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The Least Σ-jump Inversion Theorem for n-families

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Abstract: Studying the Σ-reducibility of families introduced by [Kalimullin and Puzarenko 2009] we show that for every set $X \geqslant_T \emptyset'$ there is a family of sets $\mathcal{F}$ which is the Σ-least countable family whose Σ-jump is Σ-equivalent to $X \oplus \overline{X}$. This fact will be generalized for the class of n-families (families of families of . . . of sets).
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1 Introduction

In this paper study a reducibility among families of sets introduced in [Kalimullin and Puzarenko 2009]. We will say that a family $\mathcal{F}_0 \subseteq 2^\omega$ is Σ-reducible to a family $\mathcal{F}_1 \subseteq 2^\omega$ if for every admissible set $\mathbb{A}$

$$\mathcal{F}_1 \text{ is } \Sigma\text{-definable in } \mathbb{A} \implies \mathcal{F}_0 \text{ is } \Sigma\text{-definable in } \mathbb{A}.$$

A family $\mathcal{F} \subseteq 2^\omega$ is Σ-definable in $\mathbb{A}$ if there is a Σ-formula Φ such that

$$\mathcal{F} = \{\{x \in \omega : \Phi(x, y)\} : y \in Y\},$$

for some Σ-definable subset $Y \subseteq \mathbb{A}$. This reducibility can be reformulated in terms of enumeration operators:

Theorem 1. *[Kalimullin and Puzarenko 2009] For families $\mathcal{F}_0$ and $\mathcal{F}_1$, the following conditions are equivalent:*

1. $\mathcal{F}_0 \leqslant_\Sigma \mathcal{F}_1$;
2. $\mathcal{F}_0 \cup \{\emptyset\} = \{\Theta(C \oplus B \oplus E(\mathcal{F}_1)) : C \in K_{\mathcal{F}_1}\}$ *for some enumeration operator* Θ *and some set* $B \in K_{\mathcal{F}_1}$*, where* $E(\mathcal{F}) = \{u : \exists X \in \mathcal{F}\,[D_u \subseteq X]\}$*, and* $K_{\mathcal{F}_1}$ *is the class of sets of the form* $\{\langle n, m\rangle\} \oplus A_1 \oplus \ldots \oplus A_m$*,* $A_i \in \mathcal{F}_1$.

On the other hand, $\leqslant_\Sigma$ is a natural extension of the enumeration and Turing reducibilities, since $A \leqslant_e B \iff \{A\} \leqslant_\Sigma \{B\}$.

Let us highlight that Σ-reducibility among families is equivalent to the Σ-definability relation between special structures $\mathfrak{M}_\mathcal{F}$ [Kalimullin and Puzarenko 2009]. See the end of this section for the detailed definition of $\mathfrak{M}_\mathcal{F}$. Following [Montalbán 2009], [Puzarenko 2009], [Stukachev 2009] we can view the Σ-jumps of families as the jumps of the corresponding structures.

Definition 2. For a structure $\mathfrak{M}$, define the jump of $\mathfrak{M}$ to be the structure $\mathcal{J}(\mathfrak{M}) = (\mathbb{HF}(\mathfrak{M}), U_\Sigma)$, where U_Σ is a ternary Σ-predicate on $\mathbb{HF}(\mathfrak{M})$ universal for the class of all binary Σ-predicates on $\mathbb{HF}(\mathfrak{M})$, is called a *$\Sigma$-jump*.

For any n-family $\mathcal{F}$ instead of $\mathcal{J}(\mathfrak{M}_\mathcal{F})$ we simply write $\mathcal{J}(\mathcal{F})$. The Σ-jump does not depend on the choice of a universal Σ-predicate, up to Σ-equivalence. Furthermore, this Σ-jump on structures having Turing (enumeration) degrees acts in the same way as a Turing (enumeration) jump (see [Puzarenko 2009]). As in the classical case, the Σ-jump operation satisfies the following:

1. $\mathfrak{A} \leqslant_\Sigma \mathcal{J}(\mathfrak{A})$;
2. $\mathfrak{A} \leqslant_\Sigma \mathfrak{B} \Rightarrow \mathcal{J}(\mathfrak{A}) \leqslant_\Sigma \mathcal{J}(\mathfrak{B})$.

We define $\mathcal{J}^n(\mathfrak{A})$ by induction on $n \in \omega$ as follows: $\mathcal{J}^0(\mathfrak{A}) = \mathfrak{A}$, $\mathcal{J}^{n+1}(\mathfrak{A}) = \mathcal{J}(\mathcal{J}^n(\mathfrak{A}))$. It was shown in [Puzarenko 2009] that for any structures $\mathfrak{M}$ and $\mathfrak{A}$ on a finite signature, $\mathfrak{M}$ is Σ_{m+1}-definable in $\mathfrak{A}$ iff $\mathfrak{M} \leqslant_\Sigma \mathcal{J}^m(\mathfrak{A})$.

