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On directed interval arithmetic and its applications

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Abstract: We discuss two closely related interval arithmetic systems: i) the system of directed (generalized) intervals studied by E. Kaucher, and ii) the system of normal intervals together with the outer and inner interval operations. A relation between the two systems becomes feasible due to introduction of special notations and a so-called normal form of directed intervals. As an application, it has been shown that both interval systems can be used for the computation of tight inner and outer inclusions of ranges of functions and consequently for the development of software for automatic computation of ranges of functions.

Key Words: computer arithmetic, error analysis, interval algebraic manipulation
Category: G.1.0, I.1.1

1 Introduction

We briefly recall some well-known facts about the interval arithmetic for compact intervals on the real line $\mathbb{R}$. Given $a^{(-)}, a^{(+)} \in \mathbb{R}$, with $a^{(-)} \leq a^{(+)}$, a (proper) interval is defined by $[a^{(-)}, a^{(+)}] = \{x \mid a^{(-)} \leq x \leq a^{(+)}\}$. The set of all intervals is denoted by $I(\mathbb{R})$. Thus $a^{(s)} \in \mathbb{R}$ with $s \in \Lambda = \{+, -\}$ is the left or right endpoint of $A \in I(\mathbb{R})$ depending on the value of s. In what follows the binary variable s will be sometimes expressed as a product of two binary variables from Λ, defined by $++ = -- = +$, $+- = -+ = -$. An interval $A = [a^{(-)}, a^{(+)}]$ with $a^{(-)} = a^{(+)}$ is called degenerate.

Let $A = [a^{(-)}, a^{(+)}]$, $B = [b^{(-)}, b^{(+)}] \in I(\mathbb{R})$. Inclusion in $I(\mathbb{R})$ has the well-known set-theoretic meaning. In terms of the end-points we have

$$A \subseteq B \Longleftrightarrow (b^{(-)} \leq a^{(-)}) \wedge (a^{(+)} \leq b^{(+)}), \quad A, B \in I(\mathbb{R}).$$

Denote $Z = \{A \in I(\mathbb{R}) \mid a^{(-)} \leq 0 \leq a^{(+)}\}$, $Z^* = \{A \in I(\mathbb{R}) \mid a^{(-)} < 0 < a^{(+)}\}$, $I(\mathbb{R})^* = I(\mathbb{R}) \setminus Z^*$. The function "sign" $\sigma : I(\mathbb{R})^* \rightarrow \Lambda$ is defined for $A \in I(\mathbb{R})^* \setminus \{0\}$ with $\sigma(A) = \{+, \text{ if } a^- \geq 0;\ -, \text{ if } a^+ \leq 0\}$, and for zero argument by $\sigma([0,0]) = \sigma(0) = +$. In particular, σ is well defined over $\mathbb{R}$. The sign σ is not defined for intervals from Z^*. Note that the intervals from Z^* comprise both positive and negative numbers.

The operations $+, -, \times, /$ are defined by

$$A * B = \{a * b \mid a \in A, b \in B\},\ * \in \{+, -, \times, /\},\ A, B \in I(\mathbb{R}), \tag{1}$$

assuming in the case "$* = /$" that $B \in I(\mathbb{R}) \setminus Z$ [Moore 1966], [Sunaga 1958]. The following expressions hold true for $A = [a^{(-)}, a^{(+)}]$, $B = [b^{(-)}, b^{(+)}]$

$$A + B = [a^{(-)} + b^{(-)},\ a^{(+)} + b^{(+)}],\ A, B \in I(\mathbb{R}), \tag{2}$$

$$A \times B = \begin{cases} [a^{(-\sigma(B))}b^{(-\sigma(A))},\ a^{(\sigma(B))}b^{(\sigma(A))}], & A, B \in I(\mathbb{R})^*, \\ [a^{(\delta)}b^{(-\delta)},\ a^{(\delta)}b^{(\delta)}], \quad \delta = \sigma(A), & A \in I(\mathbb{R})^*,\ B \in Z^*, \\ [a^{(-\delta)}b^{(\delta)},\ a^{(\delta)}b^{(\delta)}], \quad \delta = \sigma(B), & A \in Z^*,\ B \in I(\mathbb{R})^*, \end{cases} \tag{3}$$

$$A \times B = [\min\{a^{(-)}b^{(+)}, a^{(+)}b^{(-)}\},\ \max\{a^{(-)}b^{(-)}, a^{(+)}b^{(+)}\}], \quad A, B \in Z^*, \tag{4}$$

$$1/B = [1/b^{(+)}, 1/b^{(-)}],\ B \in I(\mathbb{R}) \setminus Z. \tag{5}$$

For a degenerate interval $A = [a, a] = a \in R$, expression (3) gives $a \times B = [ab^{(-\sigma(a))},\ ab^{(\sigma(a))}]$. For $a = -1$ we obtain the operator negation $(-1) \times B = -B = [-b^{(+)}, -b^{(-)}]$. The operations subtraction and division defined by (1) can be expressed as composite operations by $A - B = A + (-B)$, $A/B = A \times (1/B)$. In terms of end-points we have

$$A - B = [a^{(-)} - b^{(+)}, a^{(+)} - b^{(-)}],\ A, B \in I(\mathbb{R}), \tag{6}$$

$$A/B = \begin{cases} [a^{(-\sigma(B))}/b^{(\sigma(A))}, a^{(\sigma(B))}/b^{(-\sigma(A))}], & A \in I(\mathbb{R})^*, B \in I(\mathbb{R}) \setminus Z, \\ [a^{(-\delta)}/b^{(-\delta)}, a^{(\delta)}/b^{(-\delta)}], \quad \delta = \sigma(B), & A \in Z^*, B \in I(\mathbb{R}) \setminus Z. \end{cases} \tag{7}$$

The algebraic properties of $(I(\mathbb{R}), +, \times, /, \subseteq)$ are studied in [Moore 1966], [Ratschek 1969], [Ratschek 1972], [Sunaga 1958]. The subsystems $(I(\mathbb{R}), +)$, $(I(\mathbb{R})^*, \times)$ are commutative semigroups. They are not groups, that is no inverse elements w. r. t. the operations $+$, resp. $\times$ exist in general. The solutions of the equations $A + X = B$, resp. $A \times X = B$ (if existing), cannot be expressed in terms of the interval operations (1). There is no distributivity between $+$ and $\times$, except for very special cases. The interval operations (1) can be used for outer inclusions of functional ranges (which can be rough). They are of little use for the computation of inner inclusions.

