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Constructive Aspects of the Dirichlet Problem[1]

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Abstract: We examine, within the framework of Bishop's constructive mathematics, various classical methods for proving the existence of weak solutions of the Dirichlet Problem, with a view to showing why those methods do not immediately translate into viable constructive ones. In particular, we discuss the equivalence of the existence of weak solutions of the Dirichlet Problem and the existence of minimizers for certain associated integral functionals. Our analysis pinpoints exactly what is needed to find weak solutions of the Dirichlet Problem: namely, the computation of either the norm of a linear functional on a certain Hilbert space or, equivalently, the infimum of an associated integral functional.

Key Words: Dirichlet problem, Hilbert space, Bishop's constructive mathematics
Category: G.0

1 Introduction

In this paper we examine some classical proofs of the existence of weak solutions of the Dirichlet Problem, the fundamental problem of potential theory in physics. Our examination reveals both why those proofs break down in a constructive setting, where "existence" is interpreted strictly as "computability", and what needs to be done to obtain a constructive proof of the existence of weak solutions. We expect that, by analogy with the classical situation ([14], Chapter 6), a constructive existence proof for weak solutions of the Dirichlet Problem would be a first step towards a constructive (that is, algorithmic) proof of the existence of solutions of the usual, physically desirable type.

Without going into significant detail, we should point out that by **constructive mathematics** we mean mathematics developed as in Bishop's pioneering monograph [2] (see also [3]). In practice, as Richman has pointed out ([15], [16]), Bishop's constructive mathematics (**BISH**) appears to be mathematics carried out with intuitionistic, rather than classical, logic. In particular, intuitionistic logic does not allow the **Law of Excluded Middle (LEM)**,

$$P \vee \neg P,$$

or even such weaker versions as the **Limited Principle of Omniscience (LPO)**,

$$\forall \mathbf{a} \in \{\mathbf{0},\mathbf{1}\}^{\mathbf{N}} \left(\forall \mathbf{n}\,(\mathbf{a_n} = \mathbf{0}) \vee \exists \mathbf{n}\,(\mathbf{a_n} = \mathbf{1})\right),$$

where $\mathbf{N}$ is the set of natural numbers and $\{0,1\}^{\mathbf{N}}$ is the set of all binary sequences $\mathbf{a} = (\mathbf{a_n})$. In turn, the following classical[2] properties of the real number

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[2] We use the term *classical mathematics* to mean traditional mathematics—that is, mathematics with classical logic.

line $\mathbf{R}$ are seen to be essentially nonconstructive, since they entail LPO or some other logically inadmissible proposition:

$$\forall x \in \mathbf{R}\,(\mathbf{x} > \mathbf{0} \vee \mathbf{x} = \mathbf{0} \vee \mathbf{x} < \mathbf{0}),$$
$$\forall x \in \mathbf{R}\,(\mathbf{x} \geq \mathbf{0} \vee \mathbf{x} \leq \mathbf{0}).$$

Fortunately there are two constructive properties of $\mathbf{R}$ that facilitate analysis in the absence of these inadmissible classical statements: namely,

$$a < b \Rightarrow \forall x \in \mathbf{R}\,(\mathbf{a} < \mathbf{x} \vee \mathbf{x} < \mathbf{b})$$

and

$$\forall x \in \mathbf{R}\ (\neg\,(\mathbf{x} > \mathbf{0}) \Rightarrow \mathbf{x} \leq \mathbf{0}.)$$

The first of these often enables us to split arguments into two overlapping cases. Note that the converse of the second, namely

$$\forall x \in \mathbf{R}\,(\neg\,(\mathbf{x} \leq \mathbf{0}) \Rightarrow \mathbf{x} > \mathbf{0}),$$

entails **Markov's Principle,**

$$\forall \mathbf{a} \in \{\mathbf{0}, \mathbf{1}\}^{\mathbf{N}}\,(\neg \forall \mathbf{n}\,(\mathbf{a_n} = \mathbf{0}) \Rightarrow \exists \mathbf{n}\,(\mathbf{a_n} = \mathbf{1})),$$

which represents a form of unbounded search that is not derivable within intuitionistic logic; see Chapters 1 and 7 of [7].

One advantage of working within BISH, as distinct from other varieties of constructive mathematics such as recursive constructive mathematics [12], is that proofs and results in BISH can be interpreted *mutatis mutandis* either within classical mathematics or within any reasonable mathematical model of computable analysis (such as recursive constructive analysis or Weihrauch's TTE [19]). Moreover, as the work of Martin–Löf and others shows [13], we can extract algorithms from proofs within BISH, the proofs themselves showing the correctness of those algorithms.

