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Disjunctive ω-Words and Real Numbers

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Abstract: An ω-word p over a finite alphabet Σ is called *disjunctive* if every finite word over Σ occurs as a subword in p. A real number is called *disjunctive to base a* if it has a disjunctive a-adic expansion. For every pair of integers $a, b \geq 2$ such that there exist numbers disjunctive to base a but not to base b we explicitly construct very simple examples of such numbers. General versions of the following results are proved. If $(n_i)_{i\in\omega}$ is a strictly increasing sequence of positive integers with $n_{i+1} \geq 3^{n_i}$ for infinitely many i then $\sum 3^{-n_i}$ is disjunctive to base 2. The number $\sum 2^{-i!-i}$ is disjunctive to base a if a is even and not a power of 2. The sum $\sum 2^{-c^i}$ is disjunctive to base 6 if $c \geq 3$ is odd.

Key Words: ω-words, number representations, invariant properties, disjunctiveness, normality, periods of rational numbers

Category: F.m

1 Introduction

Let Σ be a finite alphabet. By Σ^* we denote the set of all finite words over Σ, and by $\Sigma^\omega := \{p : \omega \longrightarrow \Sigma\}$ the set of ω-*words* over Σ where $\omega = \{0, 1, 2, \ldots\}$.

Definition 1.1 An ω-word $p \in \Sigma^\omega$ is *disjunctive* if every finite string in Σ^* occurs as a subword in p.

We are interested in disjunctive ω-words for two reasons. The first is that "disjunctiveness" is a natural and simple weakening of "normality" and of "randomness" [see Section 2]. The second is that disjunctive ω-words appear as special cases of disjunctive ω-languages in automata-theoretic investigations. These languages are defined to be subsets of Σ^ω whose principal congruence relation is the equality [Jürgensen, Shyr, Thierrin 1983]. Since any ω-word over a finite alphabet can be considered as an expansion of a real number it is natural to consider the corresponding real numbers too if one wants to investigate ω-words with certain properties. First results concerning the connection between disjunctive ω-languages and real numbers can be found in [Jürgensen and Thierrin 1988].

Let $b \geq 2$ be an integer. The *b-adic expansion* $\nu_b(x)$ of a real number x in the interval $[0; 1)$ is the unique ω-word $p = p_0p_1p_2\ldots \in \Sigma_b^\omega$ over the alphabet $\Sigma_b := \{0, \ldots, b-1\}$ containing infinitely many digits $\neq b-1$ such that $x = \sum_{i=0}^{\infty} p_i \cdot b^{-(i+1)}$.

Definition 1.2 A real number $x \in [0; 1)$ is said to be *disjunctive (normal, random) to base b* if $\nu_b(x) \in \Sigma_b^\omega$ is disjunctive (normal, random).

We excluded the — not very interesting — ω-words ending on $(b-1)^\omega$. They are neither disjunctive nor normal nor random.

For any property of ω-words the question arises whether it is an invariant property of real numbers. For randomness the positive answer has been given quite recently [Calude and Jürgensen 1994], see also [Calude 1994]. For another proof see [Weihrauch 1995]. For normality and disjunctiveness the negative answer is part of famous result of Cassels [Cassels 1959] and of Schmidt [Schmidt 1960]. For these two properties two more problems pose themselves: the question for which pairs of integers $a, b \geq 2$ disjunctiveness (normality) to base a implies disjunctiveness (normality) to base b, and the problem to give explicitly simple counterexamples of real numbers for the other pairs of bases. It turns out that the answer to the first question is the same in both cases.

Two real numbers $a, b > 1$ are said to be *equivalent* if there are positive integers n, m such that $a^n = b^m$. Then we write $a \sim b$, else $a \not\sim b$.

Theorem 1 **A** *Assume $a \sim b$ for integers $a, b \geq 2$. Then any real number disjunctive to base a is disjunctive to base b too.*
B *Assume $a \not\sim b$ for integers $a, b \geq 2$. Then the set of real numbers which are disjunctive to base a but not disjunctive to base b has the cardinality of the continuum.*

This theorem is contained in [El-Zanati and Transue 1990]. Part B is already contained in [Schmidt 1960]. Furthermore Schmidt has proved that Theorem 1 is true too if one replaces "disjunctive" by "normal". In both papers counterexamples of real numbers for nonequivalent bases are constructed by limit processes. But still there are missing simple and natural counterexamples.

We shall give a new proof of Theorem 1 and show that for any nonequivalent pair of bases $a, b \geq 2$ there are simple and even quite prominent examples of real numbers which are disjunctive to base a but not to base b.

Before we do that we formulate a few more related results in [Section 2]. For completeness sake we give the simple proof of Theorem 1 A in [Section 3]. Then, for the construction of counterexamples we have to investigate the periods of rational numbers in [Section 4]. The following result may be of independent interest.

Theorem 4 *Let $a, b \geq 2$ be integers and let Q be the maximal divisor of b that is prime to a. There is a constant $K_{a,b}$ such that for any $n \geq 1$ and any integer c prime to b all words in Σ_a^* of length $\leq \log_a(Q) \cdot n - K_{a,b}$ occur in the periodic part of $\nu_a(\frac{c}{b^n} \bmod 1)$.*

A constant $K_{a,b}$ will be given in the proof. For special cases more precise versions of this result have been obtained by Stoneham, cf. [Stoneham 1964], [Stoneham 1973] and the references there.

In the last two sections we construct numbers that are disjunctive to a base a and not disjunctive to a base b, giving a new proof of Theorem 1 B. It turns out that there are two different cases for $a \not\sim b$ in which one can use different methods for the construction:

I. not all prime divisors of b divide a,
II. all prime divisors of b divide a (but still $a \not\sim b$).

In [Section 5] we consider Case I, i.e. the case that $\nu_a(\frac{1}{b})$ has nonzero periodic part. From the results in [Section 4] on the periods of rational numbers we deduce the following theorem.

Theorem 5 *Let $a, b \geq 2$ be integers such that the maximal divisor Q of b that is prime to a is greater than 1. If $(n_i)_{i\in\omega}$ is a strictly increasing sequence of positive integers satisfying $n_{i+1} \geq \log_b(a) \cdot Q^{n_i}$ for infinitely many i then the real number $\sum_{i=0}^{\infty} b^{-n_i}$ is disjunctive to base a.*

If e.g. $(n_i)_{i\in\omega}$ is a strictly increasing sequence of positive integers with $n_{i+1} \geq 3^{n_i}$ for infinitely many i then the number $\Sigma_{i=0}^{\infty} 3^{-n_i}$ is disjunctive to base 2.

In [Section 6] we consider Case II. We show how one can obtain strictly increasing sequences $(n_i)_{i\in\omega}$ such that $\sum b^{-n_i}$ is disjunctive to base a. By combining this method with the basic idea of [Section 4] we obtain the following two examples.

Theorem 6 *Let $b \geq 2$ be an integer. Then the number $\sum_{i=0}^{\infty} b^{-i!-i}$ is disjunctive to all bases a with $a \not\sim b$ that are divisible by all prime divisors of b.*

Obviously $\sum b^{-i!-i}$ is not disjunctive to base b.

Theorem 7 *Let $a = \prod_{p \text{ prime}} p^{d_p} \geq 2$ and $b = \prod_{p \text{ prime}} p^{e_p} \geq 2$ be integers with $a \not\sim b$ and $(e_p \neq 0 \Longrightarrow d_p \neq 0)$. Let $\vartheta := \max\{\frac{e_p}{d_p} \mid e_p \neq 0\}$ and let $c > \vartheta \cdot \log_b(a)$ be an integer which is not divisible by any prime p with $\frac{e_p}{d_p} = \vartheta$.*
Then the real number $\sum_{i=0}^{\infty} b^{-c^i}$ is disjunctive to base a but not to base b.

For example $\nu_6(\sum_{i=0}^{\infty} 2^{-c^i})$ is disjunctive if $c \geq 3$ is odd.

All numbers of the form as in Theorem 5 or in Theorem 6 are Liouville numbers and hence transcendental. The numbers described in the last theorem are not Liouville numbers. But their transcendence is a consequence of a variant of the Thue-Siegel-Roth approximation theorem.

Finally let us introduce a few notations. For a real number x and a positive integer n we denote the unique real number $y \in [0; n)$ with $x - y \in n \cdot \mathbb{Z}$ by $(x \bmod n)$. The number $\lfloor x \rfloor$ is the largest integer not greater than x, and $\lceil x \rceil$ is the smallest integer not smaller than x.

