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Some Algorithms Providing Rigourous Bounds for the Eigenvalues of a Matrix

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Abstract: Three algorithms providing rigourous bounds for the eigenvalues of a real matrix are presented. The first is an implementation of the bisection algorithm for a symmetric tridiagonal matrix using IEEE floating-point arithmetic. The two others use interval arithmetic with directed rounding and are deduced from the Jacobi method for a symmetric matrix and the Jacobi-like method of Eberlein for an unsymmetric matrix.

1 Bisection Algorithm for a Symmetric Tridiagonal Matrix

Let A be a symmetric tridiagonal matrix of order n:

$$A = A_n = \begin{pmatrix} a_1 & b_2 & & \\ b_2 & a_2 & \ddots & \\ & \ddots & \ddots & b_n \\ & & b_n & a_n \end{pmatrix}$$

Set $b_1 = 0$, and suppose $b_k \neq 0, k = 2, \dots, n$. The bisection method is based on the fact that the sequence $d_k(x)$ of principal minors of $A - xI$ is a Sturm sequence:

$$d_k(x) = \det(A_k - xI_k), k = 1, \dots, n, \quad d_0 = 1$$

In floating point arithmetic, as pointed out in [Barth, Martin, Wilkinson 1971], the direct use of this sequence is quite impossible: even for small n underflow and overflow are unavoidable. So they consider (the hypothesis $b_k \neq 0, k = 2, \dots, n$ can then be removed):

$$p_k = \frac{d_k}{d_{k-1}}, k = 1, \dots, n$$

This new sequence satisfies the recurrence relations:

$$(\mathcal{S}_x) \quad p_k := \begin{cases} a_k - x & \text{if} \quad b_k = 0 \quad \text{or} \quad p_{k-1} = -\infty \\ -\infty & \text{if} \quad p_{k-1} = 0 \\ a_k - x - \dfrac{b_k^2}{p_{k-1}} & \text{otherwise} \end{cases} \qquad \text{for } k = 1, \dots, n$$

From the Sturm property and the choice of $p_k = -\infty$ when $p_{k-1} = 0$, it comes that the number of negative terms in $(\mathcal{S}_x)$ is equal to the number of eigenvalues of A smaller than x.
Beginning with an interval containing all the eigenvalues of A, the method of bisection provides intervals as small as desired, containing the eigenvalues of A.

1.1 Setting it to work in floating point arithmetic

In floating point arithmetic there are two problems:
- to perform the calculations *without overflow*,
- to give *bounds for the error.*

[Barth, Martin, Wilkinson 1971] give bounds for all the eigenvalues. Godunov [Godunov et al. 1993], page 315, gives a similar result but uses a different sequence and guarantees no overflow.
The method presented here (using the sequence $(\mathcal{S}_x)$) guarantees the absence of overflow and provides bounds separately *for each eigenvalue.*

The set of machine-numbers (IEEE standard, available on Macintosh [Apple 1988] or PC compatible [Crawford, Gelsinger 1988]), provides representations for:

$$\begin{cases} 0, -\infty, +\infty \\ \pm 1{,}a_2 \dots a_t 2^e, a_i = 0 \text{ or } 1, -min \le e \le max, \text{(normalized numbers)} \\ \pm 0{,}a_2 \dots a_t 2^{-min}, \text{(denormalized numbers)} \end{cases}$$

t is the number of significative digits used. We suppose $min \le max$.
The positive normalized numbers lie in the interval $[\epsilon_0, M]$:

$$\epsilon_0 = 2^{-min}, \quad M = 1.1\dots1 \times 2^{max} \approx 2 \times 2^{max}$$

Notice that $min \le max \Rightarrow 1/\epsilon_0 \le 2^{max} \approx M/2$.
With *rounding to nearest* the relative error in usual operations is less than $eps = 2^{-t}$, (machine precision). For the machine-operations $\tilde{\top}$ ($\top \to +, -, \times, /$) we have:

$$x \tilde{\top} y = (x \top y)(1 + \varepsilon), |\varepsilon| \le eps$$

for normalized numbers x, y, and $x \top y$. Notice that the presence of denormalized numbers insures that an addition or substraction that gives 0 is exact.