For more information about intuitionistic logic, the foundations of constructive mathematics, and constructive analysis see [1], [3], [17], and [7]; other appropriate references are [4], [5], and [6].

We now turn to the main theme of this paper. Let Δ be the **Laplace operator on $\mathbf{R}^{\mathbf{N}}$:**

$$\Delta u = \sum_{k=1}^{N} \frac{\partial^2 u}{\partial x_k^2}.$$

The original form of the **Dirichlet Problem** is the following.

Let Ω be a bounded open Lebesgue integrable subset of $\mathbf{R}^{\mathbf{N}}$ with boundary $\partial\Omega$, and f a continuous real–valued function on $\partial\Omega$. Find a function u that is twice continuously differentiable in Ω, is continuous on the closure $\overline{\Omega}$ of Ω, and satisfies

$$\Delta u = 0 \text{ on } \Omega, \quad u = f \text{ on } \partial\Omega. \tag{1}$$

For technical reasons, we will consider instead the following form of the Dirichlet Problem.

Let Ω be a bounded open Lebesgue integrable subset of $\mathbf{R^N}$ with boundary $\partial\Omega$, and f an element of $L_2(\Omega)$. Find a function u that is twice continuously differentiable in Ω, is continuous on $\overline{\Omega}$, and satisfies

$$\Delta u = f \text{ on } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{2}$$

When f satisfies appropriate continuity conditions, these two versions of the Dirichlet Problem are equivalent, in the sense that from solutions of either one we can always construct solutions of the other; for details see ([11], page 131).

In the remainder of this paper, when we use the phrase **Dirichlet Problem**, *we shall mean version* (2) *of that problem.*

For convenience, we gather together here a number of results and definitions that we use in our later analysis.

A subset S of a metric space (X,ρ) is said to be **located** if for each point x of X the **distance from x to S**,

$$\rho(x,S) = \inf\{\rho(x,s) : s \in S\},$$

exists. Thus S is located if and only if we can compute a nonnegative number $r = \rho(x,S)$ with the following properties:

- $r \leq \rho(x,s)$ for all $s \in S$;
- for each $\varepsilon > 0$ there exists $y \in S$ such that $\rho(x,y) < r + \varepsilon$.

A subset A is **well contained** in a subset B in a metric space (X,ρ) if there exists a positive number r such that $A_r \subset T$, where

$$A_r = \{x \in X : \forall\varepsilon > 0 \exists y \in A \ (\rho(x,y) < r + \varepsilon)\}.$$

In that case we write $A \subset\subset B$.

Let Ω be a bounded located open set in the Euclidean space $\mathbf{R^N}$, and $\partial\Omega$ the boundary of Ω. For each positive integer n let

- $C^n(\Omega)$ be the space of real–valued functions that are n times uniformly differentiable on compact subset of Ω,
- $C^n(\overline{\Omega})$ be the space of real–valued functions that are uniformly differentiable and have uniformly continuous derivatives of up to n^{th} order on $\overline{\Omega}$, and
- $C_0^n(\Omega)$ be the space consisting of those elements of $C^n(\Omega)$ that have a compact support well contained in Ω.

We say that $u \in L^2(\Omega)$ is **weakly differentiable** if there exist elements $v_1, \ldots, v_N$ of $L^2(\Omega)$, called the **weak partial derivatives** of u, such that

$$\int_\Omega u \frac{\partial\varphi}{\partial x_k}\,\mathrm{dx} = -\int_\Omega \varphi \mathrm{v_k}\,\mathrm{dx} \qquad (\mathrm{k} = 1, \ldots, \mathrm{N})$$

for all $\varphi \in C_0^1(\Omega)$. We denote by $H^1(\Omega)$ the subspace of $L^2(\Omega)$ consisting of all functions that are weakly differentiable and whose weak derivatives are also members of $L^2(\Omega)$. We use the usual notations of differentiation to denote the weak derivatives, denoting the k^{th} partial derivative v_k by $\frac{\partial u}{\partial x_k}$ and the (weak or strong) gradient of u by

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_N}\right).$$

When u is differentiable, its weak derivatives coincide with its usual derivatives.

