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Analysis of the Possibility to Construct Optimal Third-degree Reference Graphs

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Abstract: The subject of this article is to present the results of the analysis of the possibilities to create optimal third-degree Reference Graphs which may be used to model ICT networks. The first part discusses the causes for the interest in the subject of the thesis. Basic definitions and terms related to the analysed type of graphs describing ICT networks are given. The second part consists of the analysis of the possibilities to construct regular third-degree structures bases on the usage of one ring of the parameters typical for Reference Graphs. A method of conduct is presented and then used for checking the potential possibility to construct optimal graphs. In the subsequent chapter the results of analysis of the possibilities to create optimal third-degree graphs with a double ring is discussed. The paper is concluded by observations and conclusions resulting from the research.

Keywords: ICT network, regular graph, chordal ring, double ring structure
Categories: C.2.0, C.2.1

1 Introduction

By observing the changes in the design process of ICT systems, it is clearly visible that their aim is to ensure proper quality, rate and reliability of information transfer [Bertin, 13, Coffman, 02, Cisco, 18]. The most expensive and the least reliable element of each stationary ICT system is its network. Therefore, a crucial course of research is to shorten the connection between users and servers as much as possible based on the above-mentioned reasons.

The aim of the authors of this paper was to check and choose a transmission network topology to connect the modules linking ICT system elements for which the transmission quality indicator (that is the probability of rejecting a service call) reaches the lowest value.

The structure of ICT systems can be described with graphs [Trudeau, 17, Diestel, 01, Deo, 17, Korzan, 78, Kotsis, 92]. Their vertices are ICT nodes and their connecting edges are the vertices — transmission channels. While designing these systems it is important not only to choose the hardware installed in the nodes, but also to select the network connections configuration [Mole, 89, Pedersen, 08]. Normally, telecommunication networks have a ring structure and optical fibres are used as a transmission medium [Xu, 01, Ramaswami, 10, Simmons, 16]. To lower the investment costs associated with the construction and utilisation of networks, a standard of node equipment is introduced, which make it possible to model the systems with regular graphs. It is necessary to study network modelled by these structures. Due to good scalability and good transmission properties chordal rings are used for this. The most basic group of such structures are third degree graphs.

Ardens's and Lee's [Arden, 81] definition of a third-degree chordal ring is as follows:

<u>Definition 1</u>

A third degree chordal ring is a regular graph in which every even node with a number i=2,4,..., N-2 is connected with a node with a number $(i - q)$ mod N which means that every odd number j = 1, 3, 5,..., N-1 is connected with a node with a number $(i + q)$ mod N, where $q \leq N/2$ is the chord length which, as a general rule, is a positive, odd value and N is a number of nodes. □

During the research on the use of regular graphs for the analysis of the above-mentioned systems, it was verified that there are structures whose parameters ensure better transmission properties of a network. The basic parameters influencing the transmission properties of a network are: an average diameter of a graph describing the networks ($D(G)$) and an average path length (d_{av}) [Bujnowski, 03, Morillo, 87, Yebra,85]. To evaluate the connection structures objectively, a notion of referential graphs has been defined. They are called Reference Graphs [Ledzinski, 18, Ledzinski, 14].

<u>Definition 2</u>

Reference Graphs are regular structures of a predetermined number of nodes, where a diameter and an average path length determined based on a random source node have equal, theoretically calculated, lower length limits.

A graph in which for every i and $j < D(G)$, a product of sets of nodes in layers $|n_i| \cap |n_j| = \emptyset$ and in layer $D(G)$ there are all remaining nodes constituting the structure, is called a perfect Reference Graph (RG_i). "Layer" means the set of all nodes distant from the referential node by the same number of edges, which create the minimum length path. A peculiar case of a perfect graph is an optimal Reference Graph marked as RG_o in which in the last layer the number of nodes is equal to the theoretically calculated maximum possible value.

On the basis of the theoretical discussion [Ledzinski, 13] a distribution of the number of nodes in subsequent layers of the third-degree Reference Graph was indicated (Table 1).

d	1	2	3	4	5	6	7	8	9	10
n_{dmax}	3	6	12	24	48	96	192	384	768	1536

Table 1: The distribution of the number of nodes n_{dmax} in subsequent layers of third-degree Reference Graphs where d is the number of a layer

The maximum number of nodes n_{dmax} in the d^{th} layer is defined by:

$$n_{dmax} = 3 \cdot 2^{(d-1)} \tag{1}$$

The numbers of nodes in the optimal graph in the function of its diameter resulting from the values given in Table 1 are presented in Table 2.