1.1.1 Calculation without overflow

In the calculation of $(\mathcal{S}_x)$ the critical term is b_k^2/p_{k-1}. To avoid overflow:
- A scaling of the matrix is performed by multiplying it by 2^p, p being chosen so that all elements become less than 1.
- The denormalized numbers occuring during the calculation of $(\mathcal{S}_x)$ are replaced by 0. This induces an absolute error $\le \epsilon_0$.

Thus, the *operations used* ($\hat{\top}$) are:

$$x \hat{\top} y = \begin{cases} 0 \quad \text{if } x \tilde{\top} y \text{ is denormalized} \\ x \tilde{\top} y \quad \text{otherwise} \end{cases}$$

and the calculated sequence is (remember that $b_1 = 0$):

$$(\hat{\mathcal{S}}_x) \quad \hat{p}_k := \begin{cases} a_k \hat{-} x & \text{if} \quad b_k = 0 \quad \text{or} \quad \hat{p}_{k-1} = -\infty \\ -\infty & \text{if} \quad \hat{p}_{k-1} = 0 \\ a_k \hat{-} x \hat{-} b_k \hat{\times} b_k \hat{/} \hat{p}_{k-1} & \text{otherwise} \end{cases} \qquad \text{for } k = 1, \ldots, n$$

The eigenvalues of A are therefore in $[-3, 3]$, and x also.
If $\hat{p}_{k-1}$ is 0 or infinite there is no problem.
Otherwise $|\hat{p}_{k-1}|$ is a normalized number $\geq \epsilon_0$:

$$|a_k - x - b_k \times b_k / \hat{p}_{k-1}| \leq |a_k| + |x| + 1/\epsilon_0 \leq 4 + 2^{min} < M$$

and this is also true for machine operations.

1.1.2 Error estimation

The scaling of the matrix does not cause any error, except for the elements that become less than ϵ_0 in absolute value and are replaced by 0. The elements b_k^2 that become less than ϵ_0 are also replaced by 0. Thus the matrix becomes $2^p A + E'$. E' is a tridiagonal symmetric matrix which elements are equal to 0 or bounded by ϵ_0 or $\sqrt{\epsilon_0}$.
A is now the scaled matrix. The classical error analysis ($\tilde{\top}$ operations) shows that the calculation of $(\hat{\mathcal{S}}_x)$ is identical to the calculation of the sequence $(\mathcal{S}_x)$ associated with a matrix $A + E''$. This remains true for the $\hat{\top}$ operations. A symmetric tridiagonal matrix E:

$$E = \begin{pmatrix} e_1 & f_2 & & \\ f_2 & e_2 & \ddots & \\ & \ddots & \ddots & f_n \\ & & f_n & e_n \end{pmatrix}$$

such that $|E''| \leq E$ will be calculated.

It will be shown that there exist small integers $na, da, nb \leq 3$, depending on k such that:

$$e_k = |a_k - x| \left((1 + eps)^{na} - 1\right) + da \times \epsilon_0$$
$$f_k = |b_k| \left((1 + eps)^{\frac{nb}{2}} - 1\right)$$

The term $da \times \epsilon_0$ comes from the operations that give 0 or a denormalized result. Let us examine the different cases that occur in the calculation of $\hat{p}_k$ and determine the values of the integers na, da and nb:

• **Case A**: $b_k = 0$ or $\hat{p}_{k-1} = -\infty$.
Then $\hat{p}_k = a_k \hat{-} x$. Let $y = a_k \tilde{-} x$. If y is a normalized number we may write:

$$y = (a_k - x)(1 + \varepsilon_1) = a_k + (a_k - x)\varepsilon_1 - x, |\varepsilon_1| \leq eps$$

So we have: $\hat{p}_k = y, e_k = |a_k - x|\, eps, f_k = 0$. $(na = 1, da = 0, nb = 0)$.