Equipped with the inner product

$$\langle u, v\rangle_{H^1(\Omega)} = \langle u, v\rangle_{L^2(\Omega)} + \langle \nabla u, \nabla v\rangle_{L^2(\Omega)}$$

and the corresponding norm

$$\|u\|_{H^1(\Omega)} = \left(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\right)^{1/2},$$

$H^1(\Omega)$ becomes a Hilbert space. The completion $H^1_0(\Omega)$ of $C^1_0(\Omega)$ in $H^1(\Omega)$ is a separable Hilbert space. The norms $\|u\|_{H^1(\Omega)}$ and $\|u\|_{H^1_0(\Omega)}$ are abbreviated as $\|u\|_H$, and $\|u\|_{L^2(\Omega)}$ as $\|u\|_2$, when it is clear from the context that no confusion can arise; similarly, we write $\langle u, v\rangle_H$ instead of either $\langle u, v\rangle_{H^1(\Omega)}$ or $\langle u, v\rangle_{H^1_0(\Omega)}$.

Our first lemma introduces a fundamental inequality, due to Poincaré ([14], Chapter 6).

Lemma 1. ***(Poincaré's inequality)*** *There exists a constant $\gamma > 0$ such that for all $v \in H^1_0(\Omega)$,*

$$\int_\Omega v^2 \,\mathrm{dx} \le \gamma^2 \int_\Omega \|\nabla \mathrm{v}\|^2 \,\mathrm{dx}.$$

It follows from Poincaré's inequality that on $H^1_0(\Omega)$ the norm

$$\|u\|_{H^1_0(\Omega)} = \|\nabla u\|_{L^2(\Omega)}$$

associated with the inner product

$$\langle u, v\rangle_{H^1_0(\Omega)} = \langle \nabla u, \nabla v\rangle_{L^2(\Omega)}$$

is equivalent to the norm $\|u\|_{H^1(\Omega)}$. When no confusion is likely, we also write $\|u\|_H$ for $\|u\|_{H^1_0(\Omega)}$.

We assume from now on that Ω is a bounded open Lebesgue integrable subset of $\mathbf{R}^\mathbf{N}$, and that the divergence theorem holds for Ω. So for any vector field w in $C(\overline{\Omega}) \cap C^1(\Omega)$ we have

$$\int_\Omega \mathrm{div}\,\mathbf{w}\,\mathrm{dx} = \int_{\partial\Omega} \mathbf{w}\cdot\mathbf{n}\,\mathrm{dS},$$

where $\mathbf{n}$ denotes the unit outward normal to $\partial\Omega$, dS indicates the $(n-1)$–dimensional area element in $\partial\Omega$, and

$$\mathrm{div}\,\mathbf{w} = \sum_{\mathbf{i}=\mathbf{1}}^{\mathbf{N}} \frac{\partial \mathbf{w_i}}{\partial \mathbf{x_i}}$$

is the divergence of the vector field $\mathbf{w}=(w_1,\ldots,w_N)$. In particular, if $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$, then taking $\mathbf{w} = \nabla \mathbf{u}$ in the divergence theorem, we obtain

$$\int_\Omega \Delta u \, \mathrm{dx} = \int_{\partial\Omega} \nabla \mathrm{u} \cdot \mathbf{n} \, \mathrm{dS}.$$

(See [10], page 13)

By a **weak solution** of the Dirichlet Problem (2) we mean a function $u \in H_0^1(\Omega)$ such that

$$\langle u, v\rangle_H = -\int_\Omega \nabla u \cdot \nabla v \, \mathrm{dx} = \int_\Omega \mathrm{fv} \tag{3}$$

for all $v \in C_0^1(\Omega)$. An approximation argument shows that $u \in H_0^1(\Omega)$ is a weak solution if and only if (3) holds for all $v \in H_0^1(\Omega)$.

Associated with the weak solvability of the Dirichlet Problem is the following minimisation problem.

Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \left(\|\nabla u\|^2 + 2uf\right) \mathrm{dx} \le \int_\Omega \left(\|\nabla \mathrm{w}\|^2 + 2\mathrm{wf}\right) \mathrm{dx}$$

for all $w \in H_0^1(\Omega)$.

For convenience we write

$$J(w) = \int_\Omega \left(\|\nabla w\|^2 + 2wf\right) \mathrm{dx}.$$

We include the following result for completeness; its classical proof is essentially constructive and is found in [14].

Proposition 2. *The following are equivalent conditions on $u \in H_0^1(\Omega)$.*

(i) $J(u) \le J(v)$ *for all* $v \in H_0^1(\Omega)$.
(ii) $-\int_\Omega \nabla u \cdot \nabla v \, \mathrm{dx} = \int_\Omega \mathrm{fv}$ *for all* $v \in H_0^1(\Omega)$.

Thus to solve the Dirichlet Problem (2) weakly, we have the alternative of trying to prove (i) of this proposition. Unfortunately, the classical approaches to proving (i) or (ii) all use constructively unacceptable principles, as we now show.

2 The Classical Approaches

The classical approach to (i) includes these key steps.