2 Known Results

We define simply normal, normal, random, and computable ω-words. For these properties and for disjunctiveness several results related to Theorem 1 are formulated.

Let Σ be a finite alphabet. For two words $v, w \in \Sigma^*$, $w = w_0 \dots w_{|w|-1}$ with $w_i \in \Sigma$

$$N_v(w) := \#\{i \in \{0, \dots, |w|-1\} \mid v = w_i \dots w_{i+|v|-1}\}$$

is the number of occurrences of v in w. For $p = p_0 p_1 \dots \in \Sigma^n \Sigma^* \cup \Sigma^\omega$

$$p[n] := p_0 \dots p_{n-1} \in \Sigma^n$$

is the prefix of p of length n. If for a finite string v and an ω-word p the limit $l := \lim_{n\to\infty}(N_v(p[n])/n)$ exists then we say that *v occurs in p with the asymptotic frequency l*. A *randomness test* on Σ^ω is a recursively enumerable subset $V \subseteq \Sigma^* \times \omega$ with $\mu(V_n) \leq |\Sigma|^{-n}$ for all $n \in \omega$ where $V_n := \{q \in \Sigma^\omega \mid (\exists i)\,(q[i], n) \in V\}$ and the measure μ on Σ^ω is defined by $\mu(w\Sigma^\omega) := |\Sigma|^{-|w|}$ for all $w \in \Sigma^*$.

Definition 2.1 An ω-word $p \in \Sigma^\omega$ is called

1. *simply normal* if every digit in Σ occurs with the asymptotic frequency $1/|\Sigma|$ in p,
2. *normal* if every finite word $w \in \Sigma^*$ occurs with the asymptotic frequency $|\Sigma|^{-|w|}$ in p,
3. *non-random* if there is a randomness test $V \subseteq \Sigma^* \times \omega$ with $p \in \bigcap V_i$, *random* it p is not non-random,
4. *computable* if the set $\{p[i] \mid i \in \omega\} \subset \Sigma^*$ is recursive (or recursively enumerable).

It is well known that computability is stronger than non-randomness, that randomness is stronger than normality, and that normality is stronger than simple normality and stronger than disjunctiveness. For all these properties and for any pair of integers $a, b \geq 2$ one may ask whether for reals $x \in [0;1)$ the expansion $\nu_b(x)$ has to have this property if $\nu_a(x)$ has it.

It is easy to see that computability is invariant. We already mentioned that the invariance of randomness has been proved by Calude & Jürgensen [Calude and Jürgensen 1994], [Calude 1994], see also [Weihrauch 1995]. By independent results of Cassels [Cassels 1959] and of Schmidt [Schmidt 1960] the other properties are not invariant.

For normality we mentioned that Theorem 1 remains true is one replaces "disjunctive" by "random". This was proved by Schmidt [Schmidt 1960]. A result which immediately implies the negative answer for the special case $a = 2$ and all non-equivalent $b \geq 2$ had been published by Cassels a few months earlier [Cassels 1959]. Later Schmidt proved a stronger result [Schmidt 1962]:

> Let A, B be two disjoint classes of possible bases with $\{2, 3, \ldots\} = A \cup B$ such that equivalent bases lie in the same class. Then the set of real numbers that are normal to all bases in A and not normal to all bases in B has the cardinality of the continuum.

The present author does not know whether a similar result is true for disjunctive numbers.

For simply normal numbers the situation is slightly different. A proof of the following theorem can be found in [Hertling 1995].

Theorem 2 A *Let $b \geq 2$ and $n \geq 1$ be integers. Then any real number simply normal to base b^n is simply normal to base b too.*
B *Let $a, b \geq 2$ be integers with $a \neq b^n$ for all integers n. Then the set of real numbers which are simply normal to base a but not simply normal to base b has the cardinality of the continuum.*

For disjunctiveness the following strengthening of Theorem 1 is proved in [El-Zanati and Transue 1990]:

> Let $a, b \geq 2$ be integers, $E \subseteq \Sigma_b^*$ be a finite nonempty set, and $C_E :=$ $\{x \in [0;1) \mid \nu_b(x)$ does not contain a word from $E\}$.
> Then C_E contains a number disjunctive to base a if and only if C_E contains a Cantor set and $a \not\sim b$.

In the following sections we show for quite simple numbers that they are disjunctive to a chosen base but not to another chosen base. But it seems to be unknown whether numbers like e or π or $\ln(2)$ or algebraic irrational numbers are disjunctive, simply normal or even normal with respect to any base. An extensive bibliography of older papers on normal and simply normal numbers can be found in [Kuipers and Niederreiter 1974], pp. 69 – 78.

3 Equivalent Bases

In this section we prove part A of Theorem 1.

Proof of Theorem 1.A. Fix integers $b \geq 2$ and $n \geq 1$. It is sufficient to prove that a real number x is disjunctive to base b^n if and only x is disjunctive to base b.

We define a bijection $h : \Sigma_b^n \longrightarrow \Sigma_{b^n}$ by $h(b_0 \dots b_{n-1}) := \sum_{i=0}^{n-1} b^i \cdot b_{n-1-i}$. It is easy to see that for a real number $x \in [0;1)$ with $\nu_b(x) = b_0 b_1 b_2 \dots$ and $\nu_{b^n}(x) = a_0 a_1 a_2 \dots$ one has $a_k = h(b_{k \cdot n} \dots b_{k \cdot n+n-1})$ for any k. If we denote by h also the induced bijective homomorphism $h : (\Sigma_b^n)^* \longrightarrow \Sigma_{b^n}^*$ we have $a_k \dots a_{k+l-1} = h(b_{k \cdot n} \dots b_{(k+l) \cdot n-1})$ for any k and l.

First, assume $x \in [0;1)$ is disjunctive to base b^n. Any word $w \in \Sigma_b^*$ can be extended to a word $w' \in \Sigma_b^*$ whose length is a multiple of n. The image $h(w')$ is a subword of $\nu_{b^n}(x)$ since $\nu_{b^n}(x)$ is assumed to be disjunctive. Since h is bijective the word w' and its subword w must be contained in $\nu_b(x)$.

Next, assume x is disjunctive to base b. Fix an arbitrary word $v \in \Sigma_{b^n}^*$. For $i = 0, \dots, n-1$ we define $w_i := h^{-1}(v)0^i$ and $w := w_0 \dots w_{n-1}$. The word w is a subword of $\nu_b(x)$ since $\nu_b(x)$ is assumed to be disjunctive. In $\nu_b(x) = b_0 b_1 b_2 \dots$ one of the subwords w_i of w starts with an index divisible by n, i.e. there exist $i \in \{0, \dots, n-1\}$ and $k \in \omega$ with $b_{k \cdot n} \dots b_{k \cdot n+|v| \cdot n+i-1} = w_i$. Hence $h(b_{k \cdot n} \dots b_{k \cdot n+|v| \cdot n-1}) = v$ is a subword of $\nu_{b^n}(x)$. □

4 Periods of Rational Numbers

It is well known that the decimal expansion of a rational number is periodic. We shall determine a bound $L_{a,b}$ such that all words of length $\leq L_{a,b}$ occur in the periodic part of the a-adic expansion of a rational number with denominator b in lowest terms. It is remarkable that for denominators b^n with fixed b the bound L_{a,b^n} grows (not slower than) linearly with n if not all prime divisors of b divide a. The basic idea is already contained in the proof of Proposition 4.6 which gives a variant of our main result for a special case.

For prime powers $b = p^n$ Stoneham [Stoneham 1964] has given precise formulas for the number of occurrences of any finite word in the periodic part of $\frac{1}{p^n}$. Later he extended his result and used it in order to construct normal numbers, cf. [Stoneham 1973] and the references there.

The following proposition gives the length of the preperiodic part and the period length of the expansion of an arbitrary rational number to an arbitrary base. Its proof can be found e.g. in [Bundschuh 1992].

Proposition 4.1 *Let $a, b \geq 2$ be integers. Set*

$$\begin{aligned} Q &:= \text{the maximal divisor of } b \text{ which is prime to } a\,, \\ d &:= \frac{b}{Q}\,, \\ \kappa &:= \min\{n \mid d | a^n\}\,, \\ \lambda &:= \text{order of } a \text{ in } (\mathbb{Z}/Q\mathbb{Z})^*\,. \end{aligned}$$

Let c be an integer prime to b. Then there are words $u \in \Sigma_a^\kappa$ and $v \in \Sigma_a^\lambda$ such that $\nu_a(\frac{c}{b} \bmod 1) = uv^\omega$.
For any integer c prime to b the numbers κ and λ are the minimal numbers with this property.

