Journal of Universal Computer Science, vol. 22, no. 7 (2016), 943-955
submitted: 23/3/15, accepted: 30/6/16, appeared: 1/7/16 © *J.UCS*

The Enumeration Spectrum Hierarchy of α-families and $\mathbf{Low}_\alpha$ Degrees

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Abstract: In this paper we introduce a hierarchy of families which can be derived from the integers using countable collections. This hierarchy coincides with the von Neumann hierarchy of hereditary countable sets in the ZFC-theory with urelements from $\mathbb{N}$. The families from the hierarchy can be coded into countable algebraic structures preserving their algorithmic properties. We prove that there is no maximal level of the hierarchy and that the collection of non-low$_\alpha$ degrees for every computable ordinal α is the enumeration spectrum of a family from the hierarchy. In particular, we show that the collection of non-low$_\alpha$ degrees for every computable limit ordinal α is a degree spectrum of some algebraic structure.

Key Words: countable family, class of families, enumeration of family, degree spectra of structure, low$_\alpha$ degree
Category: F.1.1, F.1.2, F.4.1

1 Introduction

Under the degree spectrum of a countable algebraic structure we understand the class of all Turing degrees that compute its isomorphic copy. It was shown in [Goncharov et al. 2005] that for every constructive ordinal α the classes of $\overline{\mathbf{Low}}_{\alpha+1}$ of non-low$_{\alpha+1}$ degrees ($\mathbf{x}^{(\alpha+1)} \not\leq \mathbf{0}^{(\alpha+1)}$) are the degree spectra of algebraic structures. In 2012 during a discussion Kalimullin, Montalban and Frolov found an idea allowing to construct algebraic structures with the degree spectrum $\overline{\mathbf{Low}}_\omega$ non-low$_\omega$ degrees (unpublished). In this paper we expose a uniform approach in obtaining the degree spectrum $\overline{\mathbf{Low}}_\alpha$ for limit constructive ordinals α. To do this, we consider a hierarchy of α-families which extends the notion of a family of sets of integers. It was shown in [Kalimullin and Faizrahmanov 2015], [Kalimullin and Faizrahmanov 2016] that the classes $\overline{\mathbf{Low}}_n$ are examples of spectra for which the hierarchy of n-families is proper. In this paper we will generalize this hierarchy to infinite constructive ordinals.

Definition 1. Under 0-*family* we understand a subset of $\mathbb{N}$. For an ordinal $\alpha > 0$, an α-*family* is a countable set of β-families for arbitrary $\beta < \alpha$.

In fact, the α-families are the hereditary countable sets of rank $\leqslant \alpha$ in the von Neumann hierarchy for the ZFC-theory with urelements from $\mathbb{N}$.

Whenever it is stated that a certain function or relation with ordinal arguments α, β, ... satisfies some computability property, it is presumed that these ordinals are given by ordinal notations a, b, ..., and that the corresponding function or relation for a, b, ... satisfies the property in question. In particular, the relation $\alpha < \beta$ will mean that we have $a <_O b$ (or $a <_{O^X} b$) for the corresponding ordinal notations.

We say that an integer e is an X-*enumeration index* of a 0-family $A \subseteq \mathbb{N}$ if $A = W_e^X$. Let α be an X-computable ordinal given by its $\mathcal{O}^X$-notation. An X-*enumeration index* of an α-family $\mathcal{F}$ is such an integer e that

$$\mathcal{F} = \{\mathcal{H}_\beta^i : \beta < \alpha,\ i \in \mathbb{N} \text{ and } \Phi_e^X(\beta, i)\downarrow\},$$

where $\mathcal{H}_\beta^i$ is the β-family with the X-enumeration index $\Phi_e^X(\beta, i)$ (with respect to the notation $\beta <_{O^X} \alpha$).

Definition 2. We say that an α-family $\mathcal{F}$ is X-c.e. if there is an X-enumeration index of $\mathcal{F}$ with respect to some $\mathcal{O}^X$-notation of α.

In particular, for 1-families $\mathcal{F}$ this is equivalent to the existence of a partial computable function φ such that

$$\mathcal{F} = \{W_{\varphi(i)}^X : i \in \mathbb{N} \text{ and } \varphi(i)\downarrow\}.$$

For non-empty 1-families $\mathcal{F}$ the function φ can be chosen total.

The *enumeration spectrum* $\mathbf{Sp}\,\mathcal{F}$ of α-family $\mathcal{F}$ is the class of degrees $\mathbf{x} = \deg(X)$ such that $\mathcal{F}$ is X-c.e. Extending known methods of coding countable families into algebraic structures (see, e.g., [Goncharov et al. 2005]), each α-family can be coded into a structure having an X-computable copy if and only if the α-family is X-c.e.