If y is a denormalized number then $|a_k - x| \leq \epsilon_0$ and:

$$\hat{p}_k = 0 = a_k - (a_k - x) - x$$

Thus $e_k = \epsilon_0, f_k = 0$. $(na = 0, da = 1, nb = 0)$.
Lastly if $y = 0$, there is no error and $\hat{p}_k = 0, e_k = f_k = 0.(na = da = nb = 0)$.

- **Case B**: $\hat{p}_{k-1} = 0$.

Then $\hat{p}_k = -\infty$, and $e_k = f_k = 0$. $(na = da = nb = 0)$.

- **Case C**: $\hat{p}_{k-1}$ is a normalized number.

For each operation three cases (at the most) are to be considered according to the result: 0, denormalized (replaced by 0), normalized.

∘ *Usual case C0:* In the usual case all operations give a normalized result and we get:

$$\hat{p}_k = \left((a_k - x)(1 + \varepsilon_1) - \frac{b_k^2 (1 + \varepsilon_2)}{\hat{p}_{k-1}} (1 + \varepsilon_3) \right) (1 + \varepsilon_4), |\varepsilon_i| \leq eps$$

The ε_i are the relative errors occurring in the operations. Thus:

$$\hat{p}_k = a_k + (a_k - x)(\varepsilon_1 + \varepsilon_4 + \varepsilon_1 \varepsilon_4) - x - \frac{b_k^2 (1 + \varepsilon_2)(1 + \varepsilon_3)(1 + \varepsilon_4)}{\hat{p}_{k-1}}$$

This is the exact result corresponding to:

$$a_k + (a_k - x)(\varepsilon_1 + \varepsilon_4 + \varepsilon_1 \varepsilon_4) \quad \text{and} \quad b_k^2 (1 + \varepsilon_2)(1 + \varepsilon_3)(1 + \varepsilon_4)$$

Thus in this case:

$$e_k = |a_k - x| \left(2eps + eps^2\right), (na = 2, da = 0)$$
$$f_k = |b_k| \left(\sqrt{(1 + eps)^3} - 1 \right), (nb = 3)$$

∘ *Case C1:* $b_k \tilde{\times} b_k \tilde{/} \hat{p}_{k-1}$ gives 0 or a denormalized result (and thus replaced by 0). In this case:

$$\hat{p}_k = a_k \hat{-} x$$

So we may write:

$$\hat{p}_k = a_k \hat{-} x + \left[\frac{b_k \tilde{\times} b_k}{\hat{p}_{k-1}} \right] - \frac{b_k^2 (1 + \varepsilon_2)}{\hat{p}_{k-1}}$$

and in this case we have to add ϵ_0 to the e_k obtained in case **A** (add 1 to da) and $nb = 1$.

∘ *Case C2:* $a_k - x = 0$. Excluding case *C1* we have:

$$\hat{p}_k = -\frac{b_k^2 (1 + \varepsilon_2)(1 + \varepsilon_3)}{\hat{p}_{k-1}}$$

and thus: $e_k = 0, (na = 0, da = 0), \quad f_k = |b_k| eps, (nb = 2)$.

◦ *Case C3:* $a_k \tilde{-} x$ is denormalized. This case is the same as *C2*, except that $e_k = \epsilon_0, (na = 0, da = 1, nb = 2)$.

◦ *Case C4:* All operations give a normalized result, except the last substraction. If the result is 0, then consider the usual case and set ε_4 to 0. It comes:

$$e_k = |a_k - x|eps, f_k = |b_k|eps, (na = 1, da = 0, nb = 2)$$

If the result is denormalized add ϵ_0 to e_k, $(na = 1, da = 1, nb = 2)$.