The above mentioned "deficiencies" of the system $(I(\mathbb{R}), +)$ are due to our incomplete knowledge of the interval arithmetic system. The algebraic manipulations in the set $I(\mathbb{R})$ of (proper) intervals resamble the algebraic manipulations in the set $\mathbb{R}^+ = \{x \in \mathbb{R}, x \geq 0\}$ of positive numbers. The equality $a + x = b$ possess no solution in $\mathbb{R}^+$ for $a > b$. Similarly, the equality $A + X = B$ has no solution in $I(\mathbb{R})$ if the width of A is greater than the width of B. Similarly to the algebraic completion of the set $\mathbb{R}^+$ by negative numbers (or to the completion of the reals by complex numbers), the set $I(\mathbb{R})$ can be completed by so-called *improper* intervals, having their left "end-points" greater that the right ones. The algebraic completion of $(I(\mathbb{R}), +)$ leads to a set D of *directed* (or *generalized*) intervals, which is suitable for solving interval algebraic problems. An isomorphic extension of the interval operations over D produces the system $\mathcal{K} = (D, +, \times)$, where the operations " $+$ " and " $\times$ " possess group properties [Dimitrova et al. 1991]–[Kaucher 1980], [Markov 1992]–[Markov 1993], [Shary 1993], [Shary 1995]. However, we must learn to solve algebraic problems with directed intervals and to interpret the solutions in case we need realistic solutions, that is, proper intervals, in the same way as mathematicians have learned to solve algebraic problems using real numbers (positive and negative), or later on using complex numbers.

This paper serves for a further development of the algebraic manipulation in the set of directed intervals and the interpretation of the obtained results in the set of proper intervals. We briefly introduce the system $\mathcal{K}$ using two forms of presentation for the elements of D – *component-wise* (already used by other authors) and *normal*. It has been demonstrated that the normal form actually generates two types of operations between normal intervals – the first type are the usual operations (1), which are also called "outer" interval operations and the second type are the "inner" (or "nonstandard") interval operations. The set of normal intervals together with the set of outer and inner operations $\mathcal{M} = (I(\mathbb{R}), +, +^-, \times, \times^-)$ presents an algebraic completion of the familiar interval arithmetic $(I(\mathbb{R}), +)$, which opposite to $\mathcal{K}$ makes no use of improper elements [Markov 1977]–[Markov 1992]. It has been shown that the two systems $\mathcal{K}$ and $\mathcal{M}$ are closely related and that the understanding of the relation between them can greatly increase the scope of applications of interval arithmetic. An efficient relation between the two systems becomes feasible due to the introduction of special ($\pm$)-type notations and a so-called normal form of directed intervals. The new uniform notations make it possible to develop in a consistent way software implementations of all important interval arithmetic systems and to incorporate them in a software environment supporting symbolic computations. Such an unification will be facilitated by the fact that interval algebraic relations become simple with the new notational approach. Both $\mathcal{K}$ and $\mathcal{M}$ can be used for the construction of tight inner and outer inclusions of ranges of functions, which is one of the most important application of interval arithmetic. This is shown in the last section of the paper, where several propositions are formulated that can be directly applied to the development of algorithms for automatic computation of ranges of functions [Corliss and Rall 1991]. Another vast field of applications of the interval systems $\mathcal{K}$ and $\mathcal{M}$ is the solution of algebraic problems involving interval data, where the system $\mathcal{K}$ plays an important role in the formal algebraic manipulations, whereas the system $\mathcal{M}$ can be used for the interpretation of the results in terms of proper intervals (if necessary).

2 Directed intervals and operations over them

We extend the set $I(\mathbb{R})$ up to the set $D = \{[a, b] \mid a, b \in \mathbb{R}\}$ of ordered couples. To avoid confusion with the normal intervals from $I(\mathbb{R})$, we call the elements of D directed or generalized intervals and their "endpoints" will be called components. The first component of $\mathbf{A} \in D$ is denoted by a^-, and the second by a^+, so that $\mathbf{A} = [a^-, a^+]$. The absence of brackets around $+$ and $-$ in this notation suggests that the inequality $a^- \leq a^+$ is not obligatory (as in the case with the endpoints of the proper intervals: $a^{(-)} \leq a^{(+)}$). Every directed interval $\mathbf{A} = [a^-, a^+] \in D$ defines a binary variable "direction" by

$$\tau(\mathbf{A}) = \sigma(a^+ - a^-) = \begin{cases} +, & \text{if } a^- \leq a^+, \\ -, & \text{if } a^- > a^+. \end{cases}$$

The set of all elements of D with positive direction, that is the set of proper intervals, is equivalent to $I(\mathbb{R})$; the set of directed intervals with negative direction, further called improper intervals, will be denoted by $\overline{I(\mathbb{R})}$, so that $D = I(\mathbb{R}) \cup \overline{I(\mathbb{R})}$. To every directed interval $\mathbf{A} = [a^-, a^+] \in D$ we assign a proper interval $\mathrm{pro}(\mathbf{A})$ with

$$\mathrm{pro}(\mathbf{A}) = \begin{cases} [a^-, a^+], & \text{if } \tau(\mathbf{A}) = +, \\ [a^+, a^-], & \text{if } \tau(\mathbf{A}) = -. \end{cases}$$

We have $\mathrm{pro}(\mathbf{A}) = [a^{-\tau(\mathbf{A})}, a^{\tau(\mathbf{A})}]$. Instead of $\mathrm{pro}(\mathbf{A})$ we shall also write A, if no ambiguity occurs. The interval A will be further called the projection of $\mathbf{A}$ on the set of proper intervals or, briefly, the proper projection of $\mathbf{A}$.

Denote $\overline{Z} = \{\mathbf{A} \in \overline{I(\mathbb{R})} \mid a^+ \leq 0 \leq a^-\}$, $\overline{Z}^* = \{\mathbf{A} \in \overline{I(\mathbb{R})} \mid a^+ < 0 < a^-\}$, $\mathcal{T} = Z \cup \overline{Z} = \{\mathbf{A} \in D \mid 0 \in \mathrm{pro}(\mathbf{A})\}$, $\mathcal{T}^* = Z^* \cup \overline{Z}^* = \{\mathbf{A} \in \mathcal{T} \mid (a^- < 0 < a^+) \ \vee \ (a^+ < 0 < a^-)\}$. In $D^* = D \setminus \mathcal{T}^*$ we define the function "sign" of a directed interval $\sigma : D^* \to \Lambda$, by $\sigma(\mathbf{A}) = \sigma(\mathrm{pro}(\mathbf{A}))$ (note that $\mathrm{pro}(\mathbf{A}) \in I(\mathbb{R})^*$ for $\mathbf{A} \in D^*$).