Theorem 3. *For every α-family $\mathcal{F}$ there is an algebraic structure $\mathcal{A}$ such that $\mathbf{Sp}\,\mathcal{F}$ is the degree spectrum of $\mathcal{A}$.*

Proof. To code a 0-family A let $P(A)$ be the graph starting with a central vertex v and adding a loop from v to itself of length $n+3$ for each $n \in A$. Mark also the central vertex by the label 0. Assume by induction that every β-family $\mathcal{H}$, $\beta < \alpha$, is coded into a graph $P(\mathcal{H})$. To code an arbitrary α-family $\mathcal{F}$ let $P(\mathcal{F})$ be the graph with a root vertex r and infinitely many copies of the graph $P(\mathcal{H})$ for each $\mathcal{H} \in \mathcal{F}$ whose root (or central) vertex is connected by an edge with the new root r. Mark the root vertex by the label α. Let (G, E) be the graph consisting of infinitely many disjoint copies of the graph $P(\mathcal{F})$.

To code a notation of α consider a structure $(L, <_L, P, S)$ where $(L, <_L)$ is well-ordered set of type α, P is the unary predicate indicating limit elements, and S is the successor relation on L. Fix the isomorphism h from $(L, <_L)$ onto $(\alpha, \in)$.

Let $\mathcal{A}(\mathcal{F})$ be the two-sort model

$$\mathcal{A}(\mathcal{F}) = [(L, <_L, P, S), (G, E), J],$$

where $J(a, b)$ holds if and only if $a \in L$ and the element $b \in G$ has the label $h(a)$.

If an X-enumeration index of $\mathcal{F}$ is given with respect to some $\mathcal{O}^X$-notation on α then we can get an X-computable copy of $\mathcal{A}(\mathcal{F})$, where the X-constructivisation of $(L, <_L, P, S)$ is based on the standard X-constructivisation of the well-order $\{\beta : \beta <_{\mathcal{O}^X} \alpha\}$.

Contrariwise, for given X-computable copy of $\mathcal{A}(\mathcal{F})$ we can get an an X-enumeration index of $\mathcal{F}$ with respect to the $\mathcal{O}^X$-notation based on the constructivisation of $(L, <_L, P, S) \cong \alpha$.

2 The families $\mathcal{E}_\alpha(A)$

The goal of this section is to uniformly define an $(\alpha+1)$-family $\mathcal{E}_\alpha(A)$ such that

$$\mathcal{E}_\alpha(A) \text{ is c.e. iff } A \text{ is } \emptyset^{(\alpha+1)}\text{-c.e.}$$

for every set $A \subseteq \mathbb{N}$ and every constructive ordinal α.

It is known from [Kalimullin and Faizrahmanov 2015] that such $(\alpha + 1)$-family $\mathcal{E}_\alpha(A)$ exists if $\alpha < \omega$. Namely, the families $\mathcal{E}_\alpha(A)$, $\alpha < \omega$ can be defined by the following induction:

$$\mathcal{E}_0(A) = \{\{2x, 2x+1\} : x \in \mathbb{N}\} \cup \{\{2x\} : x \in A\},$$

$$\mathcal{E}_\alpha(A) = \{\mathcal{E}_\beta(Z) : Z \in \mathcal{E}_0(A)\}, \text{ if } \alpha = \beta + 1.$$

It is easy to check that if A is X'-c.e. then $\mathcal{E}_0(A)$ is an X-c.e. 1-family. Note also that

$$x \in A \iff (\exists F \in \mathcal{E}_0(A))[2x \in F \,\&\, 2x+1 \notin F],$$

so that

$$A \text{ is } X'\text{-c.e.} \iff \text{the 1-family } \mathcal{E}_0(A) \text{ is } X\text{-c.e.}$$

Due the uniformity in the last equivalence the condition

$$A \text{ is } X^{(\alpha+1)}\text{-c.e.} \iff \text{the } (\alpha+1)\text{-family } \mathcal{E}_\alpha(A) \text{ is } X\text{-c.e.}$$

for every $A \subseteq \mathbb{N}$ implies the condition

$$A \text{ is } X^{(\alpha+2)}\text{-c.e.} \iff \text{the } (\alpha+2)\text{-family } \mathcal{E}_{\alpha+1}(A) \text{ is } X\text{-c.e.}$$

for every $A \subseteq \mathbb{N}$. To guarantee this condition for every constructive ordinal α we should correctly make an inductive definition of $\mathcal{E}_\alpha(A)$ for limit α.

Let α be a limit constructive ordinal and let **FIN** be the class of all finite subsets of $\mathbb{N}$. For each $Z \in \mathbf{FIN}$ define the class $\mathcal{D}_\alpha(Z)$ containing all functions $f : \alpha \to \mathbf{FIN}$ such that for some finite increasing sequence

$$0 = \beta_0 < \beta_1 < \ldots < \beta_n < \alpha,$$

the function f is constant in every interval $[\beta_k, \beta_{k+1})$, $k < n$, and $f(\beta) = Z$ for $\beta \in [\beta_n, \alpha)$. Let $\mathcal{D}_\alpha = \bigcup_{Z \in \mathbf{FIN}} D_\alpha(Z)$. Then we set

$$\mathcal{E}_\alpha(A) = \{\{\mathcal{E}_\beta(f(\beta)) \mid \beta < \alpha\} : f \in \mathcal{D}_\alpha(Z) \ \&\ Z \in \mathcal{E}_0(A)\}.$$