***D*(*G*)**	1	2	3	4	5	6	7	8	9	10
N_o	4	10	22	46	94	190	382	766	1534	3070

Table 2: The total number of nodes in third degree optimal Reference Graphs

The number of nodes N_o in the function of the diameter of a graph is described by the formula:

$$N_o = 3 \cdot 2^{D(G)} - 2 \tag{2}$$

The diameter of an optimal Reference Graph in the function of the number of its nodes is described by:

$$D(G)_o = log_2 \frac{N_o + 2}{3} \tag{3}$$

and the average length of the d_{avo} path is defined by:

$$d_{avo} = \frac{(D(G)_o - 1) \cdot 2^{D(G)_o} + 1}{2^{D(G)_o} - 1} \tag{4}$$

The theoretically indicated sizes of diameters and average lengths of paths in the function of the number of nodes in a graph are given in Table 3.

Nodes number	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34
***D*(*G*)**	2			3						4					
d_{av}	1.40	1.57	1.67	1.91	2.08	2.20	2.29	2.37	2.43	2.56	2.68	2.78	2.86	2.93	3.00

Table 3: The values of diameters and average path lengths in third degree optimal Reference Graphs

The aim of the analyses and research was to determine whether it is possible to build the described structure for any number of nodes of a given degree.

2 The analysis of the possibility to create third degree optimal graphs with one ring

Morillo, Comellas and Fiol [Morillo, 87] studied the problems of searching for third degree graphs with a predefined number of nodes and a minimum diameter, and of finding graphs with a maximum number of nodes and a specified diameter. They defined the notion of an optimal graph assuming that all chords connecting graph nodes have the same length.

Definition 3

An optimal graph of a node degree $d(V) = 3$ and a chord length $q < N/2$ built in accordance with Definition 1 is a chordal ring with the following features:

- the number of nodes in the d^{th} layer is defined by the formula:

$$n_d = 3d \tag{5}$$

- the total number of nodes in the graph with a $D(G)$ diameter is:

$$N = 3\frac{D(G)(D(G)+1)}{2} + 1 \tag{6}$$

- an average path length is specified by the formula:

$$d_{av} = \frac{2D(G)+1}{3}. \tag{7}$$

N	4	10	19	31	46	64	85	109
***D*(*G*)**	1	2	3	4	5	6	7	8
d_{av}	1	1.6667	2.3333	3	3.6667	4.3333	5	5.6667

Table 4: The basic parameters of optimal graphs with equal chord lengths

The comparison of data in Tables 2 and 4 shows that optimal graphs created in accordance with Definition 3 with a diameter larger than 2 cannot exist but it is potentially possible for a 10-node graph to be an optimal RG.

By Definition 1, the length of a chord of a graph with 10 nodes must be odd, so q must be 3 or 5. Yebra, Fiol, Morillo, Alegre specified the maximum number of nodes in a graph with a $D(G)$ diameter using a graphical method of graph analysis [Yebra, 85]. The method was used to prove that it is impossible to build an optimal Reference Graph with 10 nodes.

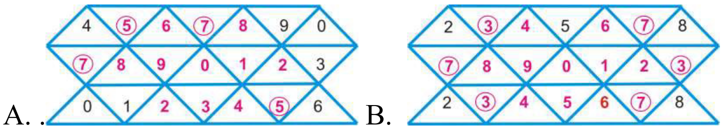


Figure 1: Mosaics exemplifying the adjacency of nodes in graphs

In Figure 1 the red colour marks the nodes with the minimum distance to the referential node (zero), and a circle shows the nodes whose distance is larger than the assumed graph diameter (D(G) = 2). In both cases (A. q = 3, B. q = 5) the structures have a diameter of 3, which means they are not Reference Graphs.

Hence the conclusion: it is impossible to build optimal RGs with equal, odd chord length outside of a full graph with 4 nodes.

Table 5 shows a graph with base parameters close to the values characteristic for optimal graphs. In this case the number of nodes constituting the nodes and the length of a chord are coprime integers.

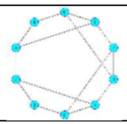
CHR(10;3)	***D(G)***	d_{av}
	3	1.888

Table 5: The parameters of the studied graphs

During the analyses it was concluded that there are merely two structures with six or eight nodes and with the features of perfect Reference Graphs, which is shown in Figure 2. However, Reference Graphs were also found with the same number of nodes which do not fulfil the conditions given in [Yebra, 85] (Figure 3).