Note that κ and λ do not depend on c. κ is the length of the preperiodic part of $\nu(\frac{c}{b} \bmod 1)$ and λ is called the *period length of $\nu_a(\frac{c}{b} \bmod 1)$* or *of $\frac{c}{b}$ in base a*. If $Q = 1$ then the periodic part is just 0^ω. If $Q > 1$ then the period is nontrivial.

For the rest of this section let $a, b \geq 2$ be integers and define Q, d, κ, and λ as in the last proposition. Write

$$Q = \prod_{p \text{ prime}} p^{e_p}\,.$$

For $p \in P_Q := \{\text{prime divisors of } Q\} = \{p \in \omega \mid p \text{ prime and } e_p \neq 0\}$ we define

$$\begin{aligned} \mu_p &:= \begin{cases} \text{order of } a \text{ in } (\mathbb{Z}/p\mathbb{Z})^* & \text{if } p \text{ is odd} \\ \text{order of } a \text{ in } (\mathbb{Z}/4\mathbb{Z})^* & \text{if } p = 2\,, \end{cases} \\ \mu &:= l.c.m.\{\mu_p \mid p \in P_Q\}\,, \\ l_p &:= \max\{l \mid p^l | (a^\mu - 1)\} \end{aligned}$$

where $l.c.m.$ means " least common multiple". The announced bound is

$$L_{a,b} := \sum_{p \in P_Q} \log_a(p) \cdot \max\{e_p - l_p, 0\}\,.$$

With

$$\mu' := \begin{cases} l.c.m.\{\mu_p \mid p \in P_Q \text{ odd}\} & \text{if } e_2 = 1 \\ \mu & \text{else} \end{cases}$$

we can formulate the main result of this section.

Theorem 3 *Let $a, b \geq 2$ be integers and let c be an integer prime to b. Set $r := \nu_a(\frac{c}{b} \bmod 1)$. For any $w \in \Sigma_a^{\lfloor L_{a,b} \rfloor}$ and any $i \in \{0, \dots, \mu' - 1\}$ there is an index $n \in \{\kappa, \dots, \kappa + \lambda - 1\}$ with $n \equiv \kappa + i \bmod \mu'$ and $r(n) \dots r(n + \lfloor L_{a,b} \rfloor - 1) = w$.*

Note that this result is almost optimal since the period length λ is equal to $\mu' \cdot a^{L_{a,b}}$ (see Corollary 4.5 below) and there are $a^{\lfloor L_{a,b} \rfloor}$ words in $\Sigma_a^{\lfloor L_{a,b} \rfloor}$.

For the proof we have to show that the residues $(\frac{c}{b} \cdot a^n \bmod 1)$ for $n \geq \kappa$ are "sufficiently uniformly" distributed in the unit interval $[0; 1)$. Fundamental is the following well known number-theoretical fact. It is closely connected with the fact that the unit groups $(\mathbb{Z}/p^n\mathbb{Z})^*$ are cyclic for odd primes p and cyclic up to a factor of order 2 for $p = 2$.

Lemma 4.2 a) *Let p be an odd prime, $n \geq 1$. Then for $m \geq 1$ and h prime to p the cyclic subgroup of $(\mathbb{Z}/p^n\mathbb{Z})^*$ generated by $1 + hp^m$ is equal to $\{x \in \mathbb{Z}/p^n\mathbb{Z} \mid x \equiv 1 \bmod p^{\min\{n,m\}}\}$.*
b) *For $n \geq 1$, $m \geq 2$ and odd h the cyclic subgroup of $(\mathbb{Z}/2^n\mathbb{Z})^*$ generated by $1 + h2^m$ is equal to $\{x \in \mathbb{Z}/2^n\mathbb{Z} \mid x \equiv 1 \bmod 2^{\min\{n,m\}}\}$.*

For completeness sake we give the proof.

Proof. Assume $x = 1 + hp^m$ with $p^m > 2$ and h prime to p. Then all powers of x are $\equiv 1 \bmod p^{\min\{n,m\}}$. By Lemma P in [Knuth 1981], p. 16 the order of x in $(\mathbb{Z}/p^n\mathbb{Z})^*$ is equal to $p^{\max\{n-m,0\}}$. As the set $\{x \in \mathbb{Z}/p^n\mathbb{Z} \mid x \equiv 1 \bmod p^{\min\{n,m\}}\}$ contains exactly $p^{\max\{n-m,0\}}$ elements the assertion follows. $\square$

Proposition 4.3 *The cyclic subgroup of $(\mathbb{Z}/Q\mathbb{Z})^*$ generated by a^μ is equal to $\{x \in \mathbb{Z}/Q\mathbb{Z} \mid x \equiv 1 \bmod \prod_{p \in P_Q} p^{\min\{e_p,l_p\}}\}$.*

Proof. Use the last lemma and the Chinese remainder theorem.

Or observe that for any power $x = a^{\mu n}$ of a^μ the number $x - 1$ is divisible by p^{l_p} and hence divisible by the product $\prod_{p \in P_Q} p^{\min\{e_p,l_p\}}$. This product is a divisor of Q. This implies $\subseteq$. On the other hand the order of a^μ in $(\mathbb{Z}/Q\mathbb{Z})^*$ must be a multiple of the order $p^{\max\{e_p-l_p,0\}}$ of a^μ in $(\mathbb{Z}/p^{e_p}\mathbb{Z})^*$ for $p \in P_Q$, since the natural map $\mathbb{Z}/Q\mathbb{Z} \longrightarrow \mathbb{Z}/p^{e_p}\mathbb{Z}$ is a ring homomorphism. Since the set $\{x \in \mathbb{Z}/Q\mathbb{Z} \mid x \equiv 1 \bmod \prod_{p \in P_Q} p^{\min\{e_p,l_p\}}\}$ contains $\prod_{p \in P_Q} p^{\max\{e_p-l_p,0\}}$ elements we obtain equality. $\square$

Corollary 4.4 *Let c be an integer prime to b, $i \in \{0, \ldots, \mu - 1\}$. Then*

$$\{c \cdot a^{\kappa+i+\mu\cdot j} \bmod b \mid j \in \omega\} = \{x \bmod b \mid x \equiv c \cdot a^{\kappa+i} \bmod d \cdot \prod_{p \in P_Q} p^{\min\{e_p,l_p\}}\}.$$

Proof. Since $c \cdot a^{\kappa+i}/d$ is a unit in $\mathbb{Z}/Q\mathbb{Z}$ Proposition 4.3 implies

$$\{c \cdot \frac{a^{\kappa+i}}{d} \cdot a^{\mu\cdot j} \bmod Q \mid j \in \omega\} = \{x \bmod Q \mid x \equiv c \cdot \frac{a^{\kappa+i}}{d} \bmod \prod_{p \in P_Q} p^{\min\{e_p,l_p\}}\}.$$

Multiplying both sides with d gives the assertion. $\square$

Corollary 4.5 $\lambda = \mu' \cdot \prod_{p \in P_Q} p^{\max\{e_p-l_p,0\}} = \mu' \cdot a^{L_{a,b}}$.

Proof. We know λ = order of a in $(\mathbb{Z}/Q\mathbb{Z})^*$. Hence μ' divides λ. Whenever $\mu' = \mu$ the assertion follows since by Proposition 4.3 the order of a^μ in $(\mathbb{Z}/Q\mathbb{Z})^*$ is $\prod_{p \in P_Q} p^{\max\{e_p-l_p,0\}}$.