Lemma 4. *There is computable function r such that for every computable limit ordinal α and finite set $W_e^{\emptyset^{(\alpha)}}$ we have $D_\alpha(W_e^{\emptyset^{(\alpha)}}) = \{f_y : y \in \omega\}$, where*

$$f_y(\beta) = W_{r(e,y,\beta)}^{\emptyset^{(\beta)}}, \beta < \alpha.$$

More precisely, the function r indeed is defined on the notations of $\beta < \alpha$ so that $D_\alpha(W_e^{\emptyset^{(\alpha)}}) = \{f_y : y \in \omega\}$ and $f_y(|b|_O) = W_{r(e,y,b)}^{\emptyset^{(b)}}, b <_O a$, where a is the $\mathcal{O}$-notation of α such that $\emptyset^{(\alpha)} = \emptyset^{(a)}$.

Proof. Let y be the canonical number of a sequence $(\beta_0, \ldots, \beta_q, F_1, \ldots, F_q)$, where

$$0 = \beta_0 < \beta_1 < \ldots < \beta_q < \alpha$$

are ordinals given by their notations $<_O a$, and $F_1, \ldots, F_q$ are finite sets.

Using the notation for $a \in \mathcal{O}$ we can uniformly extend the sequence $\{\beta_k\}_{k \leqslant q}$ to an infinite increasing sequence $\{\beta_k\}_{k \in \mathbb{N}}$ such that $\alpha = \lim_k \beta_k$.

Assuming that $\emptyset^{(\alpha)} = \{\langle\gamma, x\rangle : \gamma < \alpha \ \&\ x \in \emptyset^{(\gamma)}\}$ we can define $K^\beta = \{\langle\gamma, x\rangle : \gamma < \beta \ \&\ x \in \emptyset^{(\gamma)}\} \leq_T \emptyset^{(\beta)}$ and choose a computable function r such that

$$W_{r(e,y,\beta)}^{\emptyset^{(\beta)}} = \begin{cases} F_s, & \text{if } \beta_{s-1} \leqslant \beta < \beta_s, \quad 1 \leqslant s \leqslant q, \\ \widehat{W}_{e,s}^{K^\beta}, & \text{if } \beta_{s-1} \leqslant \beta < \beta_s \text{ for } s > q \end{cases}$$

for $\beta < \alpha$, where $\widehat{W}_{e,s}^{K^\beta}$ is the part of $W_{e,s}^{K^\beta}$ computed using the oracle $K^\beta \restriction u$ such that $\langle\gamma, x\rangle \geq u$ for all $\gamma \in [\beta, \alpha)$. Note that the c.e. relation $\gamma \in [\beta, \alpha)$ on ordinal notations is computable in $\emptyset' \leq_T \emptyset^{(\beta)}$ for $\beta > 0$.

If $W_e^{\emptyset^{(\alpha)}}$ is finite then $W_e^{\emptyset^{(\alpha)}} = \widehat{W}_e^{K^\beta}$ beginning some $\beta < \alpha$ so that the function f_y:

$$f_y(\beta) = W_{r(e,y,\beta)}^{\emptyset^{(\beta)}}, \beta < \alpha,$$

belongs to $D_\alpha(W_e^{\emptyset^{(\alpha)}})$ since $\widehat{W}_e^{K^{\beta_s}} = W_e^{\emptyset^{(\alpha)}}$ beginning some β_s.

Conversely, suppose that a function $f : \alpha \to \mathbf{FIN}$ belongs to $D_\alpha(W_e^{\emptyset^{(\alpha)}})$. Since $W_e^{\emptyset^{(\alpha)}}$ is finite we can fix an integer $p \in \mathbb{N}$ and an ordinal $\rho < \alpha$ such that $W_e^{\emptyset^{(\alpha)}} = \widehat{W}_{e,s}^{K^\gamma}$ for every $s \geqslant p$ and $\gamma \in [\rho, \alpha)$. Let

$$0 = \beta_0 < \beta_1 < \ldots < \beta_q < \alpha$$

be a sequence of ordinals such that f is constant on every interval $[\beta_s, \beta_{s+1})$, $s < q$, and $f(\beta) = W_e^{\emptyset^{(\alpha)}}$ for every $\beta \in [\beta_q, \alpha)$. Extend the sequence by one more element $\beta_{q+1} = \max(\rho, \beta_q) + 1$. If $q + 1 < p$ we also can extend it further via $\beta_{q+i} = \max(\rho, \beta_q) + i$, $1 < i \leqslant p - q$, so that in any case we have the sequence

$$0 = \beta_0 < \beta_1 < \ldots < \beta_{\max(p,q+1)} < \alpha$$

such that $\beta_{\max(p,q+1)} \in [\rho, \alpha)$. Then $f = f_y$ for the canonical index y of the sequence $(\beta_0, \ldots, \beta_{\max(p,q+1)}, F_1, \ldots, F_{\max(p,q+1)})$, where $F_s = f(\beta_s)$, $1 \leqslant i \leqslant \max(p, q+1)$.