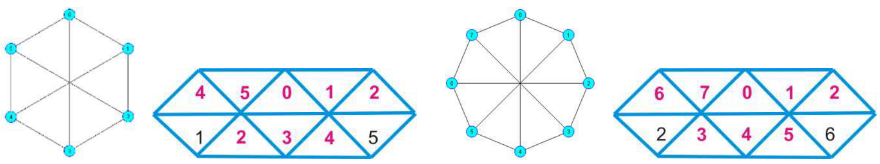


Figure 2: Reference Graphs with equal chord lengths

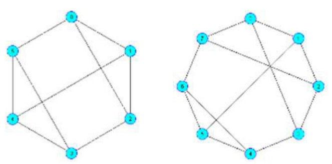


Figure 3: Reference Graphs with different chord lengths

To verify whether optimal graphs with different chord lengths exist, modified graphs CHR3m and CHR3n created in accordance with the suggestions given in [Ledzinski, 14] were tested.

Definition 4

In a modified chordal ring CHR3m every node with a number v_i (i = 0 mod 4) is connected with a node $v_{(i+q1) \bmod N}$,, node v_j (j = 1 mod 4) is connected with a node $v_{(j+q2) \bmod N}$;, node v_k (k = 2 mod 4) is connected with a node $_{(k-q2) \bmod N}$,, and a node v_l (l = 3 mod 4) is connected with a node $v_{(l-q1) \bmod N}$, where N is a number of nodes in the ring which must be divisible by 4, and q_1 and q_2 are chord lengths which must be odd and fulfil the condition $3 \leq q_i \leq N - 3$. p, q_1 i q_2 define the structure (graph) CHR3m (N; q_1,q_2).

Figure 4 contains an example of a graph of this type.

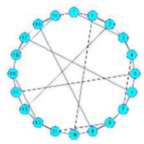


Figure 4: An example of a CHR3m(20; 3,9) graph

Definition 5

In a chordal ring CHR3n every node with a number v_i (i = 0 mod 4) is connected with a node $v_{(i+q1) \bmod N}$, node v_j (j = 2 mod 4) is connected with a node $v_{(j-q1) \bmod N}$,, v_k (k = 1 mod 4) with $v_{(k+q2) \bmod N}$, and v_l (l = 3 mod 4) with $v_{(l-q2) \bmod N}$. N is a number of nodes in the ring which must be divisible by 4, and q_1 and q_2 are chord lengths which met the condition $3 < q_i < N/2$. The graph i defined as CHR3n(N; q_1,q_2).

Figure 5 shows an example of a CHR3n graph.

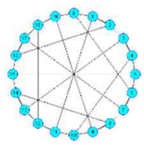


Figure 5: An example of a CHR3n(20; 6,10) graph

The above definitions prove that to build a graph of this type the number of nodes creating such structures must be divisible by 4 and none of the optimal RG graphs with a diameter of $D(G) > 1$ meets this condition.

To confirm that creating a graph with 10 nodes is not possible, an analysis was performed for various lengths of chords which do not fulfil the conditions given in Definitions 4 and 5.

During the analysis of perfect Reference Graphs, it was pointed out that the total length of the chords is affected by certain rules depending on the number of nodes in a graph. It means that it is possible to determine the upper and the lower limit of their total length determining the possibility to create a given structure.

In third degree graphs the number of chords n_{ch} is clearly indicated by:

$$n_{ch} = \frac{N}{2} \tag{8}$$

The upper limit of the above-mentioned total is:

$$\textstyle\sum_{ch\ max} = n_{ch}\frac{N}{2} = n_{ch}^2 \tag{9}$$

which corresponds with the connection of the nodes by chords with lengths corresponding with the diameter of the graph.

The lower limit determines the minimum chord length of 2:

$$\textstyle\sum_{ch\ min} = 2\frac{N}{2} = N \tag{10}$$

therefore if $\sum_{ch} < \sum_{ch\ min}$, it is not possible to build a graph.

There is one more observation: if the number of chords is described as $2k$ (it is even) then their total has to meet this condition, whereas when n_{ch} is given by $2k+1$ then the total is odd.

For example, if a graph creates 6 nodes (Figure 6), the total of chord lengths can be 9 (3 · 3) or 7 (3 + 2 · 2) (it is not possible to build a graph in which the lengths of all chords are 2).

