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Finding All Solutions of Nonlinear Systems of Equations Using Linear Programming with Guaranteed Accuracy

Yusuke Nakaya
(Oishi Lab., Graduate School of Science and Engineering, Waseda Univ., JAPAN
nakaya@oishi.info.waseda.ac.jp)

Shin'ichi Oishi
(School of Science and Engineering, Waseda Univ., JAPAN
oishi@oishi.info.waseda.ac.jp)

Abstract: A linear programming-based method is presented for finding all solutions of nonlinear systems of equations with guaranteed accuracy. In this method, a new effective linear programming-based method is used to delete regions in which solutions do not exist. On the other hand, Krawczyk's method is used to find regions in which solutions exist. As an illustrative example, all solutions of the nonlinear system of equations describing equilibrium conditions of the "high polymer liquid system", which is a well-known ill-conditioned system of equations, are identified by the method.

Key Words: Nonlinear Equations, All Solutions, Guaranteed Accuracy, Linear Programming, Krawczyk's Method

1 Introduction

In this paper, we are concerned with the problem of finding all solutions of the system of nonlinear equations of the following form:

$$f(x) = 0, \tag{1}$$

where $f : \boldsymbol{R}^n \to \boldsymbol{R}^n$ is a smooth function. Our goal is to find "all" solutions of this system in a certain region, and to guarantee that in this region there are no solutions other than these solutions. Krawczyk's method is one of the most famous methods for such a problem [1][2][3]. In previous papers [4],[5], we have presented the conditions under which one can find all solutions by Krawczyk's method with guaranteed accuracy, but, there still remain many problems for which all solutions may not be obtained by Krawczyk's method.

In this paper, we shall present a linear programming based method of finding all solutions of the system of nonlinear equations with guaranteed accuracy. This method was developed when we were solving the famous ill-conditioned problem related with equilibrium conditions of the high polymer liquid system [6], because we recognized that its all solutions are hard to obtain by Krawczyk's method.

In the first place, motivated by Yamamura's work [7], we shall present a numerical method of proving the nonexistence of solutions of nonlinear equations.

As an example, we shall show that by our method all solutions of the ill-conditioned system of nonlinear equations describing the "high polymer liquid system" can be obtained.

2 Krawczyk's Method

In this section, we introduce Kraw- czyk's method [2][4].

Notations of interval analysis used in this paper are as follows:

$I(D)$: set of all (vector) intervals belongs to $D \subset \boldsymbol{R}^n$

mid(I) : midpoint of interval I (componentwise)

The validity of Krawczyk's method is indicated by the following theorem.

Theorem 1 *Let $f : \boldsymbol{R}^n \to \boldsymbol{R}^n$ be C^1. For a given interval $I \in I(D)$, define an interval K as*

$$K := c - L^{-1}f(c) + M(I-c), \tag{2}$$

where

$$M := E - L^{-1}f'(I), \tag{3}$$

E is the $n \times n$-unit vector, c is mid(I), and L is a regular non-interval matrix, which is an approximation of $f'(c)$.

If the following condition

$$K \subset I, \|M\| < 1 \tag{4}$$

are valid, there exists a unique solution of the equation $f(x) = 0$ in I. □

According to the relation between K and I, there exists three cases:

- **case1.** $K \subset I \to$ there exists a unique solution in I,
- **case2.** $K \cap I = \phi \to$ there exists no solution in I,
- **case3.** Other $\to$ divide I, and check the relation between K and I respectively.

3 Numerical Method for Checking the Nonexistence of Solutions

In this section, we present a numerical method for checking the nonexistence of solutions using linear programming with guaranteed accuracy. We shall also show that the nonexistence of solutions in a open region or a convex polyhedral region is easily checked by using this method.