Step 1: The infimum of $J(w)$ always exists by the least–upper–bound principle, because J is bounded from below. In fact, by the inequalities of Hölder, Poincaré and Young,

$$\begin{aligned} 2\left|\int_\Omega wf\mathrm{dx}\right| &\le 2\left(\int_\Omega |w|^2\ \mathrm{dx}\right)^{\frac{1}{2}}\left(\int_\Omega |f|^2\ \mathrm{dx}\right)^{\frac{1}{2}} \\ &\le 2\gamma\left(\int_\Omega \|\nabla w\|^2\ \mathrm{dx}\right)^{\frac{1}{2}}\left(\int_\Omega |f|^2\ \mathrm{dx}\right)^{\frac{1}{2}} \\ &\le \tfrac{1}{2}\int_\Omega \|\nabla w\|^2\ \mathrm{dx} + 2\gamma^2\int_\Omega |\mathrm{f}|^2\ \mathrm{dx}, \end{aligned}$$

and therefore

$$\begin{aligned} J(w) &\ge \int_\Omega \|\nabla w\|^2\ \mathrm{dx} - \tfrac{1}{2}\int_\Omega \|\nabla \mathrm{w}\|^2\ \mathrm{dx} - 2\gamma^2\int_\Omega |\mathrm{f}|^2\ \mathrm{dx} \\ &\ge \tfrac{1}{2}\int_\Omega \|\nabla w\|^2\ \mathrm{dx} - 2\gamma^2\int_\Omega |\mathrm{f}|^2\ \mathrm{dx} \\ &\ge -2\gamma^2\int_\Omega |f|^2\ \mathrm{dx}. \end{aligned}$$

Note, incidentally, that

$$\|w\|_H^2 \le 2J(w) + 4\gamma^2\|f\|_{L^2(\Omega)}^2. \tag{4}$$

This inequality will be useful later.
Step 2: Construct a **minimising sequence** $(u_n)_{n=1}^\infty$ for J : that is, a sequence $(u_n)_{n=1}^\infty$ such that $J(u_n) \to \inf J$. Choose N so large that $J(u_n) \le \inf J(w) + 1$ for all $n \ge N$. Then for all n, using inequality (4), we have

$$\|u_n\|_H^2 \le \max\left\{\|u_n\|_H^2 :\ 1 \le n \le N\right\} + 2(\inf J(w) + 1) + 4\gamma^2\|f\|_{L^2(\Omega)}^2.$$

So the sequence (u_n) is uniformly bounded in $H_0^1(\Omega)$.
Step 3: Using the weak sequential compactness of bounded sets in $H_0^1(\Omega)$, extract a weakly convergent subsequence of (u_n).Then the weak limit u of this subsequence, still an element of $H_0^1(\Omega)$, minimises J.

The problem with this approach rests in Steps 1 and 3: neither the classical least–upper–bound principle nor the sequential compactness argument are acceptable in constructive mathematics.

The classical approach to part (ii) of Proposition 2 includes the following steps.

Step 1: Define a linear functional φ on $H_0^1(\Omega)$ by

$$\varphi(v) = -\int_\Omega vf\,\mathrm{dx}.$$

It is easy to show that φ is bounded: by the inequalities of Hölder and Poincaré,

$$\begin{aligned}|\varphi(v)| &\le \int_\Omega |v|\,|f| \, \mathrm{dx} \\ &\le \left(\int_\Omega |v|^2 \, \mathrm{dx}\right)^{\frac{1}{2}} \left(\int_\Omega |f|^2 \, \mathrm{dx}\right)^{\frac{1}{2}} \\ &\le \gamma \left(\int_\Omega \|\nabla v\|^2 \, \mathrm{dx}\right)^{\frac{1}{2}} \left(\int_\Omega |f|^2 \, \mathrm{dx}\right)^{\frac{1}{2}} \\ &= \gamma \|f\|_{L^2} \|v\|_H .\end{aligned}$$

Step 2: Apply the classical Riesz Representation Theorem to find an element u of $H_0^1(\Omega)$ such that

$$\varphi(v) = \langle u, v\rangle_H$$

for all $v \in H_0^1(\Omega)$. Then u is the desired weak solution of the Dirichlet Problem.

The problem with this approach occurs at Step 2. Constructively, a bounded linear functional φ is representable if and only if it has a norm—that is, the **norm**

$$\|\varphi\| = \sup\left\{|\varphi(v)| : \ v \in H_0^1(\Omega)\right\}$$

exists (is computable); see [3], Ch. 8, Proposition (2.3). There is no guarantee that the functional in Step 2 has a norm; indeed, it has one if and only if the desired weak solution of the Dirichlet Problem (2) exists.