1.2 Bounds for the eigenvalues

Naming $\lambda_i(X), i = 1, \dots, n$, the eigenvalues of a symmetric matrix X of order n, in increasing order, we have [Wilkinson 1965], page 101:

$$\lambda_i(A) + \lambda_1(E'') \le \lambda_i(A + E'') \le \lambda_i(A) + \lambda_n(E'')$$

Let ρ be the spectral radius of E''. We get:

$$\lambda_i(A) - \rho \le \lambda_i(A + E'') \le \lambda_i(A) + \rho$$

and also:

$$\lambda_i(A + E'') - \rho \le \lambda_i(A) \le \lambda_i(A + E'') + \rho$$

ρ is bounded by $\|E''\|_\infty$, and by $\|E\|_\infty$.
Let m_x be the number of negative terms in $(\hat{\mathcal{S}}_x)$. We have:

$$\lambda_i(A + E'') < x, i = 1, \dots, m_x$$
$$\lambda_i(A + E'') \ge x, i = m_x + 1, \dots, n$$

Naming δ_x the infinite norm of E, it comes:

$$\lambda_i(A) < x + \delta_x, i = 1, \dots, m_x$$
$$\lambda_i(A) \ge x - \delta_x, i = m_x + 1, \dots, n$$

If the interval $[x, y]$ is such that $k = m_y - m_x > 0$, we may say that:

$$\lambda_i(A) \in [x - \delta_x, y + \delta_y[\,, i = m_x + 1, \dots, m_y$$

It remains to take into account the scaling of the matrix and the initial rounding errors.
The bisection algorithm gives intervals such that $m_x < m_y$: to determine $\lambda_k(A)$ for a given k first choose $[x, y]$ such that $m_x < k \le m_y$ and calculate $z = (x \hat{+} y) \hat{/} 2$ and m_z:
• if $m_z \ge k$ the new interval will be $[x, z]$,
• if $m_z < k$ the new interval will be $[z, y]$.

1.3 Algorithm and example

The general structure of the algorithm is the same as in [Barth, Martin, Wilkinson 1971], except the final test: for each eigenvalue the bisection process is continued until the interval $[x, y]$ is such that x and y are two consecutive machine-numbers (denormalized numbers excluded), or until a fixed maximum number of steps is reached.
The machine precision used is $eps = 2^{-64} \approx 5{,}5 \times 10^{-20}$. This precision is available on micro-computers [Apple 1988], [Crawford, Gelsinger 1988]. The results have been obtained using the software [Pavec 1994].
Consider the matrix of order $n = 30$ from [Barth, Martin, Wilkinson 1971]:

$$a_i = i^4, b_i = i - 1, \quad i = 1, \ldots, 30$$

The eigenvalues lie in the interval $[0.933.. , 8.100.. \times 10^5]$. A few results are shown in the table below:

• the bounds for the absolute error '$\varDelta\lambda_i$' and relative error '$\varDelta\lambda_i/\lambda_i$' take in account the final rounding from base 2 to base 10,

• 'Steps' is the number of bisection steps performed and is less than the fixed maximum (= 100).

i	λ_i	$\varDelta\lambda_i$	$\varDelta\lambda_i/\lambda_i$	Steps
30	$8.10000\,00818\,73846\,690 \times 10^5$	2.3×10^{-13}	2.9×10^{-19}	64
20	$1.60000\,00056\,28909\,621 \times 10^5$	1.4×10^{-13}	8.8×10^{-19}	62
10	$1.00000\,02006\,27702\,490 \times 10^4$	9.3×10^{-14}	9.3×10^{-18}	64
1	$0.93340\,70848\,65963$	8.8×10^{-14}	9.5×10^{-14}	68

2 Symmetric Matrices and Jacobi's Algorithm

Jacobi's method [Rutishauser 1991] applied to a symmetric matrix A constructs a sequence of symmetric matrices A_k, similar to A, and converging to a diagonal matrix:

$$A_{k+1} = R_k^T A_k R_k$$

where R_k is a rotation (angle θ_k) matrix calculated to annihilate a non diagonal element (i, j) of A_k. $t_k = \tan\theta_k$ is obtained from $a_{ii}^{(k)}, a_{ij}^{(k)}, a_{jj}^{(k)}$.