3.1 Checking the Nonexistence of Solutions Using Linear Programming

Let us consider the system of nonlinear equations of the following form:

$$\sum_{j=1}^{l_i} f_{ij}(x_{k_{ij}}) + \sum_{k=1}^{n} a_{ik}x_k + b_i = 0, \tag{5}$$

$(1 \leq i \leq n)$. Although this type of equations has a special form, it is shown that a wide variety of equations can be reduced to this form. In the first place, we are concerned with the problem of searching all solutions of this system in a region

$$a_k \leq x_k \leq b_k, \quad (1 \leq k \leq n) \tag{6}$$

with guaranteed accuracy. We assume that the function $f_{ij}(x_k)$ is a continuous and piecewize convex or concave function of the variable x_k in the region $[a_k, b_k]$. Introducing a slack variable y_{ij}, put

$$y_{ij} = f_{ij}(x_k). \tag{7}$$

Let $F_{ij}(x_k)$ and $F'_{ij}(x_k)$ be interval inclusions of the function values of $f_{ij}(x_k)$ and $f'_{ij}(x_k)$ respectively. For a while we consider the case that f_{ij} is a convex function on the interval $[a_k, b_k]$. We can treat similarly the case that $f_{ij}(x_k)$ is concave. Since y_{ij} belongs to the range $f_{ij}([a_k, b_k])$, y_{ij} is in the polyhedral region in (6) surrounded by the following three lines:

① the line connecting the upper bounds of $F_{ij}(a_k)$ and $F_{ij}(b_k)$
② the line whose slope is the lower bound of $F'_{ij}(a_k)$ and passing the lower bound of $F_{ij}(a_k)$
③ the line whose slope is the upper bound of $F'_{ij}(b_k)$ and passing the lower bound of $F_{ij}(b_k)$

This region becomes a pentagon as shown in Figure 1.

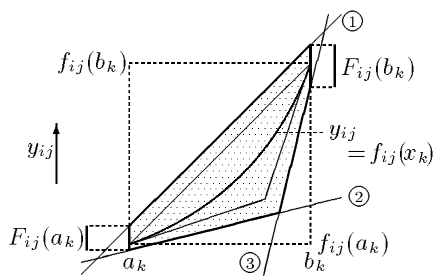

Figure 1: A region containing the range $f_{ij}([a_k, b_k])$

Noticing this, let us consider the following linear programming problem:

$$\begin{array}{l} \min \ : \ \ \text{arbitrary function}, \\ \text{subject to} \\ \quad \displaystyle\sum_{j=1}^{l_i} y_{ij} + \sum_{k=1}^{n} a_{ik} x_k + b_i = 0, \\ \quad a_k \le x_k \le b_k, \\ \quad \text{Linear inequalities for } y_{ij} : \\ \qquad\qquad \begin{pmatrix} y_{ij} \le \text{line } ① \\ y_{ij} \ge \text{line } ② \\ y_{ij} \ge \text{line } ③ \end{pmatrix}. \end{array} \tag{8}$$

It is seen that if the feasible region of this linear programming problem (8) is empty, which can be easily checked by Phase-I of the simplex method, then there are no solutions for the nonlinear equation (5) in the region (6).

3.2 Nonexistence of Solutions in an Open Region

Let us next consider the case of

$$f_{ij}(a_k) = -\infty, \tag{9}$$

where f_{ij} is a continuous and concave function on the semi-open region

$$a_k < x_k \le b_k, \tag{10}$$

for example, $f_{ij}(x_k) = \ln(x_k - a_k)$.

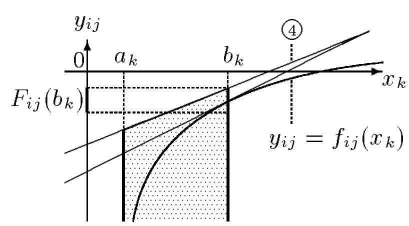

Figure 2: region of finding solutions in open interval

In this case, although it is impossible to surround the range $f_{ij}((a_k, b_k])$ with the pentagon as shown in Figure 1, it is seen from Figure 2 that y_{ij} is contained in the region

$$y_{ij} \le \ \text{line } ④, \tag{11}$$

where the line ④ is the line whose slope is the lower bound of $F'_{ij}(b_k)$ and passing the upper bound of $F'_{ij}(b_k)$. Thus, the nonexistence of solutions in (semi-)open region $(a_k, \varepsilon]$ can be checked by solving Phase-I of the following linear programming problem:

$$
\begin{aligned}
&\text{min} \ : \ \ \text{arbitrary function,} \\
&\text{subject to} \\
&\quad \sum_{j=1}^{l_i} y_{ij} + \sum_{k=1}^{n} a_{ik}x_k + b_i = 0, \\
&\quad a_k \le x_k \le b_k, \\
&\quad y_{ij} \le \ \text{line ④}.
\end{aligned}
\tag{12}
$$