The formal substitution of components for end-points and the replacement of $I(\mathbb{R})^*$ and Z^* by D^* and $\mathcal{T}^*$, resp. in (2)–(3) extends the definitions of the operations $+$, $\times$ from $I(\mathbb{R})$ up to D:

$$\mathbf{A} + \mathbf{B} = [a^- + b^-, a^+ + b^+], \quad \mathbf{A}, \mathbf{B} \in D, \tag{8}$$

$$\mathbf{A} \times \mathbf{B} = \begin{cases} [a^{-\sigma(B)} b^{-\sigma(A)},\ a^{\sigma(B)} b^{\sigma(A)}], & \mathbf{A}, \mathbf{B} \in D^*, \\ [a^{\delta} b^{-\delta},\ a^{\delta} b^{\delta}], \quad \delta = \sigma(A), & \mathbf{A} \in D^*,\ \mathbf{B} \in \mathcal{T}^*, \\ [a^{-\delta} b^{\delta},\ a^{\delta} b^{\delta}], \quad \delta = \sigma(B), & \mathbf{A} \in \mathcal{T}^*,\ \mathbf{B} \in D^*. \end{cases} \tag{9}$$

The product $\mathbf{A} \times \mathbf{B}$ is defined for $\mathbf{A}, \mathbf{B} \in \mathcal{T}^*$ by

$$\mathbf{A} \times \mathbf{B} = \begin{cases} [\min\{a^- b^+, a^+ b^-\},\ \max\{a^- b^-, a^+ b^+\}], & \mathbf{A}, \mathbf{B} \in Z^*, \\ [\max\{a^- b^-, a^+ b^+\},\ \min\{a^- b^+, a^+ b^-\}], & \mathbf{A}, \mathbf{B} \in \overline{Z}^*, \\ 0, \quad (\mathbf{A} \in Z^*,\ \mathbf{B} \in \overline{Z}^*) \ \vee \ (\mathbf{A} \in \overline{Z}^*,\ \mathbf{B} \in Z^*). \end{cases} \tag{10}$$

It has been shown that the spaces $(I(\mathbb{R}), +)$ and $(I(\mathbb{R}) \setminus Z, \times)$ are isomorphic embeddments in the spaces $(D, +)$, resp. $(D \setminus \mathcal{T}, \times)$ under the operations (8)–(10) [Kaucher 1980]. E. Kaucher gives a table form expression for the multiplication, which is equivalent to (9)–(10).

From (9) for $\mathbf{A} = [a, a] = a$, $\mathbf{B} \in D$ we have $a \times \mathbf{B} = [ab^{-\sigma(a)}, ab^{\sigma(a)}]$. This implies for the operator negation: $\mathrm{neg}(\mathbf{B}) = -\mathbf{B} = (-1) \times \mathbf{B} = [-b^+, -b^-]$. The composite operation $\mathbf{A} + (-1) \times \mathbf{B} = \mathbf{A} + (-\mathbf{B}) = [a^- - b^+, a^+ - b^-]$, for $\mathbf{A}, \mathbf{B} \in D$ is an extension in D of the subtraction $A - B$ and will be further denoted $\mathbf{A} - \mathbf{B}$ as in (6).

The systems $(D, +)$ and $(D \setminus \mathcal{T}, \times)$ are groups [Kaucher 1977], [Kaucher 1980]. Denote by $-_h\mathbf{A}$ the inverse element of $\mathbf{A} \in D$ with respect to "$+$", and by $1/_h\mathbf{A}$ the inverse element of $\mathbf{A} \in D \setminus \mathcal{T}$ with respect to "$\times$". For the inverse elements we have the component-wise presentations $-_h\mathbf{A} = [-a^-, -a^+]$, for $\mathbf{A} \in D$, and $1/_h\mathbf{A} = [1/a^-, 1/a^+]$, for $\mathbf{A} \in D \setminus \mathcal{T}$. The element $-_h\mathbf{A}$ is further called the *opposite* of $\mathbf{A}$ and the element $1/_h\mathbf{A}$ — the *inverse* of $\mathbf{A}$, symbolically $\mathrm{opp}(\mathbf{A}) = -_h\mathbf{A}$, $\mathrm{inv}(\mathbf{A}) = 1/_h\mathbf{A}$.

An important operator in D is the operator *dual element* defined by $\mathrm{dual}(\mathbf{A}) = \mathrm{dual}([a^-, a^+]) = [a^+, a^-]$. The operators negation $-\mathbf{A} = [-a^+, -a^-]$, opposite element $-_h\mathbf{A} = [-a^-, -a^+]$ and dual element $\mathrm{dual}(\mathbf{A} = [a^+, a^-]$ are interrelated by:

$$\mathrm{dual}(\mathbf{A}) = -_h(-\mathbf{A}) = -(-_h\mathbf{A}), \tag{11}$$

that is $\mathrm{dual}(\mathbf{A}) = \mathrm{opp}(\mathrm{neg}(\mathbf{A})) = \mathrm{neg}(\mathrm{opp}(\mathbf{A}))$. The equalities (11) suggest that there might exist an operator $\mathrm{rec}(\mathbf{A})$ in $D \setminus \mathcal{T}$, which (by analogy to the operator $-\mathbf{A}$ in (11)) possibly satisfies:

$$1/_h(\mathrm{rec}(\mathbf{A})) = \mathrm{rec}(1/_h\mathbf{A}) = \mathrm{dual}(\mathbf{A}), \tag{12}$$

that is $\mathrm{dual}(\mathbf{A}) = \mathrm{inv}(\mathrm{rec}(\mathbf{A})) = \mathrm{rec}(\mathrm{inv}(\mathbf{A}))$. The unique such operator is the *reciprocal* operator $\mathrm{rec}(\mathbf{A}) = 1/\mathbf{A} = \mathrm{dual}\ (1/_h\mathbf{A}) = 1/_h(\mathrm{dual}\ (\mathbf{A})) = [1/a^+, 1/a^-]$, for $\mathbf{A} \in D \setminus \mathcal{T}$. Both the opposite and the negative elements play an important role in the substructure $(D, +)$, and so do symmetrically the inverse and the reciprocal elements in the subsystem $(D \setminus \mathcal{T}, \times)$. The composition of these operators with the basic operations (8)–(10) generates a rich set of compound operations.

The operation $\mathbf{A} \times (1/\mathbf{B})$ for $\mathbf{A} \in D$, $\mathbf{B} \in D \setminus \mathcal{T}$ is further denoted by $\mathbf{A}/\mathbf{B}$ (it is an extension in D of the operation A/B, defined by (7)); we have

$$\mathbf{A}/\mathbf{B} = \begin{cases} [a^{-\sigma(B)}/b^{\sigma(A)}, a^{\sigma(B)}/b^{-\sigma(A)}], & \mathbf{A} \in D^*, \mathbf{B} \in D \setminus \mathcal{T}, \\ [a^{-\delta}/b^{-\delta}, a^{\delta}/b^{-\delta}],\ \delta = \sigma(B), & \mathbf{A} \in \mathcal{T}^*,\ \mathbf{B} \in D \setminus \mathcal{T}. \end{cases}$$