2.1 Setting it to work on matrices of intervals

In order to obtain rigourous bounds for the eigenvalues, we work on *matrices of intervals.*
The elementary operations $(+, -, \times, /, \surd)$ on intervals [Moore 1979] are performed using directed rounding provided by the IEEE standard, without using the extensions developped in [Kulisch, Miranker 1981].

A number or a matrix x represented by an interval is written $[x]$.

Let $[A]$ be the initial symmetric matrix of intervals. Starting with $[A]_1 = [A]$, we generate a sequence of symmetric matrices $[A]_k$ such that all symmetric matrix B belonging to $[A]$ is similar to a symmetric matrix B_k belonging to $[A]_k$.
Thus the eigenvalues of B are the eigenvalues of a symmetric matrix B_k belonging to $[A]_k$.
For an interval matrix $[A]_k$, in order to restrict the growth of the intervals, a machine-number t_k is calculated on the *midpoint* of the intervals $[a]_{ii}^{(k)}$, $[a]_{ij}^{(k)}$ and $[a]_{jj}^{(k)}$, using ordinary floating point arithmetic.
The corresponding rotation is:

$$R_k = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \qquad c = 1/\sqrt{1+t_k^2}, s = t_k c$$

Using interval arithmetic to evaluate c and s, we get an interval matrix $[R]_k$ containing R_k and we set:

$$[A]_{k+1} = [R]_k^T [A]_k [R]_k$$

If B is a symmetric matrix belonging to $[A]$ then

$$B_k = R_{k-1}^T \dots R_1^T B R_1 \dots R_{k-1}$$

is similar to B and belongs to $[A]_k$.
Notice that the sequence $[A]_k$ does not necessarily contain the sequence generated by Jacobi's method, and that the element $[a]_{ij}^{(k)}$ is not annihilated but replaced by a *small* interval.
These rotations are performed in a cyclic way, among the elements that do not contain 0.

2.2 Bounds for the eigenvalues

Let D_k be the diagonal matrix which diagonal elements are the midpoint of the elements of $[A]_k$, and $[C]_k$ a matrix such that:

$$[A]_k \subseteq D_k + [C]_k$$

The theorem of Gershgorin [Wilkinson 1965] shows that the eigenvalues of a symmetric matrix belonging to $[A]$ are in the union of the closed intervals:

$$[d_i - r_i, d_i + r_i], \quad r_i = \sum_{j=1}^{n} \left| [c]_{ij} \right|$$

where $\left| [c]_{ij} \right|$ is:

$$\left| [c]_{ij} \right| = \max_{t \in [c]_{ij}} |t|$$

2.3 Examples

• Condider the matrix [Wilkinson, Reinsch, 1991], page 223:

$$A = \begin{pmatrix} 10 & 1 & 2 & 3 & 4 \\ 1 & 9 & -1 & 2 & -3 \\ 2 & -1 & 7 & 3 & -5 \\ 3 & 2 & 3 & 12 & -1 \\ 4 & -3 & -5 & -1 & 15 \end{pmatrix}$$

This matrix is stored as zero-length intervals. After a sweep of the lower triangle the Gershgorin's disks are (radius ≤ 2.1):