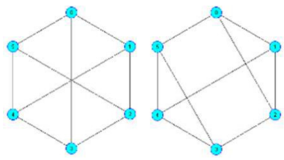


Figure 6: The analysed 6-node graphs

Based on the 6-node graphs a way of checking a potential possibility of constructing optimal graphs was presented.

As mentioned above, a 6-node graph is a structure in which the number of chords is an odd value, therefore their total length is also odd.

In this particular case $\sum_{ch\ max} = 9$, $\sum_{ch\ min} = 6$.

To check the possibility to construct an optimal RG graph it is necessary to choose the parameter values a and b (multipliers corresponding to the chords of a given length) so that the equations are fulfilled:

$$\textstyle\sum_{ch} = 3a + 2b \quad where \sum_{ch} = 2e + 1 \in (\sum_{ch\ min}, \dots, \sum_{ch\ max}), \\ n_{ch} = a + b = 3 \tag{11}$$

Since the resulting total is an odd number, then parameter a also has to be odd.

In this case it was necessary to check 2^4, i.e. 16 cases (the parameters can have values from 0 to 3, and the chords can have two lengths: 2 and 3).

In the first step from a combination of values of the parameters *a* and *b* only the ones which meet the condition of number of chords allowed in a graph are chosen. For these parameter values the totals $\sum_{ch}$ are calculated. Then the combinations whose value $\sum_{ch}$ is lower than the presupposed value $\sum_{ch\ min} + 1$ are eliminated. The last step is to show the combinations whose total is and odd number.

Table 6 presents the stages of searching for graphs which will undergo further tests of optimality.

a	***b***	S_{ch}
0	0	0
0	1	2
0	2	4
0	3	6
1	0	3
1	1	5
1	2	7
1	3	9
2	0	6
2	1	8
2	2	10
2	3	12
3	0	9
3	1	11
3	2	13
3	3	15

Table 6: The illustration of the stages of searching for the parameters of the analysed graphs

In Table 6 yellow is used to mark the parameter combinations that meet the chord number condition; red is used to make the combinations whose total is lower than the presupposed value $\sum_{chmin} + 1$; blue is used to mark the combinations which give an even value $\sum_{ch}$; orange is used to mark the combinations that meet the conditions.

The same procedure was used for 8-node graphs which resulted in the parameter values shown in Table 7; L_{ch} is the chord length.

L_{ch}		
4	**3**	**2**
a	*b*	*c*
0	0	4
0	2	2
0	4	0
1	0	3
1	2	1
2	0	2
2	2	0
3	0	1
4	0	0

Table 7: Parameter values for 8-node graphs

Creating a graphical representation of the constructed graph on the basis of Table 7 is difficult, therefore, to make it simpler it was proposed to use the following procedure, using an example of searching for graphs created with 8 nodes.

It was assumed that all graphs are built on the basis of a ring connecting all the nodes. The value *N* is entered, denoting the number of nodes, which shows that the number of edges of the graph $n_{ch} = 1.5\ N$.

A table of size *N* x *N* is created.

N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**
0	0	1	2	3	4	3	2	1
1	1	0	1	2	3	4	3	2
2	2	1	0	1	2	3	4	3
3	3	2	1	0	1	2	3	4
4	4	3	2	1	0	1	2	3
5	3	4	3	2	1	0	1	2
6	2	3	4	3	2	1	0	1
7	1	2	3	4	3	2	1	0

The values in the cells of the table determine the length of chords connecting individual nodes. The cells marked in red correspond to the edges forming the ring and the connections of the nodes with each other. Due to the table symmetry the cells marked in yellow are going to be analysed.

The number of the cells is determined by the formula:

$$C_n = (N-3)\frac{N}{2} \ or \ C_n = \frac{N!}{(N-2)!\,2!} - N \tag{12}$$

An auxiliary table is created.

2	2					
3	3	2				
4	4	3	2			
5	3	4	3	2		
6	2	3	4	3	2	
7		2	3	4	3	2
N	**0**	**1**	**2**	**3**	**4**	**5**

Based on the data from Table 7, defining the values of the parameters, the edges connecting the nodes are chosen.

For example, when $a = 1$, $b = 2$, $c = 1$, the graph has chords of length 2 and 4 and two chords of length 3.

Let the first chord connect nodes 0 and 3. Because the nodes can be now connected with other nodes, the rows and columns describing the length of the chords and corresponding to these nodes become unavailable (blue colour).

2	2					
3	3	2				
4	4	3	2			
5	3	4	3	2		
6	2	3	4	3	2	
7		2	3	4	3	2
N	**0**	**1**	**2**	**3**	**4**	**5**

The next chord of the length 3 is chosen, connecting two unconnected nodes, e.g. 1 and 6.