In [Kalimullin and Puzarenko 2009] some unexpected properties of the family InfCE of all infinite c.e. sets under Σ-reducibility were found. In particular, for a family $\mathcal{F}$, $\mathcal{F} \leqslant_\Sigma$ InfCE iff the following conditions hold:

1. all sets in $\mathcal{F}$ are c.e.;
2. the index set $\{e : W_e \in \mathcal{F}\}$ is Σ^0_3;
3. there exists a computable cover $\widehat{\mathcal{F}}$ of $\mathcal{F}$, i.e., a computable family $\widehat{\mathcal{F}} \subseteq \mathcal{F}$ such that for any $W \in \mathcal{F}$, we have $W \subseteq V$ for some $V \in \widehat{\mathcal{F}}$.

In this paper, we show that the family InfCE has yet another natural property: InfCE is the Σ-least family among all countable families, whose Σ-jump computes $\emptyset''$, i.e., it is the least jump inversion of the Turing degree of $\emptyset''$. Moreover, each set $A \geqslant_T \emptyset'$ has such jump inversion. We show also, that each family $\mathcal{F} \geqslant_\Sigma \overline{\emptyset'}$ has the Σ-least jump inversion in the extended class of n-families.

The notation and terminology follows from Rogers [Rogers 1967] and [Ershov 1996]. We now formally introduce the generalized notion of n-families and fix the precise way of their coding into the structures.

Definition 3. A 0-family is a subset of ω. For an integer $n > 0$, an n-family is a countable set of $(n-1)$-families.

We consider the empty set as an 0-family.

According to [Kalimullin and Faizrahmanov 2016] the definition of computably enumerable n-families is inductive: an n-family $\mathcal{F}$ is computably enumerable if it's elements, $(n-1)$-families, are uniformly computably enumerable. We give this definition generalized to an arbitrary admissible set (see [Ershov 1996]):

Definition 4. 1. A Σ_s-formula $\Phi(\overline{z}, y)$, $s \in \omega$, *defines* a 0-family $X \subseteq \mathrm{Nat}(\mathbb{A})$ in an admissible set $\mathbb{A}$ if there is a tuple $\overline{c} \in \mathbb{A}^k$ such that

$$X = \{m \in \mathrm{Nat}(\mathbb{A}) : \mathbb{A} \models \Phi(\overline{c}, m)\}.$$

In this case, we will write $X = \mathcal{F}^{0,\mathbb{A}}_{\Phi(\overline{c})}$.

2. A Σ_s-formula $\Phi(\overline{z}, x, y)$ *defines* an 1-family $\mathcal{F}$, if there are a nonempty Σ_s-subset $E \subseteq \mathbb{A}$ and a tuple $\overline{c} \in \mathbb{A}^k$ such that

$$\mathcal{F} = \{\mathcal{F}^{0,\mathbb{A}}_{\Phi(\overline{c},x)} : x \in E\}.$$

In this case, we will write $\mathcal{F} = \mathcal{F}^{1,\mathbb{A}}_{\Phi(\overline{c}),E}$.

3. A Σ_s-formula $\Phi(\overline{z}, x_1, \ldots, x_{n+2}, y)$ *defines* an $(n+2)$-family $\mathcal{F}$, if there are a nonempty Σ_s-subset $E \subseteq \mathbb{A}^{n+2}$ and a tuple $\overline{c} \in \mathbb{A}^k$ such that

$$\mathcal{F} = \{\mathcal{F}^{n+1,\mathbb{A}}_{\Phi(\overline{c},x),E^{(x)}} : x \in \mathrm{Pr}_1(E)\},$$

where

$$Pr_1(E) = \{x : \exists y_1 \ldots \exists y_{n+1}\, (x, y_1, \ldots, y_{n+1}) \in E\},$$

$$E^{(x)} = \{(y_1, \ldots, y_{n+1}) : (x, y_1, \ldots, y_{n+1}) \in E\}.$$

In this case, we will write $\mathcal{F} = \mathcal{F}^{n+2,\mathbb{A}}_{\Phi(\overline{c}),E}$.

An n-family $\mathcal{F}$ is Σ_s-*definable* (Σ-*definable* for the case $s = 1$) in $\mathbb{A}$ if some Σ_s-formula defines $\mathcal{F}$ in $\mathbb{A}$.

This definition extends the definition given in [Kalimullin and Puzarenko 2009].