From (11) and (12)) we have $-_h\mathbf{A} = -\mathrm{dual}(\mathbf{A})$, $1/_h\mathbf{A} = 1/\mathrm{dual}(\mathbf{A})$. The elements $-_h\mathbf{A}$, $1/_h\mathbf{A}$ generate the operations $\mathbf{A} + (-_h\mathbf{B}) = \mathbf{A} + (-\mathrm{dual}(\mathbf{B})) = \mathbf{A} - \mathrm{dual}(\mathbf{B})$, $\mathbf{A} \times (1/_h\mathbf{B}) = \mathbf{A} \times (1/\mathrm{dual}(\mathbf{B})) = \mathbf{A}/\mathrm{dual}(\mathbf{B})$, which are denoted by $\mathbf{A} -_h \mathbf{B}$, resp. $\mathbf{A}/_h\mathbf{B}$:

$$\mathbf{A} -_h \mathbf{B} = \mathbf{A} - \mathrm{dual}(\mathbf{B}) = [a^- - b^-, a^+ - b^+],\ \mathbf{A}, \mathbf{B} \in D,$$

$$\mathbf{A}\ /_h\ \mathbf{B} = \mathbf{A}/\mathrm{dual}(\mathbf{B}) = \begin{cases} [a^{-\sigma(B)}/b^{-\sigma(A)}, a^{\sigma(B)}/b^{\sigma(A)}], & \mathbf{A}, \mathbf{B} \in D \setminus \mathcal{T}, \\ [a^{-\delta}/b^{\delta}, a^{\delta}/b^{\delta}],\ \delta = \sigma(B), & \mathbf{A} \in \mathcal{T},\ \mathbf{B} \in D \setminus \mathcal{T}. \end{cases}$$

From the last equality we obtain $\mathbf{A}/\mathbf{B} = \mathbf{A}/_h(\mathrm{dual}(\mathbf{B})) = \mathbf{A}/_h(-_h((-1) \times \mathbf{B}))$, which shows that division "$/$" can be expressed by the operations "$\times$", "$-_h$" and "$/_h$". Therefore we may not include division in the list of basic operations of the algebraic system $\mathcal{K} = (D, +, \times)$ thus obtained, as we should do in the case with the familiar system $(I(\mathbb{R}), +, \times, /)$. The system $\mathcal{K}$ involves the compound operations subtraction $\mathbf{A} - \mathbf{B}$ and division $\mathbf{A}/\mathbf{B}$, the operator $\mathrm{dual}(\mathbf{A})$, the operations $\mathbf{A} - \mathrm{dual}(\mathbf{B})$, $\mathbf{A}/\mathrm{dual}(\mathbf{B})$, and their dual operations $\mathrm{dual}(\mathbf{A}) - \mathbf{B}$, $\mathrm{dual}(\mathbf{A})/\mathbf{B}$. Similarly, we can compose $\mathbf{A} + \mathrm{dual}(\mathbf{B})$, $\mathbf{A} \times \mathrm{dual}(\mathbf{B})$, $\mathrm{dual}(\mathbf{A}) + \mathbf{B}$, $\mathrm{dual}(\mathbf{A}) \times \mathbf{B}$, etc.

For the operator dual element we shall further use the notation $\mathbf{A}_- = \mathrm{dual}(\mathbf{A})$. Assuming $\mathbf{A}_+ = \mathbf{A}$, we can introduce the notation $\mathbf{A}_\alpha = \{\mathbf{A}, \text{if } \alpha = +; \mathbf{A}_-, \text{if } \alpha = -\}$. Using this notation we can formulate a simple distributive relation in D^*, which is more convenient than the one formulated in table form in [Kaucher 1980].

Proposition 1. *Conditionally Distributive Law for directed intervals. For* $\mathbf{A} \in D^*, \mathbf{B} \in D^*, \mathbf{C} \in D^*, \mathbf{A} + \mathbf{B} \in D^*$ *we have*

$$(\mathbf{A} + \mathbf{B}) \times \mathbf{C}_{\sigma(A+B)} = (\mathbf{A} \times \mathbf{C}_{\sigma(A)}) + (\mathbf{B} \times \mathbf{C}_{\sigma(B)}).$$

Note that in the above expression we take $\mathbf{C}$ or $\mathrm{dual}(\mathbf{C})$ dependent on the signs of the intervals A, B and $A + B$.

3 Directed intervals in normal form

We introduce another form of presentation for the directed intervals $\mathbf{A} = [a^-, a^+] \in D$, which we call normal form. The set D is equivalent to the direct product $I(\mathbb{R}) \otimes \Lambda$, $\Lambda = \{+, -\}$. Note that the space $I(\mathbb{R}) \otimes \Lambda$ involves degenerate intervals with negative direction, which have not been defined yet. We shall stipulate that such elements coincide with the (same) degenerate intervals with positive direction. Hence a directed interval $\mathbf{A}$ can be presented as a couple consisting of a normal (proper) interval and a sign showing its direction, that is $\mathbf{A} = [A; \alpha] = [a^{(-)}, a^{(+)}; \alpha]$ with $A = [a^{(-)}, a^{(+)}] \in I(\mathbb{R})$ and $\alpha = \tau(\mathbf{A}) \in \Lambda$. For $A = a \in \mathbb{R}$ we have by definition $[a; \alpha] = [a; -\alpha]$.

We may use the following formulae for transition from the component-wise form $[a^-, a^+]$ to normal form $[a^{(-)}, a^{(+)}; \alpha]$ and vice verse

$$\alpha = \sigma(a^+ - a^-),\ a^{(-)} = a^{-\alpha},\quad a^{(+)} = a^{\alpha},\ a^- = a^{(-\alpha)},\quad a^+ = a^{(\alpha)}.$$

We can write, of course, other equivalent expressions like $a^{(-)} = \min\{a^-, a^+\}$, $a^{(+)} = \max\{a^-, a^+\}$, but such formulae are not suitable for algebraic manipulations. We shall next find an expression for the sum $\mathbf{C} = \mathbf{A} + \mathbf{B}$ of two directed intervals $\mathbf{A}$, $\mathbf{B}$ involving normal form presentation. Denoting the length of the interval by $\omega(A) = a^{(+)} - a^{(-)}$, we obtain for the direction $\tau(\mathbf{C})$ of the sum $\mathbf{C} = [c^-, c^+] = [a^- + b^-, a^+ + b^+]$

$$\begin{aligned}\tau(\mathbf{C}) &= \sigma(a^+ + b^+ - a^- - b^-) = \sigma((a^{(\alpha)} - a^{(-\alpha)}) + (b^{(\beta)} - b^{(-\beta)}))\\ &= \sigma\left(\alpha\omega(A) + \beta\omega(B)\right)\\ &= \begin{cases}\alpha,\ \alpha = \beta,\\ \alpha,\ \alpha = -\beta,\ \omega(A) > \omega(B),\\ \beta,\ \alpha = -\beta,\ \omega(A) < \omega(B),\\ +,\ \alpha = -\beta,\ \omega(A) = \omega(B).\end{cases}\end{aligned} \tag{13}$$

In the expression $\sigma\left(\alpha\omega(A) + \beta\omega(B)\right)$ the symbols $\alpha, \beta \in \Lambda$ preceding the real positive numbers $\omega(A)$, resp. $\omega(B)$, should be interpreted as signs of these numbers, that is as $\alpha, \beta = \pm 1$.