2	2					
3	3	2				
4	4	3	2			
5	3	4	3	2		
6	2	3	4	3	2	
7		2	3	4	3	2
N	**0**	**1**	**2**	**3**	**4**	**5**

After the connection is made it becomes visible that there is no cell available corresponding to the chord of length 4, therefore it is not possible to create a graph this way.

Another chord of length 3 is chosen, e.g. the one connecting nodes 1 and 4.

2	2					
3	3	2				
4	4	3	2			
5	3	4	3	2		
6	2	3	4	3	2	
7		2	3	4	3	2
N	**0**	**1**	**2**	**3**	**4**	**5**

In this case it is possible to connect the nodes 2 and 6 with the chord of length 4 and the nodes 5 and 7 with the chord of length 2.

2	2					
3	3	2				
4	4	3	2			
5	3	4	3	2		
6	2	3	4	3	2	
7		2	3	4	3	2
N	**0**	**1**	**2**	**3**	**4**	**5**

Since all the nodes are connected with chords, the graph construction process is finished, and the resulting graph is shown in Figure 7.

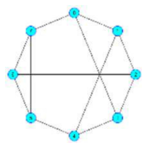


Figure 7: The resulting 8-node graph

In order to check which of the structures given in Table 6 fulfilling the equation (11) make it possible to build perfect Reference Graphs, tests were performed. For that a piece of software utilising an algorithm created by the authors of this paper was used. The software compares the calculated totals of unevenness coefficients of the analysed graph in reference to the theoretically determined values of patterns [Bujnowski, 19].

Definition 6

The unevenness coefficient w_{spi} is the parameter determining the number of uses of a given edge in sets of parallel paths connecting vertices of a graph. The parallel paths have the same length and consist of different configurations of edges connecting the same nodes. Individual edges can be a part of multiple paths, even those that connect the same nodes.

w_{spi} coefficient is described by the formula:

$$w_{spi} = \sum_{i=1}^{D(G)} u_{io} \tag{13}$$

where *D(G)* is the diameter of the graph and u_{io} values calculated the formula:

$$u_{io} = \frac{u_k}{k} \tag{14}$$

where u_k means the number of uses of a particular edge in the sets of parallel paths of count *k* [Bujnowski,19].

Based on the theoretical discussion and the tests it was concluded that the totals of unevenness coefficients for reference graphs are strictly defined and related to the number of nodes creating a graph and with the length of its diameter resulting from this number. Table 8 contains examples of this totals.

Nodes number	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34
$\sum w_{spiN}$	42	88	150	252	378	528	702	900	1122	1416	1742	2100	2490	2912	3366
	$D(G) = 2$			$D(G) = 3$						$D(G) = 4$					

Table 8: Total values of the coefficients of w_{spi} for the third-degree nodes [Bujnowski,19]

By analysing the determined summary total values $\sum w_{spiN}$ contained in the table 8, it was found that they can be calculated theoretically for any Reference Graph using a formula:

$$\sum w_{spiN} = N \cdot (N-1) \cdot d_{av} \tag{15}$$

where d_{av} is an average path length in the graph.

On the basis of the research it was concluded that the totals of the unevenness coefficients specified for the structures other than reference graphs are always larger than the values typical for the latter which is caused by diameters and average path lengths in reference graphs always reaching their minimum values.

Figure 8 shows two examples of 8-node graphs. The first, A, is a reference graph with a diameter $D(G) = 2$ and $d_{av} = 1.57$, whereas in graph B it is $D(G) = 3$ i $d_{av} = 1.71$. The totals of the coefficients are respectively $w_{spi\mathrm{A}} = 88$ and $w_{spi\mathrm{B}} = 96$.

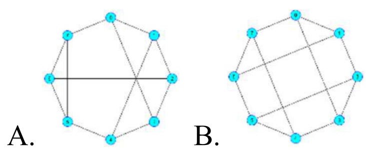


Figure 8: Examples of the analysed graphs

The data presented above validate the use of the above-mentioned parameter for the discussed purpose, which is to search for Reference Graphs.