Lemma 5. *There is a partial computable function g such that for every computable ordinal α and every integer e the value $g(e, \alpha)$ is the enumeration index of the $(\alpha+1)$-family $\mathcal{E}_\alpha(W_e^{\emptyset^{(\alpha+1)}})$.*

Proof. Fix a computable function f such that for every m and X

$$\mathcal{E}_0(W_m^{X'}) = \{W_{f(i,m)}^X : i \in \mathbb{N}\}.$$

For example, we can set

$$W_{f(i,m)}^X = \begin{cases} \{2x, 2x+1\}, & \text{if } i = \langle 0, x\rangle; \\ \{2x\}, & \text{if } i = \langle s+1, x\rangle \text{ and } x \in W_{m,t}^{X'_t} \\ & \text{for all } t > s; \\ \{2x, 2x+1\}, & \text{otherwise,} \end{cases}$$

where $\{X'_t\}_{t\in\omega}$ is an X-computable enumeration of X'.

For $\alpha = 0$ the value $g(e, 0)$ is the index of the computable function

$$\Phi_{g(e,0)}(\beta, i) = \begin{cases} f(i, e), & \text{if } \beta = 0, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

It is easy to see that $g(e, 0)$ is the enumeration index of the 1-family $\mathcal{E}_0(W_e^{\emptyset'})$.

Let $\alpha > 0$. Assume by induction that $g(e, \alpha)$ is defined such that $g(e, \alpha)$ is the enumeration index of the $(\alpha+1)$-family $\mathcal{E}_\alpha(W_e^{\emptyset^{(\alpha+1)}})$ for every e.

Since

$$\mathcal{E}_{\alpha+1}(W_e^{\emptyset^{(\alpha+2)}}) = \{\mathcal{E}_\alpha(W_{f(i,e)}^{\emptyset^{(\alpha+1)}}) : i \in \mathbb{N}\}$$

the index $g(e, \alpha+1)$ defined by

$$\Phi_{g(e,\alpha+1)}(\beta, i) = \begin{cases} g(f(i,e),\alpha), & \text{if } \beta = \alpha+1, \\ \text{undefined}, & \text{otherwise} \end{cases}$$

is the enumeration index of $\mathcal{E}_{\alpha+1}(W_e^{\emptyset^{(\alpha+2)}})$.

Suppose now that α is limit ordinal and the function $g(e, \beta)$ is defined for $\beta < \alpha$ such that $g(e, \beta)$ is the enumeration index of the $(\beta+1)$-family $\mathcal{E}_\beta(W_e^{\emptyset^{(\beta+1)}})$ for every e and $\beta < \alpha$.

Since $\emptyset^{(\beta)} \leqslant_T \emptyset^{(\beta+1)}$ by Lemma 4 there is a partial computable function r such that $D_\alpha(W_{f(i,e)}^{\emptyset^{(\alpha)}}) = \{f_y : y \in \mathbb{N}\}$, where

$$f_y(\beta) = W_{r(e,y,\beta)}^{\emptyset^{(\beta+1)}},\ \beta < \alpha,$$

so that we have

$$\begin{aligned} \mathcal{E}_\alpha(W_e^{\emptyset^{(\alpha+1)}}) = \{\{\mathcal{E}_\beta(f(\beta)) \mid \beta < \alpha\} : f \in \mathcal{D}_\alpha(Z) \ \&\ Z \in \mathcal{E}_0(W_e^{\emptyset^{(\alpha+1)}})\} = \\ \{\{\mathcal{E}_\beta(f(\beta)) \mid \beta < \alpha\} : f \in \mathcal{D}_\alpha(W_{f(i,e)}^{\emptyset^{(\alpha)}}) \ \&\ i \in \mathbb{N}\} = \\ \{\{\mathcal{E}_\beta\left(W_{r(f(i,e),y,\beta)}^{\emptyset^{(\beta+1)}}\right) \mid \beta < \alpha\} : i, y \in \mathbb{N}\}. \end{aligned}$$

Define a partial computable function v such that for every y, i and e the value $v(y, i, e)$ is the enumeration index of the α-family

$$\{\mathcal{E}_\beta\left(W_{r(f(i,e),y,\beta)}^{\emptyset^{(\beta+1)}}\right) \mid \beta < \alpha\}$$

by letting

$$\Phi_{v(y,i,e)}(\gamma, j) = \begin{cases} g(r(f(i,e),y,\beta),\beta), & \text{if } \gamma = \beta+1, \\ \text{undefined}, & \text{if } \gamma = 0 \text{ or } \gamma \text{ is a limit ordinal.} \end{cases}$$

Then the value of $g(e, \alpha)$ defined by

$$\Phi_{g(e,\alpha)}(\beta, \langle y, i\rangle) = \begin{cases} v(y,i,e), & \text{if } \beta = \alpha, \\ \text{undefined}, & \text{otherwise} \end{cases}$$

is the enumeration index of $\mathcal{E}_\alpha(W_e^{\emptyset^{(\alpha+1)}})$.