Denoting for brevity $\gamma = \tau(\mathbf{C}) = \tau(\mathbf{A} + \mathbf{B})$, as given by (13), we have $c^{(-)} = c^{-\gamma} = a^{-\gamma} + b^{-\gamma} = a^{(-\alpha\gamma)} + b^{(-\beta\gamma)}$, $c^{(+)} = c^{\gamma} = a^{\gamma} + b^{\gamma} = a^{(\alpha\gamma)} + b^{(\beta\gamma)}$, so that

$$\begin{aligned}\mathbf{A} + \mathbf{B} &= [a^{(-)}, a^{(+)}; \alpha] + [b^{(-)}, b^{(+)}; \beta]\\ &= [a^{(-\alpha\gamma)} + b^{(-\beta\gamma)}, a^{(\alpha\gamma)} + b^{(\beta\gamma)}; \gamma].\end{aligned} \tag{14}$$

Expression (14) implies that for $\alpha = \beta\ (= \gamma)$ the normal part $\mathrm{pro}(\mathbf{A} + \mathbf{B}) = [a^{(-\alpha\gamma)} + b^{(-\beta\gamma)}, a^{(\alpha\gamma)} + b^{(\beta\gamma)}]$ of the sum $\mathbf{A} + \mathbf{B}$ is equal to $[a^{(-)} + b^{(-)}, a^{(+)} + b^{(+)}] = A + B = \mathrm{pro}(\mathbf{A}) + \mathrm{pro}(\mathbf{B})$. For $\alpha \neq \beta$ the normal part of $\mathbf{A} + \mathbf{B}$ is

$$\begin{aligned}\mathrm{pro}(\mathbf{A} + \mathbf{B})|_{\alpha \neq \beta} &= \begin{cases}[a^{(-)} + b^{(+)}, a^{(+)} + b^{(-)}], \text{if } a^{(-)} + b^{(+)} \leq a^{(+)} + b^{(-)},\\ [a^{(+)} + b^{(-)}, a^{(-)} + b^{(+)}], \text{if } a^{(-)} + b^{(+)} > a^{(+)} + b^{(-)},\end{cases}\\ &= \begin{cases}[a^{(-)} + b^{(+)}, a^{(+)} + b^{(-)}], \omega(A) \geq \omega(B),\\ [a^{(+)} + b^{(-)}, a^{(-)} + b^{(+)}], \omega(A) < \omega(B),\end{cases}\end{aligned} \tag{15}$$

which is the proper interval with end-points $a^{(-)} + b^{(+)}$ and $a^{(+)} + b^{(-)}$. The interval (15) is called inner (or nonstandard) sum of the proper intervals A, B

and is denoted by $A +^- B$ [Markov 1980]. In contrast, the sum $A+B$ is sometimes called *outer sum* of A, B. We have

$$\mathrm{pro}(\mathbf{A}+\mathbf{B}) = \begin{cases} A+B, & \tau(\mathbf{A}) = \tau(\mathbf{B}), \\ A +^- B, & \tau(\mathbf{A}) \neq \tau(\mathbf{B}). \end{cases}$$

We thus see that for the presentation of the proper projection of a sum of two directed intervals by means of the proper projections of these intervals we need two types of summation of (proper) intervals: an outer summation ("+") and an inner summation ("$+^-$"). For $A, B \in I(\mathbb{R})$ we have $A +^- B \subseteq A + B$. We can characterize the sum $A +^- B$ in the case when A, B are nondegenerate, resp. the four numbers $c_{\alpha\beta} = a^{(\alpha)} + b^{(\beta)}$, $\alpha, \beta \in \Lambda$, are all different, as follows. Arranging the numbers $c_{\alpha\beta}$ in increasing order and renaming them by $c_i, i = 1,2,3,4$, so that $c_1 < c_2 < c_3 < c_4$, we have $A + B = [c_1, c_4]$, $A +^- B = [c_2, c_3]$.

Introducing the notation $+^+ = +$, we can summarize both cases $\alpha = \beta$, $\alpha \neq \beta$ by writing $C = A +^{\alpha\beta} B$, showing that C is either an outer or an inner sum of A and B. We thus obtain the simple expressions

$$\begin{aligned} [A;\alpha] + [B;\beta] &= [A +^{\alpha\beta} B; \tau([A;\alpha] + [B;\beta])], \ A, B \in I(\mathbb{R}), \alpha, \beta \in \Lambda, \\ \mathbf{A} + \mathbf{B} &= [A +^{\tau(\mathbf{A})\tau(\mathbf{B})} B; \tau(\mathbf{A}+\mathbf{B})], \ \mathbf{A}, \mathbf{B} \in D, \end{aligned} \tag{16}$$

wherein the direction $\tau(\mathbf{A}+\mathbf{B}) = \gamma$ is given by (13).

To present a difference $\mathbf{C} = \mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}) = [a^- - b^+, a^+ - b^-]$ in normal form we first compute $\tau(\mathbf{C}) = \tau(\mathbf{A} - \mathbf{B}) = \tau(\mathbf{A} + (-\mathbf{B})) = \sigma(\alpha\omega(A) + \beta\omega(-B)) = \sigma(\alpha\omega(A) + \beta\omega(B)) = \tau(\mathbf{A} + \mathbf{B})$. Note that $\omega(-(B)) = \omega(B)$ and $\tau(\mathbf{B}) = \tau(-\mathbf{B}) = \beta$. Further, using the transition formulae, we compute $c^{(-)} = c^{-\gamma} = a^{-\gamma} - b^{\gamma} = a^{(-\alpha\gamma)} - b^{(\beta\gamma)}$, $c^{(+)} = c^{\gamma} = a^{\gamma} - b^{-\gamma} = a^{(\alpha\gamma)} - b^{(-\beta\gamma)}$, so that

$$\begin{aligned} \mathbf{A} - \mathbf{B} &= [a^{(-)}, a^{(+)}; \alpha] - [b^{(-)}, b^{(+)}; \beta] \\ &= [a^{(-\alpha\gamma)} - b^{(\beta\gamma)}, a^{(\alpha\gamma)} - b^{(-\beta\gamma)}; \tau(\mathbf{A} - \mathbf{B})], \end{aligned} \tag{17}$$

where $\tau(\mathbf{A} - \mathbf{B}) = \tau(\mathbf{A} + \mathbf{B}) = \gamma$ is given by (13).