Let us assume $\mu' \neq \mu$. Then $e_2 = 1$ and $\mu = 2 \cdot \mu'$. Fix a prime $p \in P_Q$. By Lemma 4.2 the order of a^μ in $(\mathbb{Z}/p^{e_p}\mathbb{Z})^*$ is equal to $p^{\max\{e_p-l_p,0\}}$. The order of $a^{\mu'}$ in $(\mathbb{Z}/p^{e_p}\mathbb{Z})^*$ must be the same or twice this number. But since p divides $a^{\mu'} - 1$ by Lemma 4.2 the order of $a^{\mu'}$ in $(\mathbb{Z}/p^{e_p}\mathbb{Z})^*$ must be a power of p. Hence it is $p^{\max\{e_p-l_p,0\}}$ if p is odd. This is true for $p=2$ too because $e_2=1$ implies $(\mathbb{Z}/2^{e_2}\mathbb{Z})^*=\{1\}$ and $l_2\geq 1$ and hence $2^{\max\{e_2-l_2,0\}}=1$ We obtain

$$\begin{aligned}\text{order of } a^{\mu'} \text{ in } (\mathbb{Z}/Q\mathbb{Z})^* &= \prod_{p\in P_Q} \text{order of } a^{\mu'} \text{ in } (\mathbb{Z}/p^{e_p}\mathbb{Z})^* \\ &= \prod_{p\in P_Q} p^{\max\{e_p-l_p,0\}}\,.\end{aligned}$$

This finishes the proof in the case $\mu'\neq\mu$ too. □

Proof of Theorem 3. Let c be an integer prime to b and set $r:=\nu_a(\frac{c}{b} \bmod 1)$. Let $w\in\Sigma_a^{\lfloor L_{a,b}\rfloor}$ be an arbitrary word. Then for $m\in\omega$ we have $r(m)\dots r(m+\lfloor L_{a,b}\rfloor-1)=w$ if and only if $(c\cdot a^m \bmod b)$ lies in the interval $b\cdot I_w\subseteq[0;b)$ where

$$I_w:=\{x\in[0;1)\mid \nu_a(x) \text{ begins with } w\}\,.$$

This interval has a left closed end and a right open end. The interval $b\cdot I_w$ has length

$$b\cdot a^{-\lfloor L_{a,b}\rfloor}\geq b\cdot a^{-L_{a,b}}=d\cdot\prod_{p\in P_Q}p^{\min\{e_p,l_p\}}\,.$$

Hence by Corollary 4.4 for any $i\in\{0,\dots,\mu'-1\}\subseteq\{0,\dots,\mu-1\}$ there is a $j\in\omega$ such that $(c\cdot a^{\kappa+i+\mu j} \bmod b)$ lies in this interval. Since r is periodic with period length λ, i.e.

$$c\cdot a^{\kappa+m}\equiv c\cdot a^{\kappa+m+\lambda} \bmod b \quad \text{for all } m\in\omega\,,$$

the number $n:=\kappa+(i+\mu j \bmod \lambda)\in\{\kappa,\dots,\kappa+\lambda-1\}$ is the desired index. It satisfies $n\equiv\kappa+i \bmod \mu'$ because μ' divides μ and λ. □

Theorem 4 *Let $a,b\geq 2$ be integers and let Q be the maximal divisor of b that is prime to a. There is a constant $K_{a,b}$ such that for any $n\geq 1$ and any integer c prime to b all words in Σ_a^* of length $\leq\log_a(Q)\cdot n-K_{a,b}$ occur in the periodic part of $\nu_a(\frac{c}{b^n} \bmod 1)$.*

Proof. Because of

$$\begin{aligned}L_{a,b^n} &= \sum_{p\in P_Q}\log_a(p)\cdot\max\{ne_p-l_p,0\}\\ &\geq \sum_{p\in P_Q}\log_a(p)\cdot(ne_p-l_p)\\ &= \log_a(Q)\cdot n-\sum_{p\in P_Q}\log_a(p)\cdot l_p\end{aligned}$$

Theorem 3 gives the assertion with

$$K_{a,b}:=\sum_{p\in P_Q}\log_a(p)\cdot l_p\,.$$

□

Note that $K_{a,b} < \mu$ because $a^{K_{a,b}} = \prod_{p \in P_Q} p^{l_p}$ divides $a^\mu - 1$. The multiplicative constant $\log_a(Q)$ in Theorem 4 is optimal because for sufficiently large n the period length of $\nu_a(\frac{c}{b^n} \bmod 1)$ is Q^n times a constant while Σ_a^l contains a^l elements. But the additive constant $K_{a,b}$ given in the proof can be improved in special cases.

Proposition 4.6 *Let p be an odd prime and $a \geq 2$ be a primitive root modulo p with $a^{p-1} \not\equiv 1 \bmod p^2$. For any $n \geq 1$ and any integer c prime to p set $r := \nu_a(\frac{c}{p^n} \bmod 1)$ and*

$$L'_n := \log_a(p) \cdot n - \log_a(2)\,.$$

Then for any $w \in \Sigma_a^{\lfloor L'_n \rfloor}$ there is an index $m < (p-1)p^{n-1}$ with $r(m) \dots r(m + \lfloor L'_n \rfloor - 1) = w$.

Note that in this case $\kappa = 0$, i.e. r is purely periodic, $\lambda = (p-1)p^{n-1}$, $L_{a,p^n} = \log_a(p) \cdot (n-1)$ and the constant of Corollary 4 is equal to $K_{a,b} = \log_a(p) > \log_a(2)$.

Proof. The group $(\mathbb{Z}/p^n\mathbb{Z})^*$ has order $(p-1)p^{n-1}$. Thus for a primitive root a modulo p the condition $a^{p-1} \not\equiv 1 \bmod p^2$ implies by Lemma 4.2 a) that a generates $(\mathbb{Z}/p^n\mathbb{Z})^*$. Hence for c prime to p we have

$$\{c \cdot a^j \bmod p^n \mid j \in \omega\} = (\mathbb{Z}/p^n\mathbb{Z})^* = \{x \bmod p^n \mid p \not\equiv 0 \bmod p\}\,.$$

This is an improvement of Corollary 4.4 for this special case. Since for any $w \in \Sigma_a^{\lfloor L'_n \rfloor}$ the interval

$$p^n \cdot I_w = p^n \cdot \{x \in [0;1) \mid \nu_a(x) \text{ begins with } w\}$$

has length

$$p^n \cdot a^{-\lfloor L'_n \rfloor} \geq 2$$

it contains a number $(c \cdot a^j \bmod p^n)$. For this integer j we have $r(j) \dots r(j + \lfloor L'_n \rfloor - 1) = w$. Since $(p-1)p^{n-1}$ is the period length, the number $m := j \bmod (p-1)p^{n-1}$ is the desired index. □

An example is given by $a = 2$ and $p = 3$. For c not divisible by 3 the 2-adic expansion of $c/3^n$ is purely periodic and has period length $2 \cdot 3^{n-1}$. By the last proposition it contains all words over $\{0, 1\}$ of length $\leq \log_2(3) \cdot n - 1$.

5 Nonequivalent Bases: Case I

The following result gives a constructive proof of Theorem 1.B in the case that b has a prime divisor not dividing a.

Theorem 5 *Let $a, b \geq 2$ be integers such that the maximal divisor Q of b that is prime to a is greater than 1. If $(n_i)_{i \in \omega}$ is a strictly increasing sequence of positive integers satisfying*

$$n_{i+1} \geq \log_b(a) \cdot Q^{n_i} \tag{1}$$

for infinitely many i then the real number $\sum_{i=0}^{\infty} b^{-n_i}$ is disjunctive to base a.

If the sequence $(n_i)_{i\in\omega}$ satisfies (1) for almost all i the number $\sum_{i=0}^{\infty} b^{-n_i}$ is certainly not disjunctive to base b. Or one could add the condition

$$n_{i+1} \geq n_i + 2 \qquad \text{for all } i\,.$$

Then the b-adic expansion does not contain the word 11. If for example $n_0 \geq 1$ and $n_{i+1} \geq 3^{n_i}$ for infinitely many i then $\sum_{i=0}^{\infty} 3^{-n_i}$ is disjunctive to base 2 but not to base 3.

We explain the basic idea for the example $a = 2$ and $b = 3$. Let us assume that $(n_i)_{i\in\omega}$ is a strictly increasing sequence. By the results in the last section, for this example especially by Proposition 4.6, we know that the periodic part plus a few more digits of $\nu_2(\sum_{i=0}^{k} 3^{-n_i}) = \nu_2(\sum_{i=0}^{k} 3^{n_k - n_i})/3^{n_k})$ contains all words over $\{0,1\}$ of length $\leq \log_2(3)\cdot k - 1$. Adding a very small term like $\sum_{i=k+1}^{\infty} 3^{-n_i}$ for an $n_{k+1} \geq 3^{n_k}$ does not change the first period and some more digits of $\nu_2(\sum_{i=0}^{k} 3^{-n_i})$. Hence also the 2-adic expansion of $\sum_{i=0}^{\infty} 3^{-n_i}$ contains all words of length $\leq \log_2(3)\cdot k - 1$. If this is true for infinitely many k then $\nu_2(\sum_{i=0}^{\infty} 3^{-n_i})$ must be disjunctive.