Lemma 6. *There is a partial computable function h such that whenever e is an enumeration index of $\mathcal{E}_\alpha(A)$, $A = W_{h(e,\alpha)}^{\emptyset^{(\alpha+1)}}$.*

Proof. Define the value of $h(e, 0)$ by letting

$$W_{h(e,0)}^{\emptyset'} = \{x : \exists i \exists t \forall y \forall s\, [W_{\Phi_{e,t}(0,i),t} \neq \emptyset \,\&\, [y \in W_{\Phi_{e,s}(0,i),s} \Rightarrow y = 2x]]\}.$$

It's easy to see that $W^{\emptyset'}_{h(e,0)} = \{x : \exists i\, [W_{\Phi_e(0,i)} = \{2x\}]\}$. Thus if e is an enumeration index of $\mathcal{E}_0(A)$ then $A = W^{\emptyset'}_{h(e,0)}$.

Suppose $\alpha = \beta + 1$. Let

$$W^{\emptyset^{(\alpha+1)}}_{h(e,\alpha)} = \{x : \exists i\, [W^{\emptyset^{(\alpha)}}_{h(\Phi_e(\beta,i),\beta)} = \{2x\}]\}.$$

Therefore if e is an enumeration index of $\mathcal{E}_\alpha(A)$ then $A = W^{\emptyset^{(\alpha+1)}}_{h(e,\alpha)}$.

Suppose now that α is limit ordinal. Let U be the partial computable function defined by

$$U(e, \alpha, \beta, j, i) = \Phi_{\Phi_e(\alpha,j)}(\beta, i).$$

Define the value of $h(e, \alpha)$ by letting

$$W^{\emptyset^{(\alpha+1)}}_{h(e,\alpha)} = \{x : \exists i \exists \gamma < \alpha \forall \beta \in (\gamma, \alpha) \forall i\, [W^{\emptyset^{(\beta+1)}}_{h(U(e,\alpha,\beta,j,i),\beta)} = \{2x\}]\}.$$

Let e be an enumeration index of $\mathcal{E}_\alpha(A)$. Every α-family

$$\{\mathcal{E}_\beta(f(\beta)) \mid \beta < \alpha\},$$

where $f \in \bigcup_{Z \in \mathcal{E}_0(A)} \mathcal{D}_\alpha(Z)$, has the enumeration index $\Phi_e(\alpha, j)$ for some j. Then every $(\beta + 1)$-family in this α-family has the enumeration index $U(e, \alpha, \beta, j, i)$ for some i. Therefore $x \in A$ iff

$$\exists i \exists \gamma < \alpha \forall \beta \in (\gamma, \alpha) \forall i\, [W^{\emptyset^{(\beta+1)}}_{h(U(e,\alpha,\beta,j,i),\beta)} = \{2x\}].$$

Hence $A = W^{\emptyset^{(\alpha+1)}}_{h(e,\alpha)}$.

Corollary 7. *Let $\alpha < \omega_1^{CK}$. Then $A \in \Sigma^0_1(X^{(\alpha+1)})$ iff $\mathcal{E}_\alpha(A)$ X-c.e.*

3 Main Result

It was shown in [Kalimullin and Faizrahmanov 2016] that for every integer n the class $\overline{\mathbf{Low}}_n$ is the enumeration spectrum of m-family for some integer m. Namely, the following theorem holds.

Theorem 8. *For each integer $n > 0$ the classes $\overline{\mathbf{Low}}_{2n-1}$ and $\overline{\mathbf{Low}}_{2n-2}$ are the enumeration spectra of n-families.*

Now we will generalize this to infinite (and, therefore, to limit) computable ordinals.

Theorem 9. *For every computable ordinal α there is an $(\alpha+1)$-family with the enumeration spectrum $\overline{\mathbf{Low}}_\alpha$.*

Proof. Define the 1-family $\mathcal{W}$ by letting

$$\mathcal{W} = \{\{n\} \oplus F : F \in \mathbf{FIN},\ F \neq W_n^{\emptyset^{(\alpha)}}\}.$$

By Wehner's Theorem (see [Wehner 1998]) relativised to $\emptyset^{(\alpha)}$ the family $\mathcal{W}$ is X-c.e. iff $\emptyset^{(\alpha)} <_T X$ for every $X \geq_T \emptyset^{(\alpha)}$. Suppose that $\alpha = \beta + 1$. Define the $(\alpha + 1)$-family

$$\mathcal{F} = \{\mathcal{E}_\beta(\{n\} \oplus F) : F \in \mathbf{FIN}, F \neq W_n^{\emptyset^{(\alpha)}}\}.$$

Now we show that $\overline{\mathbf{Low}}_\alpha$ is the enumeration spectrum of $\mathcal{F}$. Let $\emptyset^{(\alpha)} <_T X^{(\alpha)}$. Hence $\mathcal{W}$ is $X^{(\alpha)}$-c.e. By the Lemma 5 relativised to X we have that $\mathcal{F}$ is X-c.e. Conversely, suppose that $\mathcal{F}$ is X-c.e. Using the Lemma 6 relativised to X we obtain that $\mathcal{W}$ is $X^{(\alpha)}$-c.e. Therefore $\emptyset^{(\alpha)} <_T X^{(\alpha)}$.