We will see below that for the n-families it is enough to consider only special cases of admissible sets, namely, the hereditary finite structures $\mathbb{HF}(\mathfrak{M})$, where $\mathfrak{M}$ is some algebraic structure. Let M be the domain of $\mathfrak{M}$ and let σ be the language of $\mathfrak{M}$. The domain of $\mathbb{HF}(\mathfrak{M})$ is the class of $HF(M)$ of hereditarily finite sets over the set M is defined by induction as follows:

– $H_0(M) = \{\emptyset\}$;

– $H_{n+1}(M) = H_n(M) \cup \mathcal{P}_\omega(H_n(M) \cup M)$;

– $HF(M) = \bigcup_{n<\omega} H_n(M) \cup M$

(where $P_\omega(X)$ denotes the set of all finite subsets of X).

The hereditarily finite superstructure over $\mathfrak{M}$ is the algebraic structure $\mathbb{HF}(\mathfrak{M})$ in the signature $\sigma \cup \{U^{(1)}, \in^{(2)}, \emptyset\}$, where $U^{\mathbb{HF}(\mathfrak{M})} = M$, $\in^{\mathbb{HF}(\mathfrak{M})} \subseteq (HF(M)) \times (HF(M) \setminus M)$ is the membership relation on $\mathbb{HF}(\mathfrak{M})$, the constant symbol $\emptyset$ is interpreted as the empty set, and symbols in the signature σ are interpreted in the same way as on $\mathfrak{M}$.

Following [Kalimullin and Puzarenko 2009], we can code every n-family $\mathcal{F}$ into the admissible superstructure $\mathbb{HF}(\mathfrak{M}_\mathcal{F})$ over the special structure $\mathfrak{M}_\mathcal{F}$ defined by induction as follows.

– For an arbitrary 0-family A let $\mathfrak{M}_A$ be the structure in the signature $\sigma = \{r, I^1, R^2\}$ with the domain $M_\mathcal{F} = \omega \cup X$, $X = \{x_n : n \in A\}$, such that $R^{\mathfrak{M}_A} = \{\langle n, n+1\rangle : n \in \omega\} \cup \{\langle x_n, n\rangle : n \in A\}$, $r^{\mathfrak{M}_A} = 0$ and $I^{\mathfrak{M}_A} = \{r^{\mathfrak{M}_A}\}$.

– For an n-family $\mathcal{F} = \{\mathcal{S}_i : i \in \omega\}$, $n > 0$, let $\mathfrak{M}_A$ be the structure in the signature $\sigma = \{r, I^1, R^2\}$ with the domain $\bigcup_{k,i} |\mathfrak{M}^k_{\mathcal{S}_i}| \cup \{r^{\mathfrak{M}_\mathcal{F}}\}$ (each $\mathfrak{M}^k_{\mathcal{S}_i}$ is an isomorphic copy of $\mathfrak{M}_{\mathcal{S}_i}$ with a new domain) such that $I^{\mathfrak{M}_\mathcal{F}} = \bigcup_{k,i} I^{\mathfrak{M}^k_{\mathcal{S}_i}}$ and
$$R(x, y) \Leftrightarrow x = (\exists k, i)\,[x = r^{\mathfrak{M}_\mathcal{F}} \,\&\, y = r^{\mathfrak{M}^k_i} \ \vee\ R^{\mathfrak{M}^k_i}(x, y)]$$
for each $x, y \in |\mathfrak{M}_\mathcal{F}|$.

  Through this inductive definition, the elements of $I^{\mathfrak{M}_\mathcal{F}}$ are precisely the elements originally denoted as $r^{\mathfrak{M}_A}$ for 0-families $A \in \cdots \in \mathcal{F}$. For $i \in I^{\mathfrak{M}_\mathcal{F}}$ we denote the corresponding 0-family by A_i.

It is easy to check that every n-family $\mathcal{F}$ is Σ-definable in $\mathbb{HF}(\mathfrak{M}_\mathcal{F})$. For example, if $n = 0$ then a 0-family $A \subseteq \omega$ is defined by the formula saying that there is a sequence
$$r^{\mathfrak{M}_\mathcal{F}} = n_0, n_1, n_2, \dots, n_{x+1}, p$$
such that $R(n_i, n_{i+1})$ for all $i \leqslant x$, and $R(p, n_{x+1})$, $p \neq n_x$. Moreover, it follows from [Kalimullin and Puzarenko 2009] that the Σ-definability of $\mathcal{F}$ is equivalent to the Σ-definability of $\mathfrak{M}_\mathcal{F}$ itself.

Proposition 5. *[Kalimullin and Puzarenko 2009] An n-family $\mathcal{F}$ is Σ-definable in a countable admissible set $\mathbb{A}$ iff the structure $\mathfrak{M}_{\mathcal{F}}$ (and, therefore, $\mathbb{HF}(\mathfrak{M}_{\mathcal{F}})$) is Σ-definable in $\mathbb{A}$.*

Under Σ-interpretation of a structure $\mathfrak{M}$ in a signature σ we understand a Σ-definable structure $\mathfrak{N}$ in the language $\sigma \cup \{\sim\}$, where $\sim$ is a new congruence relation on $\mathfrak{N}$ such that $\mathfrak{N}/\sim \cong \mathfrak{M}$.