Remarks. 1. The growth condition (1) can be weakened to: $\exists C > \log_b(a)\cdot(Q-1)/Q$ with

$$n_{i+1} \geq C \cdot Q^{n_i} \tag{2}$$

for infinitely many i. In the proof we will give an even weaker more complicated condition. In special cases one can obtain still weaker growth conditions — with exponent approximately $n_i/2$ instead of n_i — by using the results in [Stoneham 1973].

2. We know more about $\nu_a(\sum_{i=0}^{\infty} b^{-n_i})$ than that it is disjunctive. If n_i satisfies (1) or (2) and is sufficiently large then all words over Σ_a of length $\leq \log_a(Q)\cdot n_i - K_{a,b}$ appear in its prefix of length Q^{n_i}.

In the following we prove Theorem 5. We assume that $a, b \geq 2$ are integers and that a is not divisible by all prime divisors of b. We use the terminology of [Section 4].

Lemma 5.1 *Let $(n_i)_{i\in\omega}$ be a strictly increasing sequence of integers. If for some m and some i*

$$n_{i+1} \geq \log_b(a)\cdot m + n_i + \log_b(\frac{ab}{b-1}) \tag{3}$$

then the prefixes of length m of $\nu_a(\sum_{j=0}^{i} b^{-n_j})$ and of $\nu_a(\sum_{j=0}^{\infty} b^{-n_j})$ are identical.

Proof. Fix an m and an i satisfying (3). Let v be the prefix of length m of $\nu_a(\sum_{j=0}^{i} b^{-n_j})$. We have to show that the difference between the infinite sum and the finite sum is so small that adding it to the finite sum does not change the prefix v of the a-adic expansion of the finite sum. We compute

$$\begin{aligned}\sum_{j=i+1}^{\infty} b^{-n_j} &\leq \frac{b}{b-1}\cdot b^{-n_{i+1}} \\ &\leq \frac{b}{b-1}\cdot b^{-\log_b(a)\cdot m - n_i - \log_b(\frac{ab}{b-1})} \\ &= a^{-m-1}\cdot b^{-n_i}\end{aligned}$$

This term is so small that only a carry could change the first m digits of the finite sum when we add it. We show that such a carry does not occurr.

Let $w := (a-1)^{\lfloor \log_a(b)\cdot n_i \rfloor +1} \in \Sigma_a^{\lfloor \log_a(b)\cdot n_i \rfloor +1}$. The interval

$$b^{n_i} \cdot I_w = b^{n_i} \cdot \{x \in [0;1) \mid \nu_a(x) \text{ begins with } w\}$$

contains no integer because its (open !) right end is the integer b^{n_i} and its length is

$$b^{n_i} \cdot a^{-\lfloor \log_a(b)\cdot n_i \rfloor -1} < 1 .$$

Hence for any integer c prime to b the a-adic expansion of $(\frac{c}{b^{n_i}} \bmod 1)$ does not contain the word w. Especially

$$\nu_a(\sum_{j=0}^{i} b^{-n_j})(m)\ldots\nu_a(\sum_{j=0}^{i} b^{-n_j})(m+|w|-1) \neq w .$$

We conclude

$$\begin{aligned}\sum_{j=0}^{i} b^{-n_j} &\leq \nu_a^{-1}(v0^\omega) + a^{-m} - a^{-m-\lfloor \log_a(b)\cdot n_i \rfloor -1} \\ &< \nu_a^{-1}(v0^\omega) + a^{-m} - a^{-m-1}\cdot b^{-n_i}\end{aligned}$$

Combining this with the estimation for the rest from above we obtain

$$\sum_{j=0}^{\infty} b^{-n_j} < \nu_a^{-1}(v0^\omega) + a^{-m} .$$

This implies that v is the prefix of length m of $\nu_a(\sum_{j=0}^{\infty} b^{-n_j})$. $\square$

In the following we assume that b has a prime divisor not dividing a. Q, d, P_Q, and μ are defined as in [Section 4]. For a positive integer n we define

$$G(n) := \min\{m \mid d^n | a^m\} + \frac{\mu}{\prod_{p\in P_Q} p^{l_p}} \cdot Q^n + \lfloor L_{a,b^n} \rfloor - \mu .$$

Corollary 5.2 *Let $(n_i)_{i\in\omega}$ be a strictly increasing sequence of positive integers. If for some sufficiently large n_i*

$$n_{i+1} \geq \log_b(a)\cdot G(n_i) + n_i + \log_b(\frac{ab}{b-1}) \tag{4}$$

holds true then the prefix of length $G(n_i)$ of $\nu_a(\sum_{j=0}^{\infty} b^{-n_j})$ contains all words over Σ_a of length $\lfloor L_{a,b^{n_i}} \rfloor$.

Proof. This is an immediate consequence of the last lemma (with $m := G(n_i)$) and of Theorem 3.

The first summand $\min\{m \mid d^{n_i}|a^m\}$ in $G(n_i)$ is the length of the preperiodic part of $\nu_a(\sum_{j=0}^{i} b^{-n_j})$. The second summand $\frac{\mu}{\prod_{p\in P_Q} p^{l_p}} \cdot Q^{n_i}$ is its period length if n_i is sufficiently large. Namely, then one has $n_i \cdot e_p \geq l_p$ for all prime divisors of Q, and one can replace μ' by μ because $n_i \cdot e_2 \neq 1$. Theorem 3 states that the subword of length

$$\textit{period length} + \lfloor L_{a,b^{n_i}} \rfloor - \mu$$

starting at index $\min\{m \mid d^{n_i}|a^m\}$ in $\nu_a(\sum_{j=0}^{i} b^{-n_j})$ contains all words over Σ_a of length $\lfloor L_{a,b^{n_i}} \rfloor$. □

Corollary 5.3 *Let $(n_i)_{i\in\omega}$ be a strictly increasing sequence of positive integers satisfying (4) for infinitely many i. Then the real number $\sum_{i=0}^{\infty} b^{-n_i}$ is disjunctive to base a.*

Proof. Immediate by the last corollary because

$$L_{a,b^{n_i}} \geq \log_a(Q) \cdot n_i - K_{a,b}$$

(see the proof of Theorem 4). □

Proof of Theorem 5. In order to finish the proof of Theorem 5 we only have to show that for sufficiently large n_i Condition (2) for a fixed constant $C > \log_b(a) \cdot (Q-1)/Q$ implies (4). Of course (1) implies (2) with $C = \log_b(a)$.

All summands in $G(n_i)$ and hence in the right side of (4) are constant or grow linearly with n_i except for $\frac{\mu}{\prod_{p\in P_Q} p^{l_p}} \cdot Q^{n_i}$. This grows exponentially with n_i. The constant factor can be estimated:

$$\frac{\mu}{\prod_{p\in P_Q} p^{l_p}} \leq \frac{l.c.m.\{p-1 \mid p \in P_Q \text{ odd}\}}{\prod_{p\in P_Q} p} \leq \frac{Q-1}{Q}\,.$$

(Note that $l_p \geq 1$ for all $p \in P_Q$ and $l_2 \geq 2$ if $2 \in P_Q$.) Hence for sufficiently large n_i Condition (2) implies (4). This finishes the proof of Theorem 5. Finally the last estimation also gives the assertion of Remark 2. □

A real number x is called a *Liouville number* if it is not rational and for each $n \in \omega$ there are integers p_n and $q_n > 0$ such that

$$|x - \frac{p_n}{q_n}| < \frac{1}{q_n^n}\,.$$

Liouville numbers are transcendental (cf. e.g. [Bundschuh 1992]).

Lemma 5.4 *Let $(n_i)_{i\in\omega}$ be a strictly increasing sequence of positive integers satisfying*

$$n_{i+1} \geq i \cdot n_i + 1 \tag{5}$$

for infinitely many i. Then $\sum_{i=0}^{\infty} b^{-n_i}$ is a Liouville number.

Proof. For any i satisfying (5) we have

$$\left|\sum_{i=0}^{\infty} b^{-n_i} - \frac{\sum_{j=0}^{i} b^{n_i-n_j}}{b^{n_i}}\right| = \left|\sum_{j=i+1}^{\infty} b^{-n_j}\right| \le b^{-n_{i+1}} \cdot \frac{b}{b-1} \le b^{-n_{i+1}+1} \le \frac{1}{(b^{n_i})^i}\,.$$

Since this is true for infinitely large i the sum $\sum_{i=0}^{\infty} b^{-n_i}$ is a Liouville number. □

Hence the numbers $\sum_{i=0}^{\infty} b^{-n_i}$ considered in Theorem 5 are Liouville numbers.

