which the image is computed is significantly reduced when compared to the original proof. Probably Zgliczyński overestimated the error (he did not use the interval arithmetic).

## 5 Periodic Points with Periods $n \geq 7$, $n \neq 9$

Lemma 4 states that the images of sets $N_i$ under $h^7$ covers horizontally the sets $N_0$ and $N_1$. It follows that for every natural $n$ there exists a periodic point of $h$ with period $7n$. In order to prove the existence of periodic points with other periods we have checked the positions of $N_0$ and $N_1$ under $h^i$, for $i = 1, \ldots, 6$.

***Lemma 5.***

1. *The set $h^1(N_0)$ covers $N_1$, i.e.,*

$$h^1(\overline{A_1A_2}) \subset E_2, h^1(\overline{A_3A_4}) \subset E_1 \cup N_0 \cup E_0, \tag{6}$$

$$h^1(\overline{A_2A_3}), h^1(\overline{A_4A_1}) \subset intW. \tag{7}$$

*The set $h^2(N_0)$ covers $N_0$, i.e.,*

$$h^2(\overline{A_1A_2}) \subset E_0, h^2(\overline{A_3A_4}) \subset E_1 \cup N_1 \cup E_2, \tag{8}$$

$$h^2(\overline{A_2A_3}), h^2(\overline{A_4A_1}) \subset intW. \tag{9}$$

*The set $h^i(N_0)$ for $i = 3, \ldots, 6$ covers both of the sets $N_0$ and $N_1$, i.e.,*

$$h^3(\overline{A_1A_2}), h^4(\overline{A_3A_4}), h^5(\overline{A_1A_2}), h^6(\overline{A_3A_4}) \subset E_2, \tag{10}$$

$$h^3(\overline{A_3A_4}), h^4(\overline{A_1A_2}), h^5(\overline{A_3A_4}), h^6(\overline{A_1A_2}) \subset E_0, \tag{11}$$

$$h^i(\overline{A_2A_3}), h^i(\overline{A_4A_1}) \subset intW \text{ for } i = 3, \ldots, 6. \tag{12}$$

2. *Images of edges of $N_1$ under $h^i$ (for $i = 1, \ldots, 6$) have empty intersection with the sets $N_0$ and $N_1$.*

$$h^1(L), h^3(L), h^5(L) \subset E_0 \text{ and } h^2(L), h^4(L), h^6(L) \subset E_2,$$

*where $L$ is any of the edges $\overline{A_5A_6}$, $\overline{A_6A_7}$, $\overline{A_7A_8}$, $\overline{A_8A_5}$.*

*Proof.* For the proof the edges $\overline{A_1A_2}$, $\overline{A_2A_3}$, $\overline{A_3A_4}$, $\overline{A_4A_1}$, $\overline{A_5A_6}$, $\overline{A_6A_7}$, $\overline{A_7A_8}$ and $\overline{A_8A_5}$ were covered by 19, 11, 42, 11, 35, 7, 16, and 7 rectangles respectively. The images of these rectangles under $h^i$ for $i = 1, \ldots, 6$ were computed. We have checked that the conditions (6)...(12) are fulfilled.

The results proved in lemmas 4 and 5 are summarized in Table 1. Using these results one can easily prove the existence of periodic points for all periods greater or equal to 7 with the exception of period 9.

| $i$ | $h^i(N_0)$ | $h^i(N_1)$ |
|---|---|---|
| 1 | $N_1$ | — |
| 2 | $N_0$ | — |
| 3 | $N_0,N_1$ | — |
| 4 | $N_0,N_1$ | — |
| 5 | $N_0,N_1$ | — |
| 6 | $N_0,N_1$ | — |
| 7 | $N_0,N_1$ | $N_0,N_1$ |

Table 1: Images of $N_0$ and $N_1$ under $h^i$ $(i = 1, \ldots, 7)$. In the second and third columns the sets which are covered horizontally by $h^i(N_0)$ and $h^i(N_1)$ are given

***Lemma 6.*** *For every integer $n \geq 7$, $n \neq 9$ there exist periodic point of $h$ with period $n$.*

*Proof.* As an example we show how to prove the existence of period-8 orbit. Let us consider the set $N_1$. As it follows from lemma 4 the image of $N_1$ under $h^7$ covers $N_0$. From lemma 5 it follows that $h(N_0)$ covers $N_1$. Hence it is clear that $h^8(N_1)$ covers $N_1$. Using similar argument as for the TS-maps one can prove that there exists a point $x$ within $N_1$ such that $h^8(x) = x$. Now it is sufficient to prove that 8 is the minimal period of $x$. But this is clear as $h^i(N_1)$ has empty intersection with $N_1$ for $i = 1, \ldots, 6$.