Suppose now that α is limit ordinal. We show that $\overline{\mathbf{Low}}_\alpha$ is the enumeration spectrum of the $(\alpha + 1)$-family

$$\mathcal{F} = \{\{\mathcal{E}_\beta(\{n\} \oplus f(\beta)) \mid \beta < \alpha\} : f \in \mathcal{D}_\alpha, \bigcup_{\beta<\alpha} f(\beta) \neq W_n^{\emptyset^{(\alpha)}}\}.$$

Let $\emptyset^{(\alpha)} <_T X^{(\alpha)}$. Denote by B the set of canonical numbers of all sequences $(\beta_0, \ldots, \beta_q, F_1, \ldots, F_q)$, where

$$0 = \beta_0 < \beta_1 < \ldots < \beta_q < \alpha$$

are ordinals given by their notations $<_O a$, $|a|_O = \alpha$, and $F_1, \ldots, F_q$ are finite sets. Since $\mathcal{W}$ is $X^{(\alpha)}$-c.e. we can fix a partial computable function u such that for every $y \in B$ and every $n \in \mathbb{N}$ the sequence $\{W_{u(y,i,n)}^{X^{(\alpha)}}\}_{i\in\mathbb{N}}$ consists of all finite sets F that satisfy the conditions

$$F \neq W_n^{\emptyset^{(\alpha)}},\ \bigcup_{k=1}^{q} F_k \subseteq F,$$

where $(F_1, \ldots, F_q)$ is second part of the sequence with number y. By Lemma 4 we can fix a partial computable function r such that

$$\mathcal{F} = \{\{\mathcal{E}_\beta(\{n\} \oplus W_{r(u(y,i,n),y,\beta)}^{X^{(\beta+1)}}) \mid \beta < \alpha\} : i, n \in \mathbb{N}, y \in B\}.$$

Hence $\mathcal{F}$ is X-c.e.

Conversely, let $\mathcal{F}$ is X-c.e. By Lemma 6 relativised to X there is a partial computable function d such that

$$\mathcal{F} = \{\{\mathcal{E}_\beta(W_{d(i,\beta)}^{X^{(\beta+1)}}) \mid \beta < \alpha\} : i \in \mathbb{N}\}.$$

Therefore, the 1-family

$$\mathcal{W} = \{\bigcup_{\beta<\alpha} W_{d(i,\beta)}^{X^{(\beta+1)}} : i \in \mathbb{N}\}$$

is $X^{(\alpha)}$-c.e., and hence $\emptyset^{(\alpha)} <_T X^{(\alpha)}$.

Corollary 10. *For every computable ordinal α there is a structure $\mathcal{A}$ whose degree spectrum is equal to $\overline{\mathbf{Low}}_\alpha$.*

Note that, the last result for successor ordinal α was known from a general framework from [Goncharov et al. 2005]. The use of α-families allows to get the result for limit ordinals.

4 Further results and discussions

It was shown in [Kalimullin and Faizrahmanov 2016] that the n-families, $n \in \mathbb{N}$, form a hierarchy on their enumeration spectra. Namely, for every integer $n > 0$ there is an n-family $\mathcal{F}$ such that $\mathbf{Sp}\,\mathcal{F}$ is not enumeration spectrum of an m-family, $m < n$. Furthermore, the classes $\overline{\mathbf{Low}}_n$ are examples of spectra for which the hierarchy of m-families is proper.

Theorem 11. *For every n the class $\overline{\mathbf{Low}}_{2n}$ is not an enumeration spectrum of an n-family.*

On other hand, by Theorem 8 $\overline{\mathbf{Low}}_{2n}$ is an enumeration spectrum of an $(n+1)$-family. To show that there is no maximal level of the hierarchy of α-families, $\alpha < \omega_1^{CK}$, we prove the following Lemma.

Lemma 12. *There is a computable function g such that for all $\alpha < \omega_1^{CK}$, $X \subseteq \mathbb{N}$ and $\delta \in (0,1) \cap \mathbb{Q}$*

$$\mu\{Y : (X \oplus Y)^{(\alpha)} = \Phi_{g(\alpha,\delta)}^{X^{(\alpha)} \oplus Y}\} > \delta,$$

where μ is the uniform probability measure on the Cantor space.