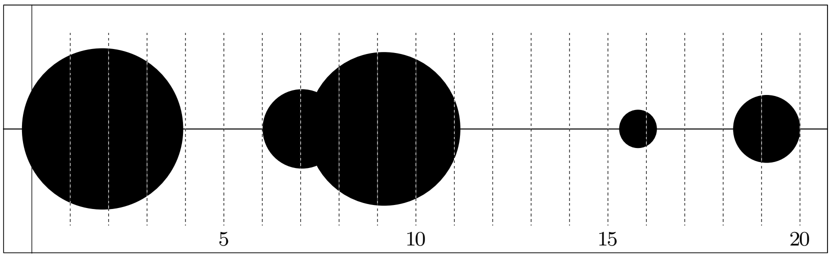

Figure 1: Gershgorin disks after a sweep

Going on with the sweeps until all non-diagonal elements contain 0, the radii of all the disks (*Machine precision*: 2^{-64}) become successively less than:

$$0.073, \quad 1.67 \times 10^{-5}, \quad 2.25 \times 10^{-16}$$

• The matrix $B = 8J - 5J^2 + J^3$ of order 44 [Rutishauser 1991], page 209, where J denotes the tridiagonal matrix with $J_{ii} = 2, J_{i,i+1} = 1$, has eleven eigenvalues in the interval $[4, 4.163]$, the length of which is $1/100$ of the total length of the spectrum.
After 7 sweeps all the eigenvalues are separated, except two that lie in the interval $[4.0032, 4.0065]$.

3 Eberlein's Jacobi-like Method for Unsymmetric Real Matrices

For an unsymmetric real matrix A Eberlein gives a Jacobi-like method [Eberlein 1962]:

$$A_{k+1} = T_k^{-1} A_k T_k$$

A_k generally converges to a block-diagonal matrix, with blocks of order 1 or 2. The blocks of order two have complex conjugate eigenvalues $a \pm ib$:

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

T_k is the product of a rotation R_k and of a matrix S_k:

$$S_k = \begin{pmatrix} \cosh y & -\sinh y \\ -\sinh y & \cosh y \end{pmatrix}$$

working in a coordinate plane (i, j).

3.1 Bounds for the eigenvalues

After some sweeps of Eberlein's method we expect a matrix of the form:

$$\Delta + E$$

where Δ is a block-diagonal matrix, and E a matrix with small elements. The following theorem, that is a particular case of the theorem of Bauer-Fike [Golub, Van Loan 1983], page 342, gives disks of the complex plane containing the eigenvalues of $\Delta + E$:

Theorem. *Let $A = \Delta + E$, where Δ is a block-diagonal with blocks of order 1 or 2.*

$$\Delta = diag\,(\Delta_1, \Delta_2, \ldots, \Delta_p)$$

$$\Delta_r = \alpha_r \quad \text{or} \quad \begin{pmatrix} \alpha_r & -\beta_r \\ \beta_r & \alpha_r \end{pmatrix}$$

Δ can be diagonalised with a unitary block-diagonal matrix X which has the same structure as Δ. Let $F = X^{-1}EX$. Then the eigenvalues of A are in the disks of the complex plane, which centers are the eigenvalues of Δ and the radii:

$$r_i = \sum_{j=1}^{n} |f_{ij}|$$

If the union U of k of these disks is isolated from the others then U contains exactly k eigenvalues of A.

Proof. The blocks of order 1 of X are equal to 1 and the blocks of order 2 to:

$$X_r = \frac{1}{\sqrt{2}} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}, \quad X_r^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}$$

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of Δ and

$$\Delta' = X^{-1}\Delta X = diag(\lambda_1, \ldots, \lambda_n)$$

If λ is an eigenvalue of A, then $A - \lambda I$ is singular, and

$$X^{-1}(A - \lambda I)X = \Delta' - \lambda I + F$$

also. There exists a complex vector x, $||x||_\infty = 1$, such that:

$$(\Delta' - \lambda I)x = -Fx$$

Thus

$$(\lambda_i - \lambda)\, x_i = -\sum_{j=1}^{n} f_{ij} x_j, i = 1, \ldots, n$$

Choosing i such that $|x_i| = 1$ we get:

$$|\lambda_i - \lambda| \leq \sum_{j=1}^{n} |f_{ij}| = r_i$$

r_i is easily expressed in terms of E.