A method used by numerical analysts to solve the Dirichlet Problem approximately is the **Ritz–Galerkin method**, in which solutions of the Dirichlet Problem in finite–dimensional subspaces of are constructed as approximations to the solution of the general problem. We now look at this approach.

Select an orthonormal basis $(v_n)_{n=1}^\infty$ of $H_0^1(\Omega)$, and let

$$H_n = \mathrm{span}\{\mathrm{v}_1, \dots, \mathrm{v}_\mathrm{n}\}.$$

be the n–dimensional subspace of $H_0^1(\Omega)$ generated by $\{v_1, \dots, v_n\}$. Since φ is uniformly continuous on the unit ball B_n of H_n, and S_n is totally bounded (as is any ball in a finite–dimensional normed space),

$$\sup\{|\varphi(v)| : \ v \in S_n\}$$

exists ([3], Ch. 4, (4.3)). In other words, the bounded linear functional φ, restricted to H_n, has a norm. By the constructive Riesz Representation Theorem ([3], Ch. 8, (2.3)), there exists $u_n \in H_n$ such that

$$\varphi(v) = -\langle v, u_n\rangle \quad (v \in H_n)$$

—that is,

$$-\int_\Omega \nabla u_n \cdot \nabla v \, \mathrm{dx} = \int_\Omega \mathrm{vf} \, \mathrm{dx} \quad (\mathrm{v} \in \mathrm{H_n}).$$

If the Dirichlet Problem (2) has a weak solution u, then (u_n) will converge to u. To see this, let $u = \sum_{i=1}^{\infty} \alpha_i v_i$, and let $P_n u = \sum_{i=1}^{n} \alpha_i v_i$ be the projection of u in H_n. For all $v \in H_n$ we have

$$-\int_{\Omega} \nabla (P_n u) \cdot \nabla v \,\mathrm{dx} = -\int_{\Omega} \nabla \mathrm{u} \cdot \nabla \mathrm{v} \,\mathrm{dx} = \int_{\Omega} \mathrm{vf} \,\mathrm{dx}$$

and therefore

$$\int_{\Omega} \nabla (P_n u - u_n) \cdot \nabla v \,\mathrm{dx} = 0.$$

Taking $v = P_n u - u_n$, we obtain

$$\int_{\Omega} \| \nabla (P_n u - u_n) \| \,\mathrm{dx} = 0.$$

So $P_n u = u_n$, and therefore

$$\| u_n - u \|_H = \| P_n u - u \|_H \to 0 \text{ as } n \to \infty.$$

Classically, the weak solution u always exists, and we can therefore use the approximations u_n to solve the Dirichlet Problem numerically. But constructively, to justify such a numerical approach we would have to be able to construct—in other words, to compute in principle—the exact solution u in advance. This leads us back to the problem of the existence of the norm of the functional φ.

3 Minimising Sequences

We now examine what happens *if* the infimum of the functional J exists and we can therefore construct a minimising sequence for J. We first show that any such minimising sequence is a weakly Cauchy sequence relative to the inner product on $H_0^1(\Omega)$. Our proof is a modification of the one on pages 131–137 of [11].

Proposition 3. *Suppose that*

$$L = -\inf_{w \in H_0^1(\Omega)} J(w)$$

exists, and let (u_n) *be a minimising sequence for* J *in* $H_0^1(\Omega)$:

$$\lim_{n \to \infty} J(u_n) = -L.$$

Then

$$\lim_{n \to \infty} \int_{\Omega} \nabla u_n \cdot \nabla v \,\mathrm{dx} + \int_{\Omega} \mathrm{vf} \,\mathrm{dx} = 0.$$

Proof. For convenience write

$$D(u,v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{dx},$$
$$M_n = \int_\Omega \nabla u_n \cdot \nabla v \, \mathrm{dx} + \int_\Omega \mathrm{vf} \, \mathrm{dx}.$$

If $v \in H_0^1(\Omega)$ and $\varepsilon \in \mathbf{R}$, then $u_n + \varepsilon v \in H_0^1(\Omega)$ and so

$$-L \leq J(u_n + \varepsilon v) = J(u_n) + \varepsilon^2 D(v,v) + 2\varepsilon M_n.$$

Thus

$$-\varepsilon^2 D(v,v) - 2\varepsilon M_n \leq J(u_n) + L,$$

so that

$$\begin{aligned}(J(u_n) + L)\, D(v,v) &\geq -\left(\varepsilon^2 D(v,v)^2 + 2\varepsilon M_n D(v,v) + M_n^2 - M_n^2\right)\\ &= -\left((\varepsilon D(v,v) + M_n)^2 - M_n^2\right)\end{aligned}$$