Proof. It was shown in [Stillwell 1972] that there is a computable function f such that for every set X and every rational $\delta \in (0,1)$

$$\mu\{Y : (X \oplus Y)' = \Phi_{f(\delta)}^{X' \oplus Y}\} > \delta.$$

Thus, we can fix a computable function d such that

$$\mu\{Y : \Phi_e^{(X \oplus Y)'} = \Phi_{d(\delta,e)}^{X' \oplus Y}\} > \delta$$

for all X, e, δ. Define $\Phi_{g(0,\delta)}^Z = Z$ for every Z. Fix a computable function h such that $(\Phi_e^Z)' = \Phi_{h(e)}^{Z'}$ for every Z, e if Φ_e^Z is total. Suppose $\alpha = \beta + 1$. Assume by induction that the Lemma holds for β. Let

$$g(\alpha,\delta) = d\left(\frac{1+\delta}{2}, h\left(g\left(\beta, \frac{1+\delta}{2}\right)\right)\right).$$

For every set X and rational $\delta \in (0,1)$ let $\mathcal{C}_0 \subseteq 2^{\mathbb{N}}$ be the class defined by

$$\mathcal{C}_0 = \{Y : (X \oplus Y)^{(\alpha)} = \left(\Phi^{X^{(\beta)} \oplus Y}_{g(\beta, \frac{1+\delta}{2})}\right)'\}.$$

Since $(X \oplus Y)^{(\alpha)} = ((X \oplus Y)^{(\beta)})'$ and by induction assumption we have $\mu\mathcal{C}_0 > \frac{1+\delta}{2}$. Let $\mathcal{C}_1 \subseteq 2^{\mathbb{N}}$ be the class defined by

$$\mathcal{C}_1 = \{\Phi^{(X^{(\beta)} \oplus Y)'}_{h(g(\beta, \frac{1+\delta}{2}))} = \Phi^{X^{(\alpha)} \oplus Y}_{g(\alpha,\delta)}\}.$$

By definition of the function d, $\mu\mathcal{C}_1 > \frac{1+\delta}{2}$. Since

$$\left(\Phi^{X^{(\beta)} \oplus Y}_{g(\beta, \frac{1+\delta}{2})}\right)' = \Phi^{(X^{(\beta)} \oplus Y)'}_{h(g(\beta, \frac{1+\delta}{2}))}$$

for every $Y \in \mathcal{C}_0$ we have

$$(X \oplus Y)^{(\alpha)} = \Phi^{X^{(\alpha)} \oplus Y}_{g(\alpha,\delta)}$$

for every $Y \in \mathcal{C}_0 \cap \mathcal{C}_1$ and $\mu(\mathcal{C}_0 \cap \mathcal{C}_1) > \delta$.

Suppose now that α is limit ordinal. Assume by induction that the Lemma holds for $\beta < \alpha$. Let $\{\beta_i\}_{i>0}$ be an injective combutable sequence of all ordinals less than α. Define the value of $g(\alpha, \delta)$ by letting

$$\Phi^{X^{(\alpha)} \oplus Y}_{g(\alpha,\delta)}(\langle u, \beta_i \rangle) = \Phi^{X^{(\beta_i)} \oplus Y}_{g\left(\beta_i, \frac{2^i - 1 + \delta}{2^i}\right)}(u)$$

and $\Phi^{X^{(\alpha)} \oplus Y}_{g(\alpha,\delta)}(\langle u, \gamma \rangle) = 0$ if γ is not a (notation of) ordinal lesser α. For every X, $\delta \in (0,1) \cap \mathbb{Q}$, $i > 0$ let $\mathcal{C}_i \subseteq 2^{\mathbb{N}}$ be the class defined by

$$\mathcal{C}_i = \{Y : (X \oplus Y)^{(\beta_i)}(u) = \Phi^{X^{(\beta_i)} \oplus Y}_{g\left(\beta_i, \frac{2^i - 1 + \delta}{2^i}\right)}(u)\}.$$

By induction assumption $\mu\mathcal{C}_i > \frac{2^i - 1 + \delta}{2^i}$. Since

$$(X \oplus Y)^{(\alpha)}(\langle u, \beta_i \rangle) = (X \oplus Y)^{(\beta_i)}(u)$$

for every Y,i, u we have

$$(X \oplus Y)^{(\alpha)}(\langle u, \beta_i \rangle) = \Phi^{X^{(\alpha)} \oplus Y}_{g(\alpha,\delta)}(\langle u, \beta_i \rangle)$$

for every $Y \in \bigcap_{i>0} \mathcal{C}_i$. On other hand

$$\mu \bigcap_{i>0} \mathcal{C}_i > 1 - \sum_{i=1}^{\infty} \left(1 - \frac{2^i - 1 + \delta}{2^i}\right) = 1 - \sum_{i=1}^{\infty} \frac{1 - \delta}{2^i} = \delta.$$

Therefore $g(\alpha, \delta)$ satisfies the Lemma.

Now we will show that for a given computable ordinal α each *almost c.e.* α-family has an enumeration in some fixed level of the hyperarithmetical hierarchy. Since $\Delta_1^1 = \bigcup_{\alpha<\omega_1^{CK}} Low_\alpha$, it implies that there is no maximal level of the hierarchy of α-families.