6 Nonequivalent Bases: Case II

In this section we assume that $a, b \ge 2$ are integers with $a \not\sim b$ but that each prime divisor of b divides a. We shall show how one can construct sequences $(n_i)_{i\in\omega}$ such that the sum $\sum_{i=0}^{\infty} b^{-n_i}$ is disjunctive to base a but not to base b.

We write $a = \prod_{p \text{ prime}} p^{d_p}$, $b = \prod_{p \text{ prime}} p^{e_p}$. Our assumption is $(e_p \neq 0 \Longrightarrow d_p \neq 0)$. Hence

$$\vartheta := \max\{\frac{e_p}{d_p} \mid e_p \neq 0\}$$

exists. Note that $a \not\sim b$ implies $\log_a(b) < \vartheta$. By the next lemma for any $n \in \omega$ only a finite block of digits in $\nu_a(b^{-n})$ can be nonzero.

Lemma 6.1 *For any $n \in \omega$*

$$\nu_a(b^{-n})(j) \neq 0 \Longrightarrow \lceil n \cdot \log_a(b)\rceil - 1 \le j \le \lceil n \cdot \vartheta\rceil - 1\,.$$

Proof. If $\nu_a(b^{-n})(j) \neq 0$ then $a^{-(j+1)} \le b^{-n}$ and b^n does not divide a^j. Because of

$$\min\{k \mid b^n \text{ divides } a^k\} = \lceil n \cdot \vartheta\rceil \tag{6}$$

the assertion follows. □

Hence, if a sequence $(n_i)_{i\in\omega}$ grows sufficiently fast then the blocks of possibly nonzero digits of the summands b^{-n_i} do not overlap in $\nu_a(\sum_{i=0}^{\infty} b^{-n_i})$. This can be used for the construction of disjunctive numbers.

Proposition 6.2 *Let $(n_i)_{i\in\omega}$ be a strictly increasing sequence of positive integers satisfying*

$$\lceil n_{i+1} \cdot \log_a(b)\rceil - 1 \ge \lceil n_i \cdot \vartheta\rceil \tag{7}$$

for almost all i. Furthermore assume that the topological closure of the set

$$M := \{\left(\frac{a^j}{b^{n_i}} \bmod 1\right) \mid i \in \omega,\ \lceil n_i \cdot \log_a(b)\rceil - 1 \le j \le \lceil n_i \cdot \vartheta\rceil - 1\}$$

contains an interval of positive length. Then the real number $\sum_{i=0}^{\infty} b^{-n_i}$ is disjunctive to base a but not to base b.

Proof. The b-adic expansion $\nu_b(\sum_{i=0}^{\infty} b^{-n_i})$ cannot be disjunctive because (7) and $\log_a(b) < \vartheta$ imply that the sequence (n_i) grows exponentially fast.

Let i_0 be a constant such that (7) holds true for all $i \geq i_0$. Because of (6) nonzero digits in $\nu_a(\sum_{j=0}^{i_0} b^{-n_j})$ appear only in the prefix of length $\lceil n_{i_0} \cdot \vartheta \rceil$. By Lemma 6.1 for $i > i_0$ the blocks of possibly nonzero digits in $\nu_a(b^{-n_i})$ do not overlap with this prefix or with each other. Hence for $i > i_0$ we obtain

$$\nu_a(\sum_{l=0}^{\infty} b^{-n_l})(j) = \nu_a(b^{-n_i})(j)$$

for $\lceil n_i \cdot \log_a(b) \rceil - 1 \leq j \leq \lceil n_i \cdot \vartheta \rceil - 1$, i.e. the block of possibly nonzero digits in $\nu_a(b^{-n_i})$ is a subword of $\nu_a(\sum_{j=0}^{\infty} b^{-n_j})$.

Let $w \in \Sigma_a^*$ be an arbitrary word. By our second assumption there is an $i > i_0$ and a $j \in \{\lceil n_i \cdot \log_a(b) \rceil - 1, \ldots, \lceil n_i \cdot \vartheta \rceil - 1\}$ such that $\nu_a(\frac{a^j}{b^{n_i}} \bmod 1)$ contains the word $w1$. Then $w1$ is a subword of the block of possibly nonzero digits in $\nu_a(b^{-n_i})$ and hence a subword of $\nu_a(\sum_{l=0}^{\infty} b^{-n_l})$. So this must be disjunctive. $\square$

Remark. It is easy to see that the closure of the set M in Proposition 6.2 automatically contains the unit interval $[0; 1]$ if it contains an interval of positive length. Namely, if the closure of M contains an interval of positive length then it contains an interval $I_v := \{x \in [0; 1) \mid \nu_a(x)$ begins with $v\}$ for some $v \in \Sigma_a^*$. Then for any $w \in \Sigma^*$ there are a number n_i and a $j \geq \lceil n_i \cdot \log_a(b) \rceil - 1$ such that $\nu_a(\frac{a^j}{b^{n_i}} \bmod 1)$ begins with $vw1$. Hence $\nu_a(\frac{a^{j+|v|}}{b^{n_i}} \bmod 1)$ begins with $w1$. By Lemma 6.1 we see $j + |v| + |w| \leq \lceil n_i \cdot \vartheta \rceil - 1$. Hence also $(\frac{a^{j+|v|}}{b^{n_i}} \bmod 1)$ lies in M. Since this is true for all $w \in \Sigma^*$ the set M is dense in $[0; 1]$.

By the next proposition there are indeed uncountably many sequences $(n_i)_{i \in \omega}$ satisfying the assumptions of Proposition 6.2. One can even choose them to grow arbitrarily fast or to fulfill further conditions like $n_{i+1} \geq i \cdot n_i + 1$ for infinitely many i (then $\sum_{j=0}^{\infty} b^{-n_j}$ is a Liouville number by Lemma 5.4) or $n_{i+1} \geq n_i + 2$ for all i (then $\nu_b(\sum_{j=0}^{\infty} b^{-n_j})$ does not contain the word 11). The next proposition states that for any word w with

$$I_w \cap (\frac{1}{a}; 1) \neq \emptyset$$

where $I_w := \{x \in [0; 1) \mid \nu_a(x)$ begins with $w\}$ there are arbitrarily large integers n such that $\nu_a(\frac{1}{b^n})$ begins with $0^{\lceil n \cdot \log_a(b) \rceil - 1} w$.

Proposition 6.3 *Let $a, b > 1$ be real numbers with $a \not\sim b$. Then the set*

$$\{\frac{a^{\lceil n \cdot \log_a(b) \rceil - 1}}{b^n} \mid n \geq 1\}$$

is a dense subset of the interval $(\frac{1}{a}; 1)$.

Proof. It is easy to see that for an irrational number $x > 0$ the sequence $(m \cdot x)_{m \in \omega}$ is dense modulo 1, i.e. the set $\{(m \cdot x \bmod 1) \mid m \in \omega\}$ is a dense subset of $[0; 1)$ (in fact the sequence $(m \cdot x)_{m \in \omega}$ is uniformly distributed modulo 1, see

[Kuipers and Niederreiter 1974]). Hence for irrational $x > 0$ the set $\{m \cdot x - n \mid m, n \in \omega\}$ is a dense subset of $\mathbb{R}$.

Our assumption $a \not\sim b$ is equivalent to "$\log_b(a)$ is irrational". Thus, by setting $x := \log_b(a)$ and multiplying with $\ln(b)$ we conclude that $\{m \cdot \ln(a) - n \cdot \ln(b) \mid m, n \in \omega\}$ is dense in $\mathbb{R}$. Exponentiating we see that $\{\frac{a^m}{b^n} \mid m, n \in \omega\}$ is a dense subset of the positive real numbers. The number $\frac{a^m}{b^n}$ is in $(\frac{1}{a}; 1)$ if and only if $n \neq 0$ and $m = \lceil n \cdot \log_a(b) \rceil - 1$. Hence the assertion follows. □

The last proposition could be used for the construction of numbers disjunctive to a base a but not to b also in the case that not all prime divisors of b divide a, which was considered in the last section. In that case one cannot use Proposition 6.2. Instead one has to use the fact that $\nu_a(\frac{c}{b^n})$ is periodic. But this idea will probably not lead to a better construction than the one in Theorem 5. Hence we do not pursue this idea.

The next results are obtained by combining Proposition 6.2 and Lemma 4.2 resp. Proposition 4.3.