3.3 Nonexistence of Solutions in a Convex Polyhedron

Let us consider the case that the region is given by a polyhedron. As an example, let us consider the case that the inequality

$$x_i \ge x_j \tag{13}$$

holds between x_i and x_j, so that the region becomes a triangle. In this case, the nonexistence of solutions can also be checked by adding the condition (13) to the linear programming problem. Similarly, it is easy to see that the nonexistence of solutions in arbitrary convex polyhedron D $(\subset \boldsymbol{R}^n)$ can be checked by our method.

4 Numerical Example

In this section, a numerical example is reported to illustrate the usefulness of our method. In the example, new linear programming based method described in previous sections is used for checking the nonexistence of solution. Generally, it is known that it is difficult to apply the simplex method with guaranteed accuracy. However, doing Phase-I with no roundings on all steps by using rational arithmetic, we can calculate with guaranteed accuracy. On the other hand, Krawczyk's method is used for proving the existence of solution.

Let us consider finding all sets of x_1, y_1 and y_2 of the following nonlinear equation related with the high polymer liquid system:

$$
\begin{cases}
f(x_1, y_1) = f(x_2, y_2) \\
g(x_1, y_1) = g(x_2, y_2) \\
h(x_1, y_1) = h(x_2, y_2)
\end{cases},
\tag{14}
$$

where $f, g,$ and h are

$$
\begin{aligned}
f(x,y) = \ln(1-x) + &\left(\alpha + \frac{\beta_0(1-\gamma)}{(1-\gamma x)^2}\right)x^2 + x \\
&- \frac{1}{2}\left(\left(\frac{1}{m_1} + \frac{1}{m_2}\right)x + \left(\frac{1}{m_1} - \frac{1}{m_2}\right)y\right),
\end{aligned}
$$

$$
\begin{aligned}
g(x,y) = {} & m_1(1-x)^2\left(\alpha+\frac{\beta_0}{(1-\gamma x)^2}\right) \\
& +\ln\left(\frac{x+y}{2}\right)+m_1x-(m_1-1) \\
& -\frac{m_1}{2}\left(\left(\frac{1}{m_1}+\frac{1}{m_2}\right)x+\left(\frac{1}{m_1}-\frac{1}{m_2}\right)y\right), \\
h(x,y) = {} & m_2(1-x)^2\left(\alpha+\frac{\beta_0}{(1-\gamma x)^2}\right) \\
& +\ln\left(\frac{x-y}{2}\right)+m_2x-(m_2-1) \\
& -\frac{m_2}{2}\left(\left(\frac{1}{m_1}+\frac{1}{m_2}\right)x+\left(\frac{1}{m_1}-\frac{1}{m_2}\right)y\right),
\end{aligned}
$$

and $m_1 = 125.2, m_2 = 5116, \alpha = -0.1091, \gamma = 0.2481, \beta_0 = 0.86$, $x_2 = 0.28$, and $x_1 < x_2$.

This nonlinear equation is too ill-conditioned to find all solutions. In the case of applying Krawczyk's method, the relation between $K(I)$ and I does not hold **case 1, 2** of Krawczyk's method, which is described in section 2, and the region is just divided to many small regions. As a result, the solutions of this problem have not been obtained by Krawczyk's method within reasonable time and memory space. On the other hand, this nonlinear equation has logarithmic function, so that the region becomes an open and a convex polyhedral region. Therefore, this nonlinear equation is applicable for our method.

By our method, regions in which solutions do not exist are deleted very effectively. As a result, three regular solutions shown in Table 1 are found and it is proved numerically that there exists no solutions other than those three solutions.

Table 1: Three solutions of the example

	solution
x_1	[0.124271148965879, 0.124271148965880]
y_1	[0.124271148965817, 0.124271148965818]
y_2	[0.210047170592718, 0.210047170592719]
x_1	[0.16814302489, 0.16814302490]
y_1	[0.16814300951, 0.16814300952]
y_2	[0.20947741246, 0.20947741247]
x_1	[0.215168573962, 0.215168573963]
y_1	[0.214860532426, 0.214860532427]
y_2	[0.210934524116, 0.210934524117]

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