The procedure discussed before was used in reference to 10-node graphs. In order to check the possibility to build an optimal RG, parameter values *a, b, c* and *d* were chosen corresponding to the chords of length 2, 3, 4 and 5 which fulfil the equation:

$$\sum_{ch} = 5a + 4b + 3c + 2d,\ (11 < \sum_{ch} \leq\ 25),\ a + b + c + d = 5 \qquad (16)$$

Σ_{ch} is an odd number so in order to meet this condition, parameters *a* and *c* must be present in every equation but only one of them can have an odd value. In this case it was necessary to check 6^4 (1296) instances.

Using the algorithm described earlier and after considering the number of chords condition, 56 combinations were left for further analysis, whereby in only 28 of them Σ_{ch} was an odd number.

It is not possible to create a graph				**Values *D*(*G*) and d_{av} depend on the node number**								***D*(*G*) =3, d_{av} = 1.88**				**Graph**
a	*b*	*c*	*d*	*a*	*b*	*c*	*d*	*a*	*b*	*c*	*d*	*a*	*b*	*c*	*d*	
2	0	1	2	0	0	1	4	1	0	2	2	1	0	4	0	A
2	0	3	0	0	0	3	2	1	1	0	3	1	2	2	0	B
3	1	0	1	0	1	1	3	1	1	2	1	3	0	2	0	C
4	0	1	0	0	1	3	1	1	2	2	0	5	0	0	0	D
				0	2	1	2	1	3	0	1					
				0	2	3	0	1	4	0	0					
				0	3	1	1	2	1	1	1					
				0	4	1	0	2	2	1	0					
				0	0	5	0	3	0	0	2					
				1	0	0	4	3	2	0	0					

Table 9: The values of parameters meeting the conditions [Yebra,85]

On the basis of the test results (Table 9) it was concluded that none of the tested 10-node graphs meets the optimality condition but the base parameters of four tested graphs (Figure 9) are the same as in the graph CHR(10;3).

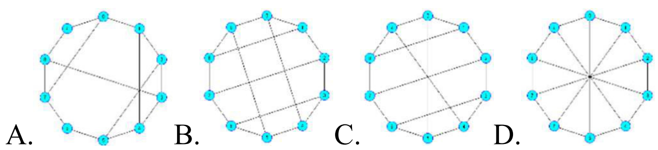


Figure 9: Tested 10-node graphs

The indicated Σw_{spiN} value for all graphs in the figure and for graph CHR(10; 3) is 170, which means none of them is an optimal reference graph.

Conclusion: it is not possible to create an optimal graph with 10 nodes using one ring (a Hamiltonian cycle).

3 The analysis of the possibility to create third degree optimal graphs with two rings

In the search for another possible way to configure optimal Reference Graphs NDR (*Network Double Ring)* structures were studied [Bujnowski,09, Pedersen,04, Zabludowski,12].

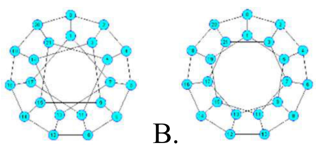
Definition 7

Two rings (each of them with $N/2$ nodes) form an NDR graph described as *NDR(2*N/2; q):*

- An outer ring, in which every o_k node is connected with two adjacent nodes of a ring $o_{k-1(\mathrm{mod}\ N/2)}$ and $o_{k+1(\mathrm{mod}\ N/2)}$;
- An inner ring, in which every $i_{k+N/2}$ node is connected with two adjacent nodes $i_{k+N/2-q\ (\mathrm{mod}\ N)}$ and $i_{k+N/2+q\ (\mathrm{mod}\ N)}$ by a chord of length q which is a multiple of the edge of the outer ring;
- Each node of the inner ring ($i_{k+N/2}$) is connected with its corresponding node of the outer ring (o_k);
- The degree of all the nodes is 3.

NDR structures can be divided into two classes [Simmons,16]:

- First in which all the edges of both rings create Hamiltonian cycles, whereby the number of nodes in the inner ring and the length of the chord used must be coprime integers (Figure 10A).
- Second in which the chord of the outer ring creates a Hamiltonian cycle with a property of $j = i \oplus 1 \bmod N/2$, whereas the inner ring consists of a specified number of separable cycles of equal length lower than $N/2$ (Fig 10B).



A. B.

Figure 10: Examples for two types of graphs: $NDR_A(2*11;3)$ and $NDR_B(2*11;2)$

By analysing the examples in Figure 10 it can be assumed that the distribution of nodes equidistant from any source node can be different depending on the node being located in the outer or the inner ring. Inherently, in Reference Graphs both the size of diameters and average path length calculated from any node must be equal, therefore it possible to consider only the case of graphs meeting this condition.