Taking v as a nonconstant function and

$$\varepsilon = -\frac{M_n}{D(v,v)},$$

we see that

$$M_n^2 \leq (J(u_n) + L)\, D(v,v)$$

and therefore

$$|M_n| \leq \sqrt{(J(u_n) + L)\, D(v,v)}.$$

Since $J(u_n) \to -L$ as $n \to \infty$, it follows that $M_n \to 0$ as $n \to \infty$. □

Corollary 4. *Under the conditions of Proposition 3,*

$$\int_\Omega (\nabla u_n - \nabla u_m) \cdot \nabla v \, \mathrm{dx} \to 0 \ \textit{as} \ \mathrm{n}, \mathrm{m} \to \infty$$

and therefore (u_n) *is a weakly Cauchy sequence in* $H_0^1(\Omega)$. □

We now define a linear functional φ on $H_0^1(\Omega)$ by

$$\varphi_f(v) = -\int_\Omega v f \, \mathrm{dx}.$$

Note that if L exists and (u_n) is a minimising sequence for J, then by Proposition 3,

$$\varphi_f(v) = \lim_{n\to\infty} \int_\Omega \nabla u_n \cdot \nabla v \, \mathrm{dx}.$$

Classically, as $H_0^1(\Omega)$ is weakly complete, there is an element $u \in H_0^1(\Omega)$ such that u_n converges to u weakly in $H_0^1(\Omega)$. This function u minimises J, and is therefore the desired weak solution of the Dirichlet Problem. Constructively, to be weakly Cauchy is not enough to guarantee the existence of a weak limit: to prove the existence of such a weak limit, we need to show that the linear functional φ_f is not just bounded but has a norm.

Proposition 5. *Suppose that*

$$L = -\inf_{w \in H_0^1(\Omega)} J(w)$$

exists, and let (u_n) *be a minimising sequence for* J *in* $H_0^1(\Omega)$. *If* (u_n) *converges weakly to* $u \in H_0^1(\Omega)$, *then the linear functional* φ_f *has a norm,* $\|\varphi_f\| = \sqrt{L}$, *and* u *is a weak solution of the Dirichlet Problem.*

Proof. Taking $v = u$ in Proposition 3, we see that

$$\int_\Omega \|\nabla u\|^2 \, \mathrm{dx} + \int_\Omega \mathrm{uf} \, \mathrm{dx} = \lim_{\mathrm{n}\to\infty} \left(\int_\Omega \nabla \mathrm{u_n} \cdot \nabla \mathrm{u} \, \mathrm{dx} + \int_\Omega \mathrm{uf} \, \mathrm{dx} \right) = 0.$$

Then

$$\varphi_f(u) = -\int uf \, \mathrm{dx} = \int_\Omega \|\nabla \mathrm{u}\|^2 \, \mathrm{dx} = \|\mathrm{u}\|_{\mathrm{H}}^2 = \langle \mathrm{u}, \mathrm{u} \rangle_{\mathrm{H}} \, .$$

It follows that u is a weak solution of the Dirichlet Problem, φ_f has a norm, and $\|\varphi_f\|^2 = \|u\|_H = -L$. Proposition 2 now shows that $J(u) = -L$. □

We have the following converse of Proposition 5.

Proposition 6. *Suppose that* φ_f *has a norm, and let* u *be the resulting weak solution of the Dirichlet Problem. Then*

$$L = -\inf_{w \in H_0^1(\Omega)} J(w)$$

exists, and any minimising sequence for J *converges weakly to* u.

Proof. It follows from Proposition 2 that L exists and $J(u) = -L$. If (u_n) is any minimising sequence for J, then by Proposition 3, for all $v \in H_0^1(\Omega)$ we have

$$\langle u_n - u, v \rangle = \langle u_n, v \rangle - \varphi_f(v) \to -\int_\Omega vf + \int_\Omega vf = 0$$

as $n \to \infty$. So (u_n) converges weakly to u. □

Now, it is tempting to believe that we can strengthen Proposition 5 by removing the hypothesis that there exist a weakly convergent minimising sequence for J : for, in order to find a weak solution of the Dirichlet Problem, will it not suffice to show that the infimum of J exists, just as it suffices to show that the norm (a supremum) of φ_f exists? To see that this is unlikely, we need only note that although the Riesz Representation Theorem guarantees that if the norm of φ_f is computable, then there is an associated vector v whose norm equals that of φ_f, we have no *a priori* guarantee that if $\inf J$ is computable, then there exists a vector v such that $\inf J = J(v)$. (In order to produce such a vector v, the classical mathematician resorts to an application of the nonconstructive result that a bounded, weakly convergent sequence contains a convergent subsequence.)