Definition 6. Let $\mathcal{F}$ be an n-family and $\mathfrak{M}$ be a structure. We say that $\mathcal{F}$ is *Σ-reducible* to $\mathfrak{M}$ (written $\mathcal{F} \leqslant_{\Sigma} \mathfrak{M}$) if $\mathfrak{M}_{\mathcal{F}}$ is Σ-definable in $\mathbb{HF}(\mathfrak{M})$. Similarly, $\mathfrak{M} \leqslant_{\Sigma} \mathcal{F}$ if $\mathfrak{M}$ is Σ-definable in $\mathbb{HF}(\mathfrak{M}_{\mathcal{F}})$. If $\mathcal{F}$ and $\mathcal{S}$ are n- and m-families correspondingly we say that $\mathcal{F}$ is *Σ-reducible* to $\mathcal{S}$ if $\mathcal{F} \leqslant_{\Sigma} \mathfrak{M}_{\mathcal{S}}$. As usual, the relation $\equiv_{\Sigma}$ holds in the case of Σ-reductions from the left to the right and from the right to the left.

Note that for an n-family $\mathcal{F}$ and the $(n+1)$-family $\{\mathcal{F}\}$ we have $\{\mathcal{F}\} \equiv_{\Sigma} \mathcal{F}$. By this reason we can view an n-family $\mathcal{F}$ as an m-family for $m > n$.

Recall that for the case $n = 0$ the standard notation is

$$Y \oplus A = \{2x : x \in Y\} \cup \{2x+1 : x \in A\}.$$

If Y is an arbitrary set and $\mathcal{F}$ is an n-family, $n > 0$, then we define the *join* of Y and $\mathcal{F}$ inductively by letting

$$Y \oplus \mathcal{F} = \{Y \oplus \mathcal{S} : \mathcal{S} \in \mathcal{F}\}.$$

For an n-family $\mathcal{F}$ and an integer k, denote by $\mathcal{F}^k$ the n-family $\{k\} \oplus \mathcal{F}$. Clearly, for every integer k and an n-family $\mathcal{F}$, we have $\mathcal{F} \equiv_{\Sigma} \mathcal{F}^k$. For n-families $\mathcal{F}, \mathcal{G}$ define the n-family

$$\mathcal{F} \oplus \mathcal{G} = \mathcal{F}^0 \cup \mathcal{G}^1.$$

It is easy to see that $\mathcal{F} \leq_{\Sigma} \mathcal{F} \oplus \mathcal{G}$, $\mathcal{G} \leq_{\Sigma} \mathcal{F} \oplus \mathcal{G}$, and

$$\mathcal{F} \leq_{\Sigma} \mathfrak{M}, \mathcal{G} \leq_{\Sigma} \mathfrak{M} \implies \mathcal{F} \oplus \mathcal{G} \leq_{\Sigma} \mathfrak{M}$$

for every structure $\mathfrak{M}$.

2 Jump and jump inversion on n-families

Example 1. ([Puzarenko 2009]). For a 0-family A the jump $\mathcal{J}(A)$ is Σ-equivalent to $\mathfrak{M}_{J(A)}$, where $J(A)$ is the the enumeration jump of A:

$$J(A) = K(A) \oplus \overline{K(A)} \text{ and } K(A) = \{n : n \in \Phi_n(A)\},$$

where $\{\Phi_n\}_{n \in \omega}$ is an effective enumeration of the enumeration operators.

Example 2. It is easy to check that for the family InfCE of all infinite c.e. sets we have $\mathfrak{J}(\text{InfCE}) \equiv_\Sigma J(J(\emptyset)) \equiv_e \overline{\emptyset''}$. Indeed, $\overline{\emptyset''}$ is computably isomorphic to $\{n : W_n \text{ is infinite}\}$, and a c.e. set W_n is infinite if and only if the (uniformly) computable set

$$V_n = \{s : W_{n,s} \neq W_{n,s+1}\}$$

is infinite, and so, if and only if $F \subseteq V_n$ for some $F \in \text{InfCE}$. The predicate $F \subseteq V_n$ can be recognized by $J(F)$.

The inverse reduction $\mathfrak{J}(\text{InfCE}) \leq_\Sigma J(J(\emptyset))$ is obvious.

Therefore, the family InfCE is a jump inversion of $J(J(\emptyset))$, i.e., $\mathfrak{J}(\text{InfCE}) \equiv_\Sigma J(J(\emptyset))$.