The possibility to build an optimal 10-node reference graph NDR belonging to the first class of the discussed structures was checked. The graphs can be described by a matrix [M] of size $N/2$,2]. The rows of the matrix define the order of occurrence of the nodes in both rings (Hamiltonian cycles).

It was assumed that the first row defines the sequence of the nodes in the outer ring, and the second — the sequence of nodes in the inner ring.

$$[\mathrm{M}] = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ \frac{N}{2} \oplus 0 \otimes q_i & \frac{N}{2} \oplus 1 \otimes q_i & \frac{N}{2} \oplus 2 \otimes q_i & \frac{N}{2} \oplus 3 \otimes q_i & \frac{N}{2} \oplus 4 \otimes q_i \end{bmatrix} \quad (17)$$

where: $\oplus$ is the adding operation mod ($N/2$), $\otimes$ — the multiplying operation mod ($N/2$).

The element of the lower ring corresponding to the node 1 in the outer ring defines the chord length of the inner ring q_i, which is the generator of the additive group mod ($N/2$).

For a graph consisting of two rings with five nodes, the generators of the groups are $q_o = 1$ and $q_i = 2$.

$$[\mathrm{M}] = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 1 & 3 \end{bmatrix} \quad (18)$$

The matrix transformation was done consisting in switching around the lower and the upper row, which means switching around the outer and the inner ring. The matrix $[\mathrm{M}^{TR}]$ was the result.

$$[\mathrm{M}^{TR}] = \begin{bmatrix} \frac{N}{2} \oplus 0 \otimes q_i & \frac{N}{2} \oplus 1 \otimes q_i & \frac{N}{2} \oplus 2 \otimes q_i & \frac{N}{2} \oplus 3 \otimes q_i & \frac{N}{2} \oplus 4 \otimes q_i \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix} \quad (19)$$

Using the formula $m_l^{\mathrm{TR}} = N/2 \oplus l \otimes q_{\mathrm{i}}$ mod ($N/2$), the values of the elements of the upper row were calculated and put in the ascending order, simultaneously changing the position of the elements of the lower row in accordance with the positions of the upper row.

$$[\mathrm{M}^{TR}] = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ \frac{N}{2} \oplus 0 \otimes q_o & \frac{N}{2} \oplus 1 \otimes q_o & \frac{N}{2} \oplus 2 \otimes q_o & \frac{N}{2} \oplus 3 \otimes q_o & \frac{N}{2} \oplus 4 \otimes q_o \end{bmatrix} \quad (20)$$

The lower row element corresponding to the $N/2 \oplus l \otimes q_o = 1$ (mod $N/2$) element in the upper row defines the chord length of the transformed outer ring — $q_o = l$.

In the example:

$$[\mathrm{M}^{TR}] = \begin{bmatrix} 0 & 2 & 4 & 1 & 3 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix} \rightarrow [\mathrm{M}^{TR}] = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 3 & 1 & 4 & 2 \end{bmatrix} \quad (21)$$

therefore $q_o = 3$.

The multiplication of the values gave the following result:

$$q_i \otimes q_o = 1 \quad (22)$$

By using the determined chord lengths to build an NDR structure, an automorphic Petersen graph [Graham, 04] (Figure 11) was created, which is an optimal Reference

Graph with 10 nodes.

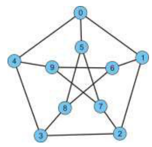


Figure 11: Petersen graph

Conclusion: for an NDR structure to be optimal, it has to be automorphic, which follows from the definition that the values of its basic parameters calculated from any node have to be equal.

By using the foregoing observations, it was checked if an optimal graph with 22 nodes actually exists.

qi_i chord lengths forming the mod (11) group generators were specified: 2, 3, 4, 5. Their respective q_o values are: 9, 8, 7, 6.

None of the calculated products$q_i \otimes q_o$ is equal to 1 (7, 2, 6, 8 mod (11)), therefore none of the NDR structures built this way is automorphic. This means that depending on the localisation of the node in the rings, average path lengths are going to be different from each other and it eliminates the possibility to build an optimal graph.

The validity of this conclusion is also proved by the below calculation results.