We end the section with some more comments on the Ritz–Galerkin method, using the notation from page 8.

A proof similar to that of Proposition 2 shows that the function u_n satisfying

$$-\int_{\Omega} \nabla u_n \cdot \nabla v \, \mathrm{dx} = \int_{\Omega} \mathrm{vf} \, \mathrm{dx} \quad (\mathrm{v} \in \mathrm{H_n}). \tag{5}$$

minimises J on H_n. We show that if $\inf_{v \in H_0^1(\Omega)} J(v)$ exists, then (u_n) is a minimising sequence for J, *even when we do not know that the Dirichlet Problem has a weak solution.* We need one more lemma to prove this.

Lemma 7. *For each $R > 0$ there exists a positive constant c (depending only on Ω, f, and R) such that if $u, v \in H_0^1(\Omega)$, $\|u\|_H \leq R$, and $\|v\|_H \leq R$, then*

$$|J(u) - J(v)| \leq c \, \|u - v\|_H \, .$$

Proof. Using the Hölder and Poincaré inequalities, for all $u, v \in H_0^1(\Omega)$ we have

$$\begin{aligned}
|J(u) - J(v)| &\leq \left| \int_{\Omega} \left(\|\nabla u\|^2 - \|\nabla v\|^2 \right) \mathrm{dx} \right| + 2 \int_{\Omega} |f| \, |u - v| \, \mathrm{dx} \\
&\leq \left| \|u\|_H^2 - \|v\|_H^2 \right| + 2 \left(\int_{\Omega} |f^2| \right)^{1/2} \left(\int_{\Omega} |u - v|^2 \right)^{1/2} \\
&\leq (\|u\|_H + \|v\|_H) \, |\|u\|_H - \|v\|_H| + 2\gamma \left(\int_{\Omega} |f^2| \right)^{1/2} \|u - v\|_H \\
&\leq 2R \, \|u - v\|_H + 2\gamma \, \|f\|_{L^2(\Omega)} \, \|u - v\|_H \, ,
\end{aligned}$$

so we can take

$$c = 2 \left(R + \gamma \, \|f\|_{L^2(\Omega)} \right) . \square \tag{6}$$

We now return to the sequence (u_n), where for each n, u_n satisfies (5). If the Dirichlet Problem (2) has a weak solution u, then the work on page 8 shows that $\|u_n - u\| \to 0$; whence, by Lemma 7, $J(u_n) \to J(u)$. In the general case, when we do not know if there is a weak solution to the Dirichlet Problem, take

$$R = \sqrt{2L + 1 + 4\gamma^2 \, \|f\|_{L^2(\Omega)}^2}$$

in Lemma 7, to obtain the corresponding positive constant c. Fix ε with $0 < \varepsilon < 1/3$, and let

$$\delta = \min \left\{ R - \sqrt{R^2 - \varepsilon}, \, c^{-1} \varepsilon \right\} .$$

Choose $v \in H_0^1(\Omega)$ such that $J(v) < -L + \varepsilon$, and then N such that $\|v - P_N v\|_H < \delta$, where P_N is the projection on H_N. By inequality (4),

$$\begin{aligned}
\|v\|_H^2 &\leq 2J(v) + 4\gamma^2 \, \|f\|_{L^2(\Omega)}^2 \\
&< 2L + 2\varepsilon + 4\gamma^2 \, \|f\|_{L^2(\Omega)}^2 < R^2 - \varepsilon
\end{aligned}$$

and therefore

$$\begin{aligned}
\|P_N v\|_H^2 &\leq (\|v\|_H + \delta)^2 \\
&< \left(\sqrt{R^2 - \varepsilon} + \delta \right)^2 \leq R^2.
\end{aligned}$$

Hence, by our choice of c,

$$|J(v) - J(P_N(v))| \leq c \, \|v - P_N v\|_H < c\delta \leq \varepsilon.$$

For all $n \leq N$, since $H_n \subset H_N$ and u_N minimises J over H_N, we now have

$$-L \leq J(u_n) \leq J(u_N) \leq J(P_N v) \leq J(v) + \varepsilon < -L + 2\varepsilon.$$

Hence (u_n) is a minimising sequence for J.

Of course, the foregoing argument depends on the existence of the infimum of J, which is implied by the existence of the norm of the linear functional φ_f.