Proposition 7. *The* 1*-family* InfCE *is the the least jump inversion for the* 0*-family* $J(J(\emptyset))$ *among countable structures, i.e.,* $J(J(\emptyset)) \leq_\Sigma \mathfrak{J}(\mathfrak{M})$ *implies* $\text{InfCE} \leq_\Sigma \mathfrak{M}$.

Proof. Suppose $J(J(\emptyset)) \leq_\Sigma \mathfrak{J}(\mathfrak{M})$ for some countable $\mathfrak{M}$. Then the index set $\{n : W_n \text{ is infinite}\}$ is Σ_2-definable in $\mathbb{HF}(\mathfrak{M})$. Then there is Δ_0-formula Φ such that

$$W_n \text{ is infinite} \iff \mathbb{HF}(\mathfrak{M}) \models \exists a \forall b\, \Phi(n, a, b).$$

Then the sequence

$$V_{n,a} = \begin{cases} W_n, & \text{if } \mathbb{HF}(\mathfrak{M}) \models \forall b\, \Phi(n, a, b); \\ \omega, & \text{otherwise,} \end{cases}$$

exhausting all infinite c.e. sets can be determined by the Σ-predicate

$$x \in V_{n,a} \iff x \in W_n \vee x \in \omega \;\&\; \exists b\, \neg\Phi(n, a, b).$$

This allows us to prove the reducibility $\text{InfCE} \leq_\Sigma \mathfrak{M}$ for every countable $\mathfrak{M}$ such that $J(J(\emptyset)) \leq_\Sigma \mathfrak{J}(\mathfrak{M})$.

Now, our goal is to extend Proposition 7 for arbitrary n-family $\mathcal{F}$. For each n-family $\mathcal{F}$, recursively define an $(n+1)$-family $\mathcal{E}(\mathcal{F})$:

$$\mathcal{E}(\mathcal{F}) = \begin{cases} \mathcal{H}_1 \cup \{\{2x\} : x \in A\}, & \text{if } n = 0 \text{ and } \mathcal{F} = A \subseteq \omega, \\ \mathcal{H}_{n+1} \cup \{\mathcal{E}(\mathcal{S}) : \mathcal{S} \in \mathcal{F}^0\}, & \text{if } n > 0, \end{cases}$$

where $\mathcal{H}_1 = \{\{2x, 2x+1\} : x \in \omega\}$ and $\mathcal{H}_{n+1} = \{\mathcal{H}_n\}$. This is similar to some definitions that appear in [Kalimullin and Puzarenko 2009] and [Faizrahmanov and Kalimullin 2016 (a), (b)].

According to the following theorem we will call $\mathcal{E}(\mathcal{F})$ as the *least Σ-jump inversion for* $\mathcal{F}$ (meaning that in fact it is an inversion of $J(\emptyset) \oplus \mathcal{F}$).

Theorem 8. *For any n-family $\mathcal{F}$ the $(n+1)$-family $\mathcal{E}(\mathcal{F})$ is the least jump inversion of $\mathcal{F}$. Namely,*

1) $\mathcal{F} \leqslant_\Sigma \mathcal{J}(\mathcal{E}(\mathcal{F}))$;

2) for each countable structure $\mathfrak{B}$ of a finite signature, $\mathcal{E}(\mathcal{F}) \leqslant_\Sigma \mathfrak{B}$ if $\mathcal{F} \leqslant_\Sigma \mathcal{J}(\mathfrak{B})$.

3) $\mathcal{J}(\mathcal{E}(\mathcal{F})) \leqslant_\Sigma J(\emptyset) \oplus \mathcal{F}$.

Proof. 1) Since we can view each n-family as an m-family for $m > n$, without loss of generality we assume that $n > 0$. Let $\mathbb{A} = \mathbb{HF}(\mathfrak{M}_{\mathcal{E}(\mathcal{F})})$.

It is easy to see that there is a Σ_2-formula Φ such that

$$\mathbb{A} \models \Phi(x_1, \ldots, x_n, m) \iff \exists i\, [R^{\mathbb{A}}(x_n, i) \,\&\, I^{\mathbb{A}}(i) \,\&\, A_i = \{2m\}] \iff$$

$$\exists t\, [R^{\mathbb{A}}(x_n, i) \,\&\, I^{\mathbb{A}}(i) \,\&\, 2m \in A_i \,\&\, 2m+1 \notin A_i],$$

where each A_i, for $i \in I^{\mathfrak{M}_{\mathcal{E}(\mathcal{F})}}$, is from the definition of $\mathfrak{M}_{\mathcal{E}(\mathcal{F})}$. Then for the Σ-subset

$$E = \{(x_1, \ldots, x_n) : R^{\mathfrak{A}}(r^{\mathfrak{A}}, x_1) \,\&\, R^{\mathfrak{A}}(x_i, x_{i+1}) \text{ for } 1 \leqslant i < n\}$$

of $\mathbb{A}^n$ we will have

$$\mathcal{F} = \{\mathcal{F}^{n,\mathbb{A}}_{\Phi(x), E^{(x)}} : x \in \mathrm{Pr}_1(E)\}.$$

Hence $\mathcal{F} \leqslant_{\Sigma_2} \mathcal{E}(\mathcal{F})$ so that $\mathcal{F} \leqslant_\Sigma \mathcal{J}(\mathcal{E}(\mathcal{F}))$.