Expression (17) implies that for $\alpha = \beta\ (= \gamma)$ the normal part $\mathrm{pro}(\mathbf{A} - \mathbf{B})$ of the difference $\mathbf{A} - \mathbf{B}$ is equal to $[a^{(-)} - b^{(+)}, a^{(+)} - b^{(-)}] = A - B = \mathrm{pro}(\mathbf{A}) - \mathrm{pro}(\mathbf{B})$. For $\alpha \neq \beta$ the proper projection $\mathrm{pro}(\mathbf{A} - \mathbf{B})$ is given by

$$\mathrm{pro}(\mathbf{A} - \mathbf{B})|_{\alpha\neq\beta} = \begin{cases} [a^{(-)} - b^{(-)}, a^{(+)} - b^{(+)}], & \omega(A) \geq \omega(B), \\ [a^{(+)} - b^{(+)}, a^{(-)} - b^{(-)}], & \omega(A) < \omega(B), \end{cases}$$

which is the proper interval with end-points $a^{(-)} - b^{(-)}$ and $a^{(+)} - b^{(+)}$. The latter interval is called *inner* (or *nonstandard*) difference of $\mathbf{A}$ and $\mathbf{B}$ and is denoted by $A -^- B$. We may now write

$$\mathrm{pro}(\mathbf{A} - \mathbf{B}) = \begin{cases} A - B, & \tau(\mathbf{A}) = \tau(\mathbf{B}), \\ A -^- B, & \tau(\mathbf{A}) \neq \tau(\mathbf{B}). \end{cases}$$

This shows that for the presentation of the proper projection of the difference of two directed intervals we need two types of subtraction: the familiar (outer) subtraction ("$-$") and the inner subtraction ("$-^-$"). For $A, B \in I(\mathbb{R})$ we have $A -^- B \subseteq A - B$.

In order to obtain an uniform expression for the difference of two directed intervals we introduce the notation $-^{+} = -$; we then summarize

$$[A;\alpha] - [B;\beta] = [A -^{\alpha\beta} B; \tau([A;\alpha] - [B;\beta])],\ A, B \in I(\mathbb{R}), \alpha, \beta \in \Lambda,$$
$$\mathbf{A} - \mathbf{B} = [A -^{\tau(\mathbf{A})\tau(\mathbf{B})} B; \tau(\mathbf{A} - \mathbf{B})],\ \mathbf{A}, \mathbf{B} \in D,$$

wherein $\tau(\mathbf{A} - \mathbf{B}) = \tau(\mathbf{A} + \mathbf{B}) = \sigma(\alpha\omega(A) + \beta\omega(B)) = \gamma$ is given by (13).

Similarly we obtain $[A;\alpha] \times [B;\beta] = [A \times^{\alpha\beta} B; \tau([A;\alpha] \times [B;\beta])]$ for any $A, B \in I(\mathbb{R})^*$, $\alpha, \beta \in \Lambda$, or

$$\mathbf{A} \times \mathbf{B} = [A \times^{\tau(\mathbf{A})\tau(\mathbf{B})} B; \tau(\mathbf{A} \times \mathbf{B})],\ \mathbf{A}, \mathbf{B} \in D^*,$$
$$\tau(\mathbf{A} \times \mathbf{B}) = \tau([A;\alpha] \times [B;\beta]) = \sigma(\alpha\chi(B) + \beta\chi(A)), \tag{18}$$

where $\chi(A) = a^{(-\sigma(A))}/a^{(\sigma(A))}$, $\times^{+} = \times$, and the inner multiplication "$\times^{-}$" is defined by

$$A \times^{-} B = \begin{cases} [a^{(-\sigma(B))}b^{(\sigma(A))}, a^{(\sigma(B))}b^{(-\sigma(A))}], & \chi(B) \geq \chi(A), \\ [a^{(\sigma(B))}b^{(-\sigma(A))}, a^{(-\sigma(B))}b^{(\sigma(A))}], & \chi(B) < \chi(A). \end{cases}$$

In order to present the proper projection of $\mathbf{A}/\mathbf{B} = \mathbf{A} \times (1/\mathbf{B})$ we need two types of interval division for proper intervals — the (outer) division "/" defined by (7) and an "inner" division "$/^{-}$" defined by

$$A /^{-} B = \begin{cases} [a^{(-\sigma(B))}/b^{(-\sigma(A))}, a^{(\sigma(B))}/b^{(\sigma(A))}], & \chi(B) \geq \chi(A), \\ [a^{(\sigma(B))}/b^{(\sigma(A))}, a^{(-\sigma(B))}/b^{(-\sigma(A))}], & \chi(B) < \chi(A). \end{cases}$$

Using the inner and outer divisions for normal intervals we can write

$$\mathbf{A}/\mathbf{B} = [A /^{\tau(\mathbf{A})\tau(\mathbf{B})} B; \tau(\mathbf{A} \times \mathbf{B})],\ \mathbf{A}, \mathbf{B} \in D^*,$$
$$\tau(\mathbf{A}/\mathbf{B}) = \tau([A;\alpha]/[B;\beta]) = \sigma(\alpha\chi(B) + \beta\chi(A)) = \tau(\mathbf{A} \times \mathbf{B}).$$

Let us make two remarks with respect to the inner operations. First, for $A, B \in I(\mathbb{R})$ we have $A *^{-} B \subseteq A * B$ for $* \in \{+, -, \times, /\}$. Second, we can interpret the result of any inner operation $A *^{-} B$, $* \in \{+, -, \times, /\}$, whenever the four numbers $c_{\alpha\beta} = a^{(\alpha)} * b^{(\beta)}$, $\alpha, \beta \in \Lambda$, are different, as follows (for multiplication and division the case when the intervals contain zero should be excluded). Rearrange these four numbers in increasing order and rename them by $c_i, i = 1, 2, 3, 4$, so that: $c_1 < c_2 < c_3 < c_4$. Then we have $A * B = [c_1, c_4]$, $A *^{-} B = [c_2, c_3]$.

We can perform all computations in $\mathcal{K}$ using normal form presentation. Let us give some examples. Multiplication by a degenerate interval a is expressed by $a \times [B;\beta] = a \times [b^{(-)}, b^{(+)}; \beta] = [ab^{(-\sigma(a))}, ab^{(\sigma(a))}; \beta]$. If $a = -1$ then $(-1) \times [b^{(-)}, b^{(+)}; \beta] = -[b^{(-)}, b^{(+)}; \beta] = [-b^{(+)}, -b^{(-)}; \beta] = -[B;\beta] = [-B;\beta]$. The opposite of $\mathbf{A} = [A;\alpha] = [a^{(-)}, a^{(+)}; \alpha]$ is the directed interval $-\mathbf{A}_{-} = [-A; -\alpha] = [-a^{(+)}, -a^{(-)}; -\alpha]$. Indeed from (14) we have $[a^{(-)}, a^{(+)}; \alpha] + [-a^{(+)}, -a^{(-)}; -\alpha] = [0, 0; \pm] = 0$. The inverse to a negative interval is the dual interval $[A;\alpha]_{-} = [A; -\alpha]$. More generally, for $\lambda \in \Lambda$ we have $\mathbf{A}_{\lambda} = [A;\alpha]_{\lambda} = [A; \lambda\alpha]$. Here the binary variable λ is an indicator for a presence/absence of the operator dual element. Similarly, the inversion of $[A;\alpha] = [a^{(-)}, a^{(+)}; \alpha]$ is the directed interval $1/\mathbf{A}_{-} = [1/A; \alpha]_{-} = [1/A; -\alpha] = [1/a^{(+)}, 1/a^{(-)}; -\alpha]$.