## 6 Periodic Points with Periods 1, 3, 5 and 9

So far we have shown that there exist periodic points with all periods but 1, 3, 5 and 9. The existence of a fixed point can be proved analytically. There exist two such points $(x_1, bx_1)$ and $(x_2, bx_2)$ where

$$x_{1,2} = \frac{b - 1 \pm \sqrt{(1-b)^2 + 4a}}{2a}.$$

One of the fixed points is embedded within the numerically observed strange attractor.

In order to decide the existence of periodic points with periods 3, 5 and 9 within the set $M = [-5, 5] \times [-5, 5]$ we have used the interval Newton method [Alefeld 94, Götz 94]. This method allows to prove the existence and uniqueness of fixed points within specific interval. It also allows to exclude the existence of a fixed point within a given interval. The idea is to divide the set $M$ into small subsets for which assumptions of the interval Newton method can be checked. Using this technique we have proved the following lemma.

***Lemma 7.*** *Let $M = [-5, 5] \times [-5, 5]$.*

1. *There exists no periodic point with period 3 within the set $M$.*
2. *There exists no periodic point with period 5 within the set $M$.*
3. *There exist 6 period-9 orbits within $M$.*

*Proof.* To prove part 1 we have covered the set $M$ by 493 rectangles. Using the interval Newton method we have proved that there are no period-3 orbits within any of these rectangles. Similarly using 4241 rectangles for the covering of $M$ we have proved that there are no period-5 orbits within $M$. For the proof of part 3 the set $M$ was covered by 2974053 rectangles. We have proved the existence of exactly 54 periodic points with period 9 within $M$ which correspond to 6 different period-9 orbits.

## 7 Conclusions

In this paper we have shown rigorously with computer assistance that

**A.** the subshift on two symbols corresponding to the deformed horseshoe is embedded in $h^2$,
**B.** the full shift on two symbols corresponding to the topological horseshoe is embedded in $h^7$,
**C.** $h$ has periodic points of all periods but 3 and 5,
**D.** $h$ has no periodic points with periods 3 and 5 within the set $[-5,5]\times[-5,5]$.

The symbolic dynamics for $h^2$ and $h^7$ is proved for invariant sets embedded in the strange attractor observed numerically. Also all the periodic orbits the existence of which is proved (apart from one of the fixed points) lie in the region where the strange attractor is observed. This indicates that the dynamics of the system is very complicated. However the existence of a strange attractor for classical values of parameters still remains an open problem.

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## References

[Alefeld 94] Alefeld, G.; "Inclusion methods for systems of nonlinear equations — the interval Newton method and modifications"; Topics in Validated Computations, J. Herzberger ed., Elsevier Science 1994, 7–26.

[Götz 94] Götz, A.; "Inclusion Methods for Systems of Nonlinear — The Interval Newton Method and Modifications"; in Topics in Validated Computations, ed. J. Herzberger, 1994, Elsevier Science B.V., 7–26.

[Galias 97] Galias, Z.; "Positive topological entropy of Chua's circuit: a computer assisted proof"; Int. J. Bifurcation and Chaos, 7, 2 (1997), 331–349.

[Galias, Zgliczyński 96] Galias, Z., Zgliczyński, P.; "Computer assisted proof of chaos in the Lorenz system", IMUJ preprint 1996/23, accepted for publication in Physica D.

[Guckenheimer, Holmes 83] Guckenheimer, J., Holmes, P.; "Nonlinear Oscillations,Dynamical Systems, and Bifurcations of Vector Fields"; Springer–Verlag, 1983.

[Hénon 76] M. Hénon; "A two dimensional map with a strange attractor"; Commun. Math. Phys., 50, 1976, 463.

[Robinson 95] Robinson, C.; "Dynamical Systems: Stability, Symbolic Dynamics, and Chaos"; CRC Press, Boca Raton, 1995.

[Knüppel 93] Knüppel, O.: "PROFIL — programmer's runtime optimized fast interval library"; Technical University Hamburg-Harburg, July 1993, Bericht 93.4.

[Zgliczyński 97a] P. Zgliczyński; "Computer assisted proof of the horseshoe dynamics in the Hénon map"; Random & Computational Dynamics, 5, 1 (1997), 1–19.

[Zgliczyński 97b] Zgliczyński, P.: "Computer assisted proof of chaos in the Rössler equations and in the Hénon map"; Nonlinearity, 10, 1 (1997), 243–252.