2) Let an n-family $\mathcal{F}$ is Σ-reducible to $\mathcal{J}(\mathfrak{B})$ for some structure $\mathfrak{B}$. Hence $\mathcal{F}^0 \leqslant_\Sigma \mathcal{J}(\mathfrak{B})$. Fix a Σ_2-subset $E \subseteq HF(\mathfrak{B})$, Σ_2-formula Θ and a tuple $\overline{c} \in HF^m(\mathfrak{B})$ such that

$$\mathcal{F}^0 = \{\mathcal{F}^{n,\mathbb{HF}(\mathfrak{B})}_{\Theta(\overline{c},x), E^{(x)}} : x \in \mathrm{Pr}_1(E)\}.$$

Let Ψ be a Δ_0-formula such that the Σ_2-formula $\exists a \forall b\, \Psi(a, b, \overline{c}, x_1 \ldots, x_n, k)$ defines the Σ_2-predicate

$$\{(x_1, \ldots, x_n) \in E^n : \Theta(\overline{c}, x_1, \ldots, x_n, k)\}$$

in $\mathbb{HF}(\mathfrak{B})$. Then there is a Σ-formula Φ such that for every $x_1, \ldots, x_n, a \in HF(\mathfrak{B})$ and $k \in \omega$ we have $\mathbb{HF}(\mathfrak{B}) \models \Phi(x_1, \ldots, x_n, \langle a, k\rangle, 2k)$ and

$$\mathbb{HF}(\mathfrak{B}) \models \Phi(x_1, \ldots, x_n, \langle a, k\rangle, 2k+1) \iff \mathbb{HF}(\mathfrak{B}) \models \exists b\, \neg\Psi(a, b, \overline{c}, x_1, \ldots, x_n, k).$$

It is easy to see that for every $x_1, \ldots, x_n, a \in HF(\mathfrak{B})$ and $k \in \omega$ we have

$$\mathcal{F}^{0,\mathbb{HF}(\mathfrak{B})}_{\Phi(x_1,\ldots,x_n,\langle a,k\rangle)} = \{2k\} \iff \mathbb{HF}(\mathfrak{B}) \models \forall b\, \Psi(a, b, \overline{c}, x_1, \ldots, x_n).$$

Thus, $\mathcal{E}(\mathcal{F}) = \{\mathcal{F}^{n+1,\mathbb{HF}(\mathfrak{B})}_{\Phi(x),C^{(x)}} : x \in \mathrm{Pr}_1(C)\} \cup \mathcal{H}_{n+1}$ for the Σ-set

$$C = \{(x_1, \dots, x_n, \langle a, k\rangle) \in \mathbb{HF}(\mathfrak{B}) : x_1, \dots, x_n, a \in HF(\mathfrak{B}), k \in \omega\}.$$

Therefore $\mathcal{E}(\mathcal{F}) \leqslant_\Sigma \mathfrak{B}$.

3) By Theorem 1 from [Stukachev 2009] there is a countable structure $\mathfrak{B}$ such that $J(\emptyset) \oplus \mathcal{F} \equiv_\Sigma \mathcal{J}(\mathfrak{B})$. Since $\mathcal{F} \leqslant_\Sigma \mathcal{J}(\mathfrak{B})$ we have $\mathcal{E}(\mathcal{F}) \leqslant_\Sigma \mathfrak{B}$. Therefore, $\mathcal{J}(\mathcal{E}(\mathcal{F})) \leqslant_\Sigma \mathcal{J}(\mathfrak{B}) \leqslant_\Sigma J(\emptyset) \oplus \mathcal{F}$. This ends the proof.

Corollary 9. *For every pair of n-families $\mathcal{F}$ and $\mathcal{G}$*

1. $\mathcal{F} \leq_\Sigma \mathcal{G} \implies \mathcal{E}(\mathcal{F}) \leq_\Sigma \mathcal{E}(\mathcal{G})$*;*

2. $\mathcal{E}(\mathcal{F} \oplus \mathcal{G}) \equiv_\Sigma \mathcal{E}(\mathcal{F}) \oplus \mathcal{E}(\mathcal{G})$*.*

Proof. Part 1 follows from the fact that $\mathcal{F} \leq_\Sigma \mathcal{G} \leq_\Sigma \mathcal{J}(\mathcal{E}(\mathcal{G}))$. Part 2 follows from the fact that $\mathcal{E}(A \oplus B) = \mathcal{H}_1 \cup \{\{2x : x \in A \oplus B\}\} = \mathcal{H}_1 \cup \{\{4x\} : x \in A\} \cup \{\{4x+2\} : x \in B\}\} \equiv_\Sigma \{X \oplus Y : X \in \mathcal{E}(A) \;\&\; Y \in \mathcal{E}(B)\} = \mathcal{E}(A) \oplus \mathcal{E}(B)$.