If the union U of k of these disks is isolated from the union V of the others, the disks associated with $A(t) = \Delta + tE$ have the same property for all $t \in [0,1]$: the centers do not change and the radii are tr_i.
The result comes then from the fact that U contains exactly k eigenvalues of $A(0)$, and V exactly $n-k$. The eigenvalues of $A(t)$ are continuous functions of t, so they cannot jump from U to V.

3.2 Setting it to work on matrices of intervals

A sequence of matrices $[A]_k$ is generated from the initial matrix $[A]$, such that each matrix $B \in [A]$ is similar to $B_k \in [A]_k$.
At each stage of the method $T_k = RS$ is calculated using ordinary floating point arithmetic and the midpoints of the elements of $[A]_k$.
Then

$$\left[T_k^{-1}\right] \text{ and } [A]_{k+1} = \left[T_k^{-1}\right][A]_k\, T_k$$

are calculated using interval arithmetic.
It remains then to decompose $[A]_k$ under the form:

$$[A]_k \subseteq \Delta_k + [C]_k$$

where Δ_k is a block-diagonal matrix and to calculate disks containing the eigenvalues. Bounds for the radii of these disks are deduced from the theorem of the preceding section.

3.3 Examples

- Consider the matrix A [Eberlein, Boothroyd 1971], page 334:

$$A = \begin{pmatrix} 6 & -3 & 4 & 1 \\ 4 & 2 & 4 & 0 \\ 4 & -2 & 3 & 1 \\ 4 & 2 & 3 & 1 \end{pmatrix}$$

A has two double eigenvalues equal to $3 \pm \sqrt{5}$ and is defective.

The three first sweeps give:

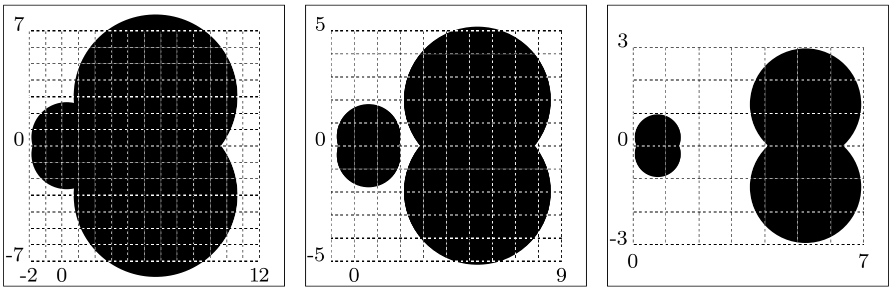

Figure 2: The three first sweeps

After 2 sweeps the disks are included within to disjoint regions.
Finally we get disks whith radii bounded by 3,98 $\times 10^{-8}$ and 2,2 $\times 10^{-8}$.

• For the matrix $B = (b_{ij})$ of order 12 [Eberlein, Boothroyd 1971], page 334, defined by:

$$\begin{cases} b_{ij} = 13 - j, \text{ if } i \leq j \\ b_{i,j} = 12 - j, \text{ if } i = j + 1 \\ b_{ij} = 0, \text{ if } i > j + 1 \end{cases}$$

after 33 sweeps the 12 disks are disjoint and the radii are < 0.001.

4 Conclusion

These algorithms are implemented in an interactive software [Pavec 1994] working on the Apple Macintosh series of micro-computers. This software is devoted to:

• teaching of mathematics [Pavec 1993]: a very simple Pascal style language lets you create interactively changeable figures containing curves, text, numerical values ...

• and teaching of elementary numerical analysis: interpolation, approximation, ordinary differential equations, numerical integration, systems of equations, eigenvalues.

A commented Pascal version of the algorithm of bisection is available by e-mail(*pavec@univ-brest.fr*).

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