4 The Existence of Weak Solutions

We have now shown that the existence of weak solutions for the Dirichlet Problem (2) is equivalent to a number of different conditions, each of which is relatively straightforward to establish directly by classical means. Although it is still open whether or not weak solutions always exist constructively, no matter how awkward the domain Ω or the function f in (2), there are some other results in [18] that hold out some hope of finding weak solutions. In order to discuss these, we need some material from the constructive theory of normed spaces.

Let (v_n) be a dense sequence in $H_0^1(\Omega)$. The corresponding **double norm** on the dual $H_0^1(\Omega)^*$ of $H_0^1(\Omega)$ is defined by

$$|||\lambda||| \equiv \sum_{n=1}^{\infty} 2^{-n} \frac{|\lambda(v_n)|}{1 + \|v_n\|_H}.$$

The following fundamental results about dual spaces and the double norm are proved in Chapter 7 of [3]. Double norms arising from different dense sequences in $H_0^1(\Omega)$ give rise to equivalent metrics on the **unit ball**

$$B^* \equiv \{\lambda \in H_0^1(\Omega)^* : \ \forall v \in H_0^1(\Omega) \ \ (|\lambda(v)| \leq \|v\|_H)\}$$

of $H_0^1(\Omega)^*$. (It is for this reason that we refer, loosely, to "the" double norm on B^*). Moreover, B^* is totally bounded relative to any double norm. For each $u \in H_0^1(\Omega)$ the mapping $\lambda \mapsto \lambda(u)$ is uniformly continuous with respect to the double norm on B^*. We denote by λ_v the bounded linear functional $u \mapsto \langle u, v\rangle$ on $H_0^1(\Omega)$; an element of $H_0^1(\Omega)^*$ has a norm if and only if it has the form λ_v, in which case $\|\lambda_v\| = \|v\|_H$. If S is a dense subset of the unit ball B of $H_0^1(\Omega)$, then the elements λ_v with $v \in S$ form a dense subset of B^*.

We now state our fundamental existence theorem.

Theorem 8. *The following conditions are equivalent.*

(i) For each $\xi \in \mathbf{R^N}$ the special Dirichlet Problem

$$\begin{aligned} \triangle u\,(x) &= e^{\mathrm{i}x\cdot\xi} \ \ \text{if } x \in \Omega, \\ u &= 0 \quad\ \ \text{on } \partial\Omega \end{aligned}$$

has a weak solution u.

(ii) *The mapping $\lambda_v \mapsto \widehat{v}(\xi)$ from S^* to $\mathbf{R}$ is uniformly continuous in the double norm.*
(iii) *S is totally bounded in $L^2(\Omega)$.*
(iv) *The Dirichlet Problem*

$$\begin{aligned} \triangle u &= f \text{ on } \Omega, \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

has a weak solution for each $f \in L^2(\Omega)$.
(v) *There exists $u \in H_0^1(\Omega)$ such that $J(u) \leq J(v)$ for all $v \in H_0^1(\Omega)$, where*

$$J(v) = \int_\Omega \left(\|\nabla v\|^2 + 2vf \right).$$

As might be expected, the proof of this result makes good use of the Fourier Transform in order to reduce the general case of the Dirichlet Problem to the special case where f has an exponential form. Part (ii) of the theorem holds classically as a special case of Rellich's Compactness Theorem (see Chapter 6 of [14]).

There is yet another constructive approach to the existence of weak solutions of (2) in [18]. Extending each $u \in H_0^1(\Omega)$ to equal 0 outside Ω, we can regard $H_0^1(\Omega)$ as a subset of the space $H_0^1(B_R)$ for any ball

$$B_R = B(0, R) \in \mathbf{R}^\mathbf{N}$$

such that $\Omega \subset\subset B_R$. Clearly, $H_0^1(\Omega)$ is a closed linear subset of $H_0^1(B_R)$. It turns out that locating the subset $H_0^1(\Omega)$ in the space $H_0^1(B_R)$ is equivalent to solving the Dirichlet Problem (2) on Ω.

Theorem 9. *The following conditions are equivalent.*

(i) *The Dirichlet Problem*

$$\begin{aligned} \triangle u &= f \text{ on } \Omega, \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

has a weak solution for each $f \in L^2(\Omega)$.
(ii) *$H_0^1(\Omega)$ is located in $H_0^1(B_R)$ for each $R > 0$ such that $\Omega \subset\subset B_R$.*

For the proof see pages 67–70 of [18], or [9].

We end with a remark on stability. As a rule, the objects of constructive mathematics vary continuously in the relevant parameters; Chapter 5 of [18] contains results showing the stability of weak solutions (when they exist) as functions of the parameters Ω and f.

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