Theorem 6 *Let $b \geq 2$ be an integer. Then the number $\sum_{i=0}^{\infty} b^{-i!-i}$ is disjunctive to all bases a with $a \not\sim b$ that are divisible by all prime divisors of b.*

Obviously $\nu_b(\sum b^{-i!-i})$ is not disjunctive. The numbers of the form as in this theorem are Liouville numbers by Lemma 5.4. Before we give the proof we formulate a result which is obtained by a similar method.

Theorem 7 *Let $a = \prod_{p \text{ prime}} p^{d_p} \geq 2$ and $b = \prod_{p \text{ prime}} p^{e_p} \geq 2$ be integers with $a \not\sim b$ and $(e_p \neq 0 \Longrightarrow d_p \neq 0)$. Let $\vartheta := \max\{\frac{e_p}{d_p} \mid e_p \neq 0\}$ and let $c > \vartheta \cdot \log_b(a)$ be an integer which is not divisible by any prime p with $\frac{e_p}{d_p} = \vartheta$.*
Then the real number $\sum_{i=0}^{\infty} b^{-c^i}$ is disjunctive to base a but not to base b.

For example $\nu_6(\sum_{i=0}^{\infty} 2^{-c^i})$ is disjunctive if $c \geq 3$ is odd, and the ω-word $\nu_{2100}(\sum_{i=0}^{\infty} 60^{-c^i})$ is disjunctive if $c \geq 2$ is odd and not divisible by 3.

For the proof of the last two theorems we need a lemma which is a special and technical application of Proposition 4.3. The basic idea is that for two integers $r, s \geq 2$ prime to each other the finite set $\{\frac{r^i}{s^n} \bmod 1 \mid i \in \omega\}$ is tending to be a dense subset of $[0; 1]$ for n tending to infinity. We write $\mathrm{ord}_y(x)$ for the order of x in $(\mathbb{Z}/y\mathbb{Z})^*$.

Lemma 6.4 *Let $r, s_1 \geq 2$, $t_1 \geq 1$ be integers with $(r, s_1) = 1$, $(r, t_1) = 1$. Let $m_b \geq 1$ be an integer and $\mathcal{N} \subseteq \omega$ be a set such that for each $m \geq m_b$ and for each $j \in \omega$ the set*

$$\{k \in \mathcal{N} \mid k \equiv j \bmod \mathrm{ord}_{s_1^m \cdot t_1}(r)\}$$

is infinite. Furthermore let $q \geq 1$ be any integer and $s_2, t_2 \geq 1$ be products of the prime divisors of r.
Then there is an $m_e \geq m_b$ such that for any $m \geq m_e$ there exists an integer q_m such that for each $j \in \omega$ the set

$$\{k \in \mathcal{N} \mid q \cdot r^k \equiv q \cdot q_m \cdot s_2^m t_2 + j \cdot s_1^{m_e} s_2^m t_1 t_2 \bmod s_1^m s_2^m t_1 t_2\}$$

is infinite. If $s_2 = 1$ then $q_m = q_0$ for all $m \geq m_e$.

Proof. Proposition 4.3 implies that there is an integer $m_1 \geq m_b$ such that for all $m \geq m_1$ and all $j \in \omega$ the set

$$\{k \in \mathcal{N} \mid r^k \equiv 1 + j \cdot s_1^{m_1} t_1 \bmod s_1^m t_1\}$$

is infinite. If k is sufficiently large then $s_2^m t_2$ divides r^k. Hence for any $j \in \omega$ the set

$$N_j := \{k \in \mathcal{N} \mid r^k \equiv 0 \bmod s_2^m t_2 \quad \text{and} \quad r^k \equiv 1 + j \cdot s_1^{m_1} t_1 \bmod s_1^m t_1\}$$

is infinite. With

$$q_m := \text{ the smallest positive integer } x \text{ with } x \cdot s_2^m t_2 \equiv 1 \bmod s_1^{m_1} t_1$$

we define

$$\tilde{N}_j := \{k \in \mathcal{N} \mid r^k \equiv q_m \cdot s_2^m t_2 + j \cdot s_1^{m_1} s_2^m t_1 t_2 \bmod s_1^m s_2^m t_1 t_2\}\,.$$

We claim that

$$\{N_j \mid j \in \omega\} = \{\tilde{N}_j \mid j \in \omega\}\,. \tag{8}$$

By the Chinese remainder theorem for any $j \in \omega$ there is a unique $y_j \in \{0, \ldots, s_1^m s_2^m t_1 t_2 - 1\}$ with

$$N_j = \{k \in N \mid r^k \equiv y_j \bmod s_1^m s_2^m t_1 t_2\}\,.$$

These numbers y_j are exactly the s^{m-m_1} numbers

$$y \in \{0, \ldots, s_1^m s_2^m t_1 t_2 - 1\} \text{ with } y \equiv 0 \bmod s_2^m t_2 \text{ and } y \equiv 1 \bmod s_1^{m_1} t_1\,. \tag{9}$$

On the other hand the s^{m-m_1} numbers

$$\tilde{y_j} := q_m \cdot s_2^m t_2 + j \cdot s_1^{m_1} s_2^m t_1 t_2 \bmod s_1^m s_2^m t_1 t_2$$

for $j \in \{0, \ldots, s^{m-m_1} - 1\}$ are pairwise different and fulfill (9). Hence Claim (8) is proved.

Multiplying the congruence equation in the definition of $\tilde{N}_j$ with q can only enlarge the set:

$$\tilde{N}_j \subseteq \{k \in \mathcal{N} \mid q \cdot r^k \equiv q \cdot q_m \cdot s_2^m t_2 + j \cdot q s_1^{m_1} s_2^m t_1 t_2 \bmod s_1^m s_2^m t_1 t_2\}\,.$$

So this set must be infinite for each $j \in \omega$. Finally there is an $m_e \geq m_1$ such that for $m \geq m_e$

$$\{j \cdot s_1^{m_e} s_2^m t_1 t_2 \bmod s_1^m s_2^m t_1 t_2 \mid j \in \omega\} \subseteq \{j \cdot q s_1^{m_1} s_2^m t_1 t_2 \bmod s_1^m s_2^m t_1 t_2 \mid j \in \omega\}\,.$$

This proves the assertion. Note that q_m does not depend on m if $s_2 = 1$. □

Proof of Theorem 6. Let

$$a = \prod_{p \text{ prime}} p^{d_p} \geq 2 \quad \text{and} \quad b = \prod_{p \text{ prime}} p^{e_p} \geq 2$$

be integers with $a \not\sim b$ and $(e_p \neq 0 \Longrightarrow d_p \neq 0)$. We write $\vartheta = \max\{\frac{e_p}{d_p} \mid e_p \neq 0\}$ in lowest terms:

$$\vartheta = \frac{\vartheta_n}{\vartheta_d} \quad \text{with} \quad (\vartheta_n, \vartheta_d) = 1\,.$$

Let

$$\mathcal{P} := \{p \mid p \text{ prime and } \frac{e_p}{d_p} = \vartheta\}, \quad P := \prod_{p \in \mathcal{P}} p^{d_p}, \quad Q := \frac{a}{P}\,.$$

Then b^{ϑ_d} divides a^{ϑ_n}, and the integer $\frac{a^{\vartheta_n}}{b^{\vartheta_d}}$ is prime to P but divisible by all prime divisors of Q. We apply Lemma 6.4 to

$$\begin{aligned} r &:= \frac{a^{\vartheta_n}}{b^{\vartheta_d}}, \\ s_1 &:= P, \\ t_1 &:= 1, \\ m_b &:= 1, \\ \mathcal{N} &:= \{\frac{(i \cdot \vartheta_d)!}{\vartheta_d} + i \mid i \in \omega\}\,. \\ s_2 &:= Q, \\ t_2 &:= 1, \\ q &:= 1\,. \end{aligned}$$

This gives us an integer m_e and for each $m \geq m_e$ an integer q_m such that the set

$$\{i \in \omega \mid \left(\frac{a^{\vartheta_n}}{b^{\vartheta_d}}\right)^{\frac{(i\cdot\vartheta_d)!}{\vartheta_d}+i} \equiv q_m \cdot Q^m + j \cdot P^{m_e} Q^m \bmod a^m\}$$

is infinite for each $j \in \omega$. For the i in this set we compute

$$\begin{aligned} \frac{a^{((i\vartheta_d)!+i\vartheta_d)\cdot\vartheta - m}}{b^{((i\vartheta_d)!+i\vartheta_d)}} &= \left(\frac{a^{\vartheta_n}}{b^{\vartheta_d}}\right)^{\frac{(i\cdot\vartheta_d)!}{\vartheta_d}+i} \cdot \frac{1}{a^m} \\ &\equiv q_m \cdot Q^m \cdot \frac{1}{a^m} + j \cdot \frac{1}{P^{m-m_e}} \bmod 1\,. \end{aligned}$$

For any fixed $m \geq m_e$ and for sufficiently large i these last numbers are contained in the set M defined in Proposition 6.2 (for $n_k := k! + k$) because for sufficiently large i we have

$$((i\vartheta_d)! + i\vartheta_d) \cdot \vartheta - m \geq \lceil ((i\vartheta_d)! + i\vartheta_d) \cdot \log_a(b) \rceil - 1\,.$$

Hence for any $m \geq m_e$ each closed interval $I \subseteq [0;1]$ of length $\geq \frac{1}{P^{m-m_e}}$ contains at least one element of M. Thus, the closure of M contains $[0;1]$. By Proposition 6.2 the number $\sum_{i=0}^{\infty} b^{-i!-i}$ is disjunctive to base a and not disjunctive to base b. □

Proof of Theorem 7. We wish to apply Lemma 6.4 and Proposition 6.2 in the same manner as in the last proof. In fact we apply Lemma 6.4 two times.