4 Relations for normal intervals derived from directed interval arithmetic

We now make the following observation: **Every proposition from the directed interval arithmetic can be reformulated in terms of normal form presentation of the directed intervals involved. It then implies a corresponding proposition for the proper projections of the participating directed intervals, that is a proposition for normal intervals using outer and inner arithmetic operations.** For directed intervals we have simple expressions and relations, due to the fact that the directed intervals form a nice algebraic structure. As we have shown the arithmetic operations between directed intervals generate both outer and inner operations for normal intervals. The set of (proper) intervals together with the set of outer and inner operations $\mathcal{M} = (I(\mathbb{R}), +, +^-, \times, \times^-, \subseteq)$ has been studied in [Dimitrova 1980]–[Dimitrova et al. 1991], [Markov 1977]–[Markov 1992]. We recall that the operations $-, -^-, /, /^-$ can be expressed via the basic operations $+, +^-, \times, \times^-$. The inner interval operations find application in the analysis of interval functions [Markov 1979], [Markov 1980], [Schröder 1981]. Inner operations are useful for the computation of inclusions (both inner and outer) of functional ranges [Bartholomew-Biggs and Zakovich 1994], [Markov 1993], [Nesterov 1993] (see the last section). We next show how some basic arithmetic relations for directed intervals generate corresponding relations between proper intervals. We consequently obtain simple arithmetic relations for the outer and inner operations.

Proposition 2. *Conditionally-associative laws for proper intervals.*
i) For $A, B, C \in I(\mathbb{R})$, and $\alpha, \beta, \gamma \in \Lambda$ we have

$$(A +^{\alpha\beta} B) +^{\gamma\tau(\mathbf{A}+\mathbf{B})} C = A +^{\alpha\tau(\mathbf{B}+\mathbf{C})} (B +^{\beta\gamma} C),$$

ii) For $A, B, C, D \in I(\mathbb{R})$, and $\alpha, \beta, \gamma, \delta \in \Lambda$ we have

$$\begin{aligned}(A +^{\alpha\beta} B) &+^{\tau(\mathbf{A}+\mathbf{B})\tau(\mathbf{C}+\mathbf{D})} (C +^{\gamma\delta} D) \\ &= (A +^{\alpha\gamma} C) +^{\tau(\mathbf{A}+\mathbf{C})\tau(\mathbf{B}+\mathbf{D})} (B +^{\beta\gamma} D),\end{aligned}$$

iii) For $A, B, C \in I(\mathbb{R})^*$ and $\alpha, \beta, \gamma \in \Lambda$ we have

$$(A \times^{\alpha\beta} B) \times^{\gamma\tau(\mathbf{A}\times\mathbf{B})} C = A \times^{\alpha\tau(\mathbf{B}\times\mathbf{C})} (B \times^{\beta\gamma} C),$$

iv) For $A, B, C, D \in I(\mathbb{R})^*$, and $\alpha, \beta, \gamma, \delta \in \Lambda$ we have

$$\begin{aligned}(A \times^{\alpha\beta} B) &\times^{\tau(\mathbf{A}\times\mathbf{B})\tau(\mathbf{C}\times\mathbf{D})} (C \times^{\gamma\delta} D) \\ &= (A \times^{\alpha\gamma} C) \times^{\tau(\mathbf{A}\times\mathbf{C})\tau(\mathbf{B}\times\mathbf{D})} (B \times^{\beta\gamma} D),\end{aligned}$$

wherein the τ-functionals are given by (13), (18).

Proof of i). Substituting $\mathbf{A} = [A;\alpha]$, $\mathbf{B} = [B;\beta]$, $\mathbf{C} = [C;\gamma]$ in $(\mathbf{A}+\mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B}+\mathbf{C})$ and using (16), we obtain $[A +^{\alpha\beta} B; \tau(\mathbf{A}+\mathbf{B})] + [C;\gamma] = [A;\alpha] + [B +^{\beta\gamma} C; \tau(\mathbf{B}+\mathbf{C})]$. Comparing the proper projection of both sides of this equality, we obtain i). □

Using i) we can change the order of execution of the operations in any expression involving additions (outer or inner) of normal intervals. The relation ii) is obtained in a similar way — this relation plays important role in the analysis for interval functions [Markov 1979], [Schröder 1981]. The relations iii), iv) are obtained in a similar way.

Proposition 3. *Conditionally-distributive law for proper intervals. For arbitrary $A, B, C \in I(\mathbb{R})^*$, such that $A + B \in I(\mathbb{R})^*$ and for any $\alpha, \beta, \gamma \in \Lambda$ we have*

$$\begin{aligned}(A +^{\alpha\beta} B) &\times^{\gamma\tau(\mathbf{A}+\mathbf{B})} C \\ &= (A \times^{\alpha\gamma\sigma(A)\sigma(A+B)} C) +^{\tau(\mathbf{A}\times\mathbf{C})\tau(\mathbf{B}\times\mathbf{C})} (B \times^{\beta\gamma\sigma(B)\sigma(A+B)} C).\end{aligned}$$

Proof. Setting $\mathbf{A} = [A;\alpha]$, $\mathbf{B} = [B;\beta]$, ; $\mathbf{C} = [C;\gamma]$ in the conditionally-distributive law for directed intervals and using (16), we obtain

$$\begin{aligned}&[A +^{\alpha\beta} B; \tau(\mathbf{A}+\mathbf{B})] \times [C;\gamma] \\ &= [A \times^{\alpha\gamma\sigma(A)\sigma(A+B)} C; \tau(\mathbf{A}\times\mathbf{C})] + [B \times^{\beta\gamma\sigma(B)\sigma(A+B)} C; \tau(\mathbf{B}\times\mathbf{C})].\end{aligned}$$

Using again (18) for the left hand-side and (16) for the right hand-side and comparing the proper projections in both sides we obtain the proof. □

The generated above basic relations for normal intervals are suitable for automatic processing, especially in computer algebra systems. The corresponding relations known by now (see e. g. [Markov 1992]) have rather complex form and are not convenient for symbolic manipulations.

