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On Four Classes of Lindenmayerian Power Series

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Abstract: We show that nonzero axioms add to the generative capacity of Lindenmayerian series generating systems. On the other hand, if nonzero axioms are allowed, nonterminals do not, provided that only quasiregular series are considered.
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1 Introduction

To define formal power series generated by L systems, Lindenmayerian series generating systems were introduced in [Honkala 95]. There four classes of morphically generated series were defined. The smallest class $\mathcal{S}(LS_0)$ consists of LS series with the zero axiom. The larger class $\mathcal{S}(LS)$ is obtained if arbitrary axioms are allowed. From these classes the classes $\mathcal{S}(ELS_0)$ and $\mathcal{S}(ELS)$ of ELS series with the zero axiom and ELS series, respectively, are obtained by allowing the use of Hadamard products. In an obvious way this corresponds to the use of nonterminals in language theory.

The inclusions $\mathcal{S}(LS_0) \subseteq \mathcal{S}(LS)$ and $\mathcal{S}(ELS_0) \subseteq \mathcal{S}(ELS)$ are clear by the definitions. It was shown in [Honkala 95] that $\mathcal{S}(LS_0)$ is properly contained in $\mathcal{S}(ELS_0)$. Hence, in the case of the zero axiom nonterminals do add to the generative capacity. The purpose of this note is to prove that $\mathcal{S}(ELS_0)$ is properly contained in $\mathcal{S}(ELS)$ and, furthermore, that the classes $\mathcal{S}(LS)$ and $\mathcal{S}(ELS)$ are equivalent, if only quasiregular power series are considered. Hence, nonzero axioms do add to the generative capacity whereas, if nonzero axioms are allowed, nonterminals do not. However, it will be seen below that in the framework of Lindenmayerian series with nonzero axioms some terminals play the role of nonterminals.

Another approach to define a power series generalization of L systems is given in [Kuich 94]. For the relationship between the two approaches see [Honkala and Kuich 95]. In the case of a complete semiring, power series generalizations of L systems are also discussed in [Honkala 94a] and [Honkala and Kuich 00].

2 Definitions

It is assumed that the reader is familiar with the basics of the theories of semirings and formal power series as developed in [Kuich and Salomaa (86)]. In this paper A will always be a commutative semiring and Σ is a finite alphabet. Suppose $h : \Sigma^* \longrightarrow A < \Sigma^* >$ is a monoid morphism. (Here $A < \Sigma^* >$ is regarded as a multiplicative monoid.) Then we extend h to a semiring morphism

$$h : A < \Sigma^* > \longrightarrow A < \Sigma^* >$$

by

$$h(P) = \sum (P, w) h(w), P \in A < \Sigma^* > .$$

Notice that the assumption of commutativeness is needed in the verification that indeed $h(r_1 r_2) = h(r_1)h(r_2)$ for $r_1, r_2 \in A < \Sigma^* >$. In the sequel we always tacitly extend a morphism $h \in \mathrm{Hom}(\Sigma^*, A < \Sigma^* >)$ to a semiring morphism $h : A < \Sigma^* > \longrightarrow A < \Sigma^* >$ as explained above. Notice that $\mathrm{Hom}(\Sigma^*, A < \Sigma^* >)$, the set of all these semiring morphisms, can be identified with the set

$$\{h : A < \Sigma^* > \longrightarrow A < \Sigma^* > \mid h \text{ is a semiring morphism and } h(a \cdot \lambda) = a \cdot \lambda \text{ for any } a \in A\}.$$

In what follows X is a denumerably infinite alphabet of variables. An *interpretation* φ over (A, Σ) is a mapping from X to $\mathrm{Hom}(\Sigma^*, A < \Sigma^* >)$. A *Lindenmayerian series generating system*, shortly, an LS system, is a 5-tuple $G = (A \ll \Sigma^* \gg, \mathcal{D}, P, \varphi, \omega)$ where A is a commutative semiring, Σ is a finite alphabet, $\mathcal{D}$ is a convergence in $A \ll \Sigma^* \gg$, P is a polynomial in $A < (X \cup \Sigma)^* >$, φ is an interpretation over (A, Σ) and ω is a polynomial in $A < \Sigma^* >$.

The series generated by an LS system is obtained by iteration. Suppose $G = (A \ll \Sigma^* \gg, \mathcal{D}, P(x_1, \ldots, x_n), \varphi, \omega)$ is an LS system and denote $h_i = \varphi(x_i)$ for $1 \leq i \leq n$. Define the sequence $(r^{(j)})$ $(j = 0, 1, \ldots)$ recursively by

$$r^