Definition 13. We say that an α-family $\mathcal{F}$ is *almost c.e.* if

$$\mu\{X : \mathcal{F} \text{ is } X\text{-c.e.}\} = 1.$$

Theorem 14. *For every $\alpha < \omega_1^{CK}$ there is a $\beta < \omega_1^{CK}$ such that every almost c.e. α-family is $\emptyset^{(\beta)}$-c.e.*

Proof. It is sufficient to define a partial computable function f such that every almost c.e. α-family is $\emptyset^{(f(\alpha))}$-c.e. Since every almost c.e. set is c.e. we can define $f(0) = 0$. Suppose $\alpha > 0$ and $\mathcal{F}$ is arbitrary almost c.e. α-family. Assume by induction that $f(\gamma)$ is defined for every $\gamma < \alpha$. Let $\mathcal{F}$ is X-c.e. Since each β-family in $\mathcal{F}$ is $\emptyset^{(f(\gamma))}$-c.e. we can uniformly choose an ordinal $\beta < \omega_1^{CK}$ such that the set

$$I = \{\langle n, \gamma\rangle : n \text{ is the } \emptyset^{(f(\gamma))}\text{-enumeration index of some } \gamma\text{-family in } \mathcal{F}, \gamma < \alpha\}$$

is $X^{(\beta)}$-c.e. uniformly by the X-enumeration index of $\mathcal{F}$. Let $f(\alpha) = \beta$. Since

$$\mu\{X : \mathcal{F} \text{ is } X\text{-c.e.}\} = 1$$

there is an integer i such that

$$\mu\{X : i \text{ is an } X\text{-enumeration index of } \mathcal{F}\} > 0.$$

Using Lebesgue's Density Theorem we can fix an integer e such that

$$\mu\{X \supset \sigma : i \text{ is an } X\text{-enumeration index of } \mathcal{F}\} > \frac{3}{4 \cdot 2^{|\sigma|}}$$

for some $\sigma \in 2^{<\omega}$. Embedding the initial segment σ into enumeration algorithms for $\mathcal{F}$ we can produce new index $e \in \omega$ such that

$$\mu\{X : e \text{ is an } X\text{-enumeration index of } \mathcal{F}\} > \frac{3}{4}.$$

By Lemma 12

$$\mu\left\{X : X^{(\beta)} = \Phi_{g(\beta,\frac{3}{4})}^{\emptyset^{(\beta)}\oplus X}\right\} > \frac{3}{4}.$$

Thus, there is a c.e. operator W such that

$$\mu\{X : I = W^{\emptyset^{(\beta)}\oplus X}\} > \frac{1}{2}.$$

By de Leeuw's et al. Theorem (see [Downey and Hirschfeldt 2010] Theorem 8.12.1) relativised to $\emptyset^{(\beta)}$ we have that I is $\emptyset^{(\beta)}$-c.e. Therefore $\mathcal{F}$ is $\emptyset^{(\beta)}$-c.e.

Corollary 15. *For every ordinal $\alpha < \omega_1^{CK}$ there is a β-family $\mathcal{F}$, $\alpha < \beta < \omega_1^{CK}$, such that* $\mathbf{Sp}\,\mathcal{F}$ *is not enumeration spectrum of an α-family.*

Proof. Fix an infinite ordinal γ such that every almost c.e. α-family is $\emptyset^{(\gamma)}$-c.e. By Theorem 9 for $\beta = \gamma^2 + 1$ there is a β-family with the enumeration spectrum $\overline{\mathbf{Low}}_{\gamma^2}$. On other hand, the enumeration spectrum of every almost c.e. α-family should contain the low$_{\gamma^2}$-degree $\mathbf{0}^{(\gamma)}$, and hence the class $\overline{\mathbf{Low}}_{\gamma^2}$ is not the enumeration spectrum of an α-family.

We finish the paper by formulating the following open questions.

1. For a given $\alpha < \omega_1^{CK}$ what is the minimal possible $\beta < \omega_1^{CK}$ such that $\overline{\mathbf{Low}}_\alpha$ is the enumeration spectrum of a β-family?

   In the paper [Kalimullin and Faizrahmanov 2016] such minimal β was found for the case of finite α. It was shown that for every integer $n > 0$ the classes $\overline{\mathbf{Low}}_{2n-2}$ and $\overline{\mathbf{Low}}_{2n-1}$ are the enumeration spectra of n-families but not enumeration spectra of $(n-1)$-families. Using the operator

   $$\mathcal{H}_0(A) = \{F \subseteq \mathbb{N} : F \text{ is finite}\} \cup \{\mathbb{N} \setminus \{x\} : x \in A\}$$

   instead of $\mathcal{E}_0$ we can improve the levels obtained in Theorem 9 by showing that for every limit α and integer n the classes $\overline{\mathbf{Low}}_{\alpha+2n-1}$ and $\overline{\mathbf{Low}}_{\alpha+2n}$ are enumeration spectra of an $(\alpha + n + 1)$-families.

2. For given computable ordinals $\alpha < \beta$ does there exist a β-family $\mathcal{F}$ such that $\mathbf{Sp}\,\mathcal{F}$ is not enumeration spectrum of an α-family?

   It follows from Corollary 15 that for every α there is a sufficiently large $\beta > \alpha$ such that the enumeration spectrum of some β-family is not an enumeration spectrum of an α-family, but the proof in fact do not give an efficient bound for β.

Acknowledgments

The research of the first author was funded by the subsidy allocated to Kazan Federal University for the state assignment in the sphere of scientific activities, project no. 1.2045.2014. The research of the second author was supported by RFBR Grant No. 15-31-20607.

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