5 Applications to functional ranges

Let $CM(T)$ be the set of all continuous and monotone functions f defined in $T = [t^{(-)}, t^{(+)}] \in I(\mathbb{R})$. The image $f(T) = \{f(t) \mid t \in T\} \in I(\mathbb{R})$ of the set T by the function f is called the range of f (over T). If $f \in CM(T)$, then for the range of f we have either $f(T) = [f(t^{(-)}), f(t^{(+)})]$ or $f(T) = [f(t^{(+)}), f(t^{(-)})]$ depending on the type of monotonicity. To every $f \in CM(T)$ corresponds a binary variable $\tau_f = \tau(f;T) \in \Lambda$, which determines the type of monotonicity of f by

$$\tau(f;T) = \begin{cases} +, & f(t^-) \le f(t^+); \\ -, & f(t^-) > f(t^+). \end{cases}$$

For $f, g \in CM(T)$, the equality $\tau_f = \tau_g$ means that the functions f, g are both isotone (nondecreasing) or are both antitone (nonincreasing) in T; $\tau_f = -\tau_g$ means that one of the functions is antitone and the other is isotone. Let $CM(T)^*$ be the set of all functions from $CM(T)$ which do not change sign in T. Obviously, if $f \in CM(T)^*$, then $|f| \in CM(T)^*$ as well and the notation $\tau_{|f|} = \tau(|f|;T)$ makes sense. Since the ranges are proper intervals, we may perform interval arithmetic manipulations over them.

Proposition 4. *Let $f, g \in CM(T)$. For $X \subseteq T$ we have*
i) if $f + g \in CM(T)$, then $(f+g)(X) = f(X) +^{\tau_f \tau_g} g(X)$,
ii) if $f - g \in CM(T)$, then $(f-g)(X) = f(X) -^{-\tau_f \tau_g} g(X)$.
Let $f, g \in CM(T)^*$. For $X \subseteq T$:
iii) if $fg \in CM(T)$, then $(fg)(X) = f(X) \times^{\tau_{|f|} \tau_{|g|}} g(X)$,
iv) if $f/g \in CM(T)$, $g(x) \neq 0$, $\forall x \in T$, then $(f/g)(X) = f(X) /^{-\tau_{|f|} \tau_{|g|}} g(X)$.

If f is continuous in $X \in I(\mathbb{R})$, then $\min_{x \in X} f(x)$, $\max_{x \in X} f(x)$ do exist. Assuming further that the interval X is fixed, we shall shortly write $\min f$, resp. $\max f$. Let f, g be continuous in X. We have:
i) $\min f + \min g \leq \min(f+g)$, $\max(f+g) \leq \max f + \max g$;
ii) $\min(f+g) \leq \min f + \max g \leq \max(f+g)$, $\min(f+g) \leq \max f + \min g \leq \max(f+g)$.

This implies that the interval $(f+g)(X) = [\min(f+g), \max(f+g)]$:
i) is contained in the interval with endpoints $\min f + \min g$, $\max f + \max g$, that is in the interval $f(X) + g(X)$;
ii) contains the interval with endpoints $\min f + \max g$, $\max f + \min g$, that is the interval $f(X) +^- g(X)$.

Symbolically, we obtain $f(X) +^- g(X) \subseteq (f+g)(X) \subseteq f(X) + g(X)$. Using similar arguments for the rest of operations we obtain:

Proposition 5. *Let the functions f, g be continuous in $T \in I(\mathbb{R})$. For $* \in \{+, -, \times, /\}$ and for every $X \subseteq T, X \in I(\mathbb{R})$ we have $f(X) *^- g(X) \subseteq (f * g)(X) \subseteq f(X) * g(X)$. (For "/" we additionally assume that g does not vanish in T.)*

The above proposition shows that the outer operations are convenient for the computation of outer inclusions, whereas the inner interval operations may serve for the computation of inner inclusions. Examples for the use of inner interval operations can be found in [Bartholomew-Biggs and Zakovich 1994], [Dimitrova and Markov 1994], [Markov 1979], [Markov 1980], [Markov 1993], [Nesterov 1993], [Schröder 1981], [Stetter 1990].

We shall now formulate an analogue of Proposition 4 for directed intervals. We first define directed range by admitting improper intervals as arguments:

Definition. Let $T \in I(\mathbb{R})$, $f \in CM(T)$. Let $\mathbf{X} = [x^-, x^+] \in D$, $\mathrm{pro}(\mathbf{X}) \subseteq T$. The *directed range* of f over $\mathbf{X}$ is the directed interval $f(\mathbf{X}) = [f(x^-), f(x^+)]$.

Proposition 6. *Let $f, g \in CM(T)$. For $\mathbf{X} \in D$, pro$(\mathbf{X}) \subseteq T$, we have*
i) if $f + g \in CM(T)$, then $(f+g)(\mathbf{X}) = f(\mathbf{X}) + g(\mathbf{X})$;
ii) if $f - g \in CM(T)$, then $(f-g)(\mathbf{X}) = f(\mathbf{X}) - g(\mathbf{X})_-$.
Let $f, g \in CM(T)^*$. For $\mathbf{X} \in D$, pro$(\mathbf{X}) \subseteq T$
iii) if $fg \in CM(T)$, then $(fg)(\mathbf{X}) = f(\mathbf{X})_{\sigma(g(X))} \times g(\mathbf{X})_{\sigma(f(X))}$;
iv) if $f/g \in CM(T)$, $0 \notin g(X)$, then $(f/g)(\mathbf{X}) = f(\mathbf{X})_{\sigma(g(X))} / g(\mathbf{X})_{-\sigma(f(X))}$.

In the last two expressions $\sigma(f(X))$ means the sign of the interval $f(X) \in I(\mathbb{R})^*$ (we can equally well write either $\sigma(f(\mathbf{X}))$ or $\sigma(f(X))$. For example, we have $g(\mathbf{X})_{\sigma(f(X))} = \{g(\mathbf{X}), \text{ if } f \geq 0; (g(\mathbf{X}))_-, \text{ if } f \leq 0\}$.

Proposition 6 gives the direction of the resulting intervals, and therefore supplies additional information (compared to Proposition 4) for the type of monotonicity (isotonicity or antitonicity) of the result $f * g$, $* \in \{+, -, \times, /\}$.

The propositions in this section can be incorporated in algorithms for automatic computation of inner and outer inclusions of ranges of functions and their derivatives. Such algorithms should use a suitable environment supporting the interval arithmetic system $\mathcal{K}$, resp. its subsystem $\mathcal{M}$.

6 Conclusion

We have shown that the set $I(\mathbb{R}) \otimes \{+,-\}$ of directed intervals can be used to establish a practical relation between the interval arithmetic $I(\mathbb{R})$ and its algebraic completion D, considered by E. Kaucher. Moreover, this relation generates the inner arithmetic operations in the set of proper intervals $I(\mathbb{R})$, which have proved to be useful for the computation of functional ranges and for the interpretation of algebraic results obtained in D by means of proper intervals, which are proper projections of directed intervals.

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