Again we write $\vartheta = \max\{\frac{e_p}{d_p} \mid e_p \neq 0\}$ in lowest terms:

$$\vartheta = \frac{\vartheta_n}{\vartheta_d} \quad \text{with} \quad (\vartheta_n, \vartheta_d) = 1\,.$$

We write the denominator ϑ_d as a product $\vartheta_d = \alpha \cdot \beta$ where α is the maximal divisor of ϑ_d that is prime to c. Furthermore

$$j_0 := \min\{j \mid \beta | c^j\}, \qquad i_0 := \mathrm{ord}_\alpha(c)\,.$$

Then ϑ_d divides $c^{j_0} \cdot (c^{i \cdot i_0} - 1)$ for any $i \in \omega$. We define $\mathcal{P}$, P, and Q as in the last proof:

$$\mathcal{P} := \{p \mid p \text{ prime and } \frac{e_p}{d_p} = \vartheta\}, \quad P := \prod_{p \in \mathcal{P}} p^{d_p}, \quad Q := \frac{a}{P}\,.$$

Note that our assumption about c implies $(P, c) = 1$. Again $\frac{a^{\vartheta_n}}{b^{\vartheta_d}}$ is an integer which is prime to P and divisible by all prime divisors of Q. By Proposition 4.3 we know that there are integers m_1 and K such that for $m \geq m_1$

$$\text{order of } \frac{a^{\vartheta_n}}{b^{\vartheta_d}} \text{ in } (\mathbb{Z}/P^m\mathbb{Z})^* = K \cdot P^{m - m_1}\,.$$

We write $K = K_1 \cdot K_2$ where K_1 is the maximal divisor of K that is prime to c (and K_2 is a product of prime divisors of c).

At first we apply Lemma 6.4 to

$$\begin{aligned} r &:= c^{i_0}, \\ s_1 &:= P, \\ t_1 &:= \alpha \cdot K_1, \\ m_b &:= 1, \\ \mathcal{N} &:= \omega, \\ s_2 &:= 1, \\ t_2 &:= K_2, \\ q &:= \frac{c^{j_0}}{\beta}\,. \end{aligned}$$

We obtain m'_e and a q'_0 (note that $s_2 = 1$) such that for each $m \geq m'_e$ and for any $j \in \omega$ the set

$$N_j^{(m)} := \{i \in \omega \mid \frac{c^{j_0}}{\beta} \cdot c^{i \cdot i_0} \equiv \frac{c^{j_0}}{\beta} \cdot q'_0 \cdot K_2 + j \cdot \alpha K P^{m'_e} \bmod \alpha K P^m\}$$

is infinite. As α divides $c^{i \cdot i_0} - 1$ it must divide $q'_0 \cdot K_2 - 1$ too. Hence this set is equal to

$$N_j^{(m)} = \{i \in \omega \mid \frac{c^{j_0} \cdot (c^{i \cdot i_0} - 1)}{\vartheta_d} \equiv \frac{c^{j_0} \cdot (q'_0 \cdot K_2 - 1)}{\vartheta_d} + j \cdot K P^{m'_e} \bmod K P^m\}\,.$$

This means that we can apply Lemma 6.4 a second time, now to:

$$\begin{aligned}
r &:= \left(\frac{a^{\vartheta_n}}{b^{\vartheta_d}}\right)^{KP^{m'_e}}, \\
s_1 &:= P, \\
t_1 &:= 1, \\
m_b &:= m'_e + m_1, \\
\mathcal{N} &:= \{\frac{c^{j_0} \cdot (c^{i \cdot i_0} - q'_0 \cdot K_2)}{\vartheta_d K P^{m'_e}} \mid i \in \omega \text{ and} \\
&\qquad\qquad \vartheta_d K P^{m'_e} \text{ divides } c^{j_0} \cdot (c^{i \cdot i_0} - q'_0 \cdot K_2)\}, \\
s_2 &:= Q, \\
t_2 &:= 1, \\
q &:= \frac{a^{\lceil c^{j_0} \cdot \vartheta \rceil}}{b^{c^{j_0}}} \cdot \left(\frac{a^{\vartheta_n}}{b^{\vartheta_d}}\right)^{\frac{c^{j_0} \cdot (q'_0 \cdot K_2 - 1)}{\vartheta_d}}.
\end{aligned}$$

Note that for $m \geq m'_e + m_1$ the order of r in $(\mathbb{Z}/P^m\mathbb{Z})^*$ is $P^{m - m'_e - m_1}$. This second application of Lemma 6.4 gives us an m_e and for each $m \geq m_e$ a q_m such that for any $j \in \omega$ the set

$$\begin{aligned}
\{i \in \omega \mid & \frac{a^{\lceil c^{j_0} \cdot \vartheta \rceil}}{b^{c^{j_0}}} \cdot \left(\frac{a^{\vartheta_n}}{b^{\vartheta_d}}\right)^{\frac{c^{j_0} \cdot (c^{i \cdot i_0} - 1)}{\vartheta_d}} \\
&\equiv \frac{a^{\lceil c^{j_0} \cdot \vartheta \rceil}}{b^{c^{j_0}}} \cdot \left(\frac{a^{\vartheta_n}}{b^{\vartheta_d}}\right)^{\frac{c^{j_0} \cdot (q'_0 \cdot K_2 - 1)}{\vartheta_d}} \cdot q_m \cdot Q^m + j \cdot P^{m_e} Q^m \bmod a^m\}
\end{aligned}$$

is infinite. For the i in this set we compute

$$\begin{aligned}
\frac{a^{\lceil c^{i \cdot i_0 + j_0} \cdot \vartheta \rceil - m}}{b^{c^{i \cdot i_0 + j_0}}} &= \frac{a^{\lceil c^{j_0} \cdot \vartheta \rceil}}{b^{c^{j_0}}} \cdot \left(\frac{a^{\vartheta_n}}{b^{\vartheta_d}}\right)^{\frac{c^{j_0} \cdot (c^{i \cdot i_0} - 1)}{\vartheta_d}} \cdot \frac{1}{a^m} \\
&\equiv T_m \cdot \frac{1}{a^m} + j \cdot \frac{1}{P^{m - m_e}} \bmod 1
\end{aligned}$$

where the term T_m does not depend on j. For any fixed $m \geq m_e$ and for sufficiently large i the numbers

$$\frac{a^{\lceil c^{i \cdot i_0 + j_0} \cdot \vartheta \rceil - m}}{b^{c^{i \cdot i_0 + j_0}}} \bmod 1$$

are contained in the set M defined in Proposition 6.2 (for $n_k := c^k$) because for sufficiently large i we have

$$\lceil c^{i \cdot i_0 + j_0} \cdot \vartheta \rceil - m \geq \lceil c^{i \cdot i_0 + j_0} \cdot \log_a(b) \rceil - 1 .$$

Hence for any $m \geq m_e$ each closed interval $I \subseteq [0;1]$ of length $\geq \frac{1}{P^{m-m_e}}$ contains at least one element of M. Thus, the closure of M contains $[0;1]$. By Proposition 6.2 (here we use the assumption $c > \vartheta \cdot \log_b(a)$) the number $\sum_{i=0}^{\infty} b^{-c^i}$ is disjunctive to base a and not disjunctive to base b. $\square$

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