By the definition of $\mathcal{E}(\cdot)$ the least double jump inversion $\mathcal{E}^2(\mathcal{F}) = \mathcal{E}(\mathcal{E}(\mathcal{F}))$ of an n-family $\mathcal{F}$ is an $(n+2)$-family. But we know from [Faizrahmanov and Kalimullin 2016 (a)] that under Turing reducibility of presentations of n-families the least double jump is an $(n+1)$-family. For example, for the case of 0-family A the least double jump $\mathcal{E}^2(A)$ has the same Turing degrees of presentations of $\mathfrak{M}_{\mathcal{E}^2(A)}$ as the degrees of presentations of $\mathfrak{M}_\mathcal{G}$, where $\mathcal{G}$ is the 1-family

$$\mathcal{G} = \{F \subseteq \omega : F \text{ is finite}\} \cup \{\overline{\{x\}} : x \in A\}.$$

Below we show that for the case of Σ-reducibility we can not have an equivalence between $\mathcal{E}^2(\mathcal{F})$ and some $(n+1)$-family even for $n = 0$.

Theorem 10. *For a set A and a 1-family $\mathcal{G}$ we have*

$$\mathcal{J}(\mathcal{G}) \leq_\Sigma J(\emptyset) \oplus \mathcal{E}(A) \implies \mathcal{J}(\mathcal{G}) \leq_\Sigma J(\emptyset)$$

and, therefore, $\mathcal{J}(\mathcal{G}) \not\equiv_\Sigma J(\emptyset) \oplus \mathcal{E}(A)$. Thus, no 1-family can be a double jump inversion of A.

Proof. (Sketch) Let us look at the jump of $\mathcal{J}(\mathcal{G}) = \mathcal{J}(\mathfrak{M}_\mathcal{G})$ for 1-families $\mathcal{G}$. Because of [Kalimullin and Puzarenko 2009], all Σ-predicates in $\mathfrak{M}_\mathcal{G}$ can be encoded in the sets

$$A_1 \oplus A_2 \oplus \cdots \oplus A_m \oplus E(\mathcal{G}),$$

where $A_i \in \mathcal{G}$ and the set $E(\mathcal{G}) = \{u : (\exists A \in \mathcal{G})\,[D_u \subseteq A]\}$ codes the $\exists$-theory of $\mathfrak{M}_\mathcal{G}$. But the family of enumeration jumps of these sets cannot fully represent the jump of the whole $\mathcal{G}$ since we need to keep the information when a jump for a tuple $A_1, \ldots, A_m$ is an extension of the jump for a tuple $A_1, \ldots, A_m, A_{m+1}$. In fact, the jump $\mathcal{J}(\mathcal{G})$ (up to Σ-equivalence) can be viewed as a structure coding the jumps of the sets $A \in E(\mathcal{G}) \oplus \mathcal{G}$ extended by the similar coding of the jumps of elements of the $\oplus$-closure of $E(\mathcal{G}) \oplus \mathcal{G}$ with an additional binary operation which maps coding places of $J(X)$, $J(Y)$ to the coding places of $J(X \oplus Y)$. Each coding instance should be generated by this binary operation from the instances coding jumps of the elements of $E(\mathcal{G}) \oplus \mathcal{G}$. The last instances should be marked by a special predicate. We omit technical details and a technical verification. Informally, such structure allows to compute all Σ-types in $\mathfrak{M}_{\mathcal{G}}$, and, therefore, to build an isomorphic copy of the original $\mathcal{J}(\mathcal{G})$.

Suppose that

$$\mathcal{J}(\mathcal{G}) \leq_\Sigma J(\emptyset) \oplus \mathcal{E}(A) = \{J(\emptyset) \oplus \{2n, 2n+1\} : n \in \omega\} \cup \{J(\emptyset) \oplus \{2n\} : n \in A\}$$

as witnessed by some Σ-formula Φ. For simplicity we assume that Φ has no parameters.