NDR(2*11; 2) – $D(G) = 5$; $d_{avi} = 2.71$ and $d_{avo} = 2.81$, $d_{av} = 2.62$;
NDR(2*11; 3) – $D(G) = 4$; $d_{avi} = 2.52$ and $d_{avo} = 2.62$, $d_{av} = 2.57$;
NDR(2*11; 4) – $D(G) = 4$; $d_{avi} = 2.62$ and $d_{avo} = 2.52$, $d_{av} = 2.57$;
NDR(2*11; 5) – $D(G) = 5$; $d_{avi} = 2.81$ and $d_{avo} = 2.71$, $d_{av} = 2.62$;

The resulting average path length d_{av} typical for a given graph was calculated from the formula:

$$d_{av} = \frac{d_{avi} + d_{avo}}{2} \tag{23}$$

where d_{avi} i d_{avo} are average path lengths calculated for the inner and the outer ring respectively.

Tests performed for 46-node graphs are also consisted with the above conclusion, and the parameter values closest to the parameters of a reference graph have isomorphic structures NDR(46;5) and NDR(46;9) – $D(G) = 5$; $d_{avi} = 3.400$ whereas $d_{avo} = 3.444$, $d_{av} = 3.422$.

By using the observation of the NDR structures automorphism, it was concluded that reaching the equality (23) is necessary to build perfect graphs, which is exemplified by the structures NDR(2*7; 2), NDR(2*7; 3) and NDR(2*13; 5) (Figure 12). Their parameters are respectively: $D(G) = 3$; $d_{av} = 2.08$ and $D(G) = 4$ $d_{av} = 2.68$.

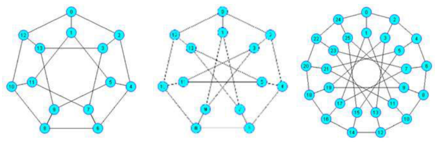
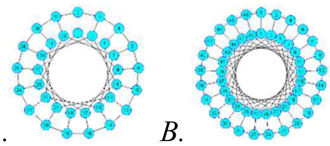


Figure 12: Automorphic perfect graphs

It is not, however, a sufficient condition, which is exemplified by the structures NDR(2*17; 5), NDR(2*25; 7) (Figure 13).



A. *B.*

Figure 13: Automorphic graphs which are not perfect

In the examples in Figure 13 the values of the diameter and the average path length calculated from any node are equal but these are not values typical for reference graphs: structure A — $D(G) = 5$; $d_{av} = 3.12$, B — $D(G) = 5$; $d_{av} = 3.49$.

While searching for the possibility to build optimal NDR graphs, the structure proposed in the paper [ ] was analysed.

<u>Definition 8</u>

NdRc (Figure 14) is a structure described as NdRc(N; q_1,q_2), formed by two rings:

- An external ring consisting of $N/2$ nodes, in which each node o_k is connected with two adjacent nodes $o_{k-1 (\mathrm{mod}\ N/2)}$ i $o_{k+1 (\mathrm{mod}\ N/2)}$;
- An internal ring also consisting of $N/2$ nodes. Each even number node $i_{2k+N/2}$ is connected with two adjacent nodes $i_{2k+N/2-q1\ (\mathrm{mod}\ N)}$ and $i_{2k+N/2+q1\ (\mathrm{mod}\ N)}$, and each odd number node $i_{2k+1+N/2}$ is connected with two nodes $i_{2k+1+N/2+q2\ (\mathrm{mod}\ N)}$ and $i_{2k+1+N/2-q2\ (\mathrm{mod}\ N)}$;
- Each node of the internal ring $i_{k+N/2}$ is connected with its corresponding node o_k of the external ring;
- Parameters q_1 and q_2 denoting chord lengths must be of even number;
- All nodes are of the third degree.

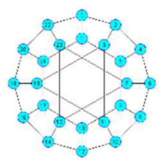


*Figure 14: An example of the structure NdRc(2*12; 2,4)*

By Definition 8 the number of nodes in individual rings must be an even number and optimal graphs do not meet this condition.

Final conclusion: except for the Petersen graph there are no other optimal NDR Reference Graphs with a number of nodes larger than 10.

4 Summary and conclusions

The paper discussed the results of the analysis of the possibility to build the network which structure corresponds to the optimal third-degree reference graphs. Theoretical discussion was presented as well as the results of the tests proving the validity of the theoretical analysis. The properties of the graphs were verified by comparing the calculated totals of the unevenness coefficients typical for a given graph in reference to the value of a theoretically indicated pattern. A method was proposed facilitating the construction of graph image based on the designation of the values of the parameters fulfilling the defined equations. On the basis of the presented results the conclusion was specified that it is not possible to build optimal Reference Graphs (for more than 4 nodes) by using one ring, whereas using a double ring structure (NDR) makes it possible to create a graph with 10 nodes only (a Petersen graph).

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