Note that the structure $\mathfrak{M}_{J(\emptyset)\oplus\mathcal{E}(A)}$ is bi-embeddable with $\mathfrak{M}_{J(\emptyset)\oplus\mathcal{H}_1} \leq_\Sigma J(\emptyset)$, where

$$J(\emptyset) \oplus \mathcal{H}_1 = \{J(\emptyset) \oplus \{2n, 2n+1\} : n \in \omega\}.$$

Moreover, they are *densely* bi-embeddable in the sense that for every finite substructure $\mathfrak{M}_0 \subseteq \mathfrak{M}_{J(\emptyset)\oplus\mathcal{E}(A)}$ there is a substructure $\mathfrak{M}_0 \subseteq \mathfrak{M}_1 \subseteq \mathfrak{M}_{J(\emptyset)\oplus\mathcal{E}(A)}$ such that $\mathfrak{M}_1 \cong \mathfrak{M}_{J(\emptyset)\oplus\mathcal{H}_1}$, and vice versa. Considering the same formula Φ in $\mathbb{HF}(\mathfrak{M}_{J(\emptyset)\oplus\mathcal{H}_1})$ we get a structure $\mathfrak{L} \leqslant_\Sigma J(\emptyset)$ densely bi-embeddable with $\mathcal{J}(\mathcal{G})$. But $J(X) \subseteq J(Y)$ implies $J(X) = J(Y)$ so that this is possible only if $\mathcal{J}(\mathcal{G}) \cong \mathfrak{L}$. Hence, $\mathcal{J}(\mathcal{G}) \leqslant_\Sigma J(\emptyset)$.

In the case when Φ has parameters we should change $\mathcal{H}_1$ by a 1-family in the form

$$\mathcal{H}_1 \cup \{n_1\} \cup \{n_2\} \cup \cdots \cup \{n_k\}$$

for appropriate choice of $n_1, \ldots, n_k \in A$ (depending on the given parameters of Φ) preserving the dense bi-embeddability property up to finitely many constants.

To prove the second part of the theorem suppose that $\mathcal{J}(\mathcal{G}) \equiv_\Sigma J(\emptyset) \oplus \mathcal{E}(A)$. Then by the first part $\mathcal{J}(\mathcal{G}) \leq_\Sigma \mathcal{J}(\emptyset)$. On the other hand, by Theorem 8

$$A \leq_\Sigma \mathcal{J}(\mathcal{E}(A)) \leq_\Sigma \mathcal{J}^2(\mathcal{G}) \leq_\Sigma \mathcal{J}^2(\emptyset) \equiv_\Sigma J^2(\emptyset),$$

so that $A \in \Sigma^0_3$.

Since $\mathcal{J}(\mathcal{E}^2(A)) \equiv_\Sigma J(\emptyset) \oplus \mathcal{E}(A)$, by Theorem 8 we have also the following corollary:

Corollary 11. *For a set a set $A \notin \Sigma^0_3$ there is no 1-family $\mathcal{G}$ such that $\mathcal{G} \equiv_\Sigma \mathcal{E}^2(A)$, so that the least double jump inversion of a 0-family A can not be replaced by a 1-family.*

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References

[Ershov 1996] Ershov, Yu. L.; "Definability and Computability"; Sib. School Alg. Log. [in Russian], Nauch. Kniga, Novosibirsk (1996).

[Faizrahmanov and Kalimullin 2016 (a)] Faizrahmanov, M., Kalimullin, I.: "The Enumeration Spectrum Hierarchy of n-Families"; Math. Log. Q., 62, 4-5 (2016) 420-426.

[Faizrahmanov and Kalimullin 2016 (b)] Faizrahmanov, M. Kh., Kalimullin, I S.; "The Enumeration Spectrum Hierarchy and Low_α Degrees"; J. Univ. Comp. Sc., 22, 7 (2016) 943-955.

[Kalimullin and Faizrahmanov 2016] Kalimullin, I. Sh., Faizrakhmanov M. Kh.: "A Hierarchy of Classes of Families and n-Low Degrees"; Algebra i Logika [in Russian], 54, 4 (2015) 536-541.

[Kalimullin and Puzarenko 2009] Kalimullin, I. S., Puzarenko, V. G.; "Reducibility on families"; Algebra Log. [in Russian], 48, 1 (2009), 31-53.

[Montalbán 2009] Montalbán, A.; "Notes on the jump of a structure". In Mathematical Theory and Computational Practice (eds K. Ambos-Spies, B. Lwe & W. Merckle), pp. 372-378. Lecture Notes in Computer Science, vol. 5635. Berlin, Germany: Springer.

[Puzarenko 2009] Puzarenko, V. G.; "A certain reducibility on admissible sets"; Sib. Mat. Zh. [in Russian], 50, 2 (2009), 415-429.

[Rogers 1967] Rogers, Hartley Jr.;"The Theory of Recursive Functions and Effective Computability"; McGraw-Hill (1967).

[Stukachev 2009] Stukachev, A. I.; "A jump inversion theorem for the semilattices of Σ-degrees"; Sib. Élektron. Mat. Izv., 6 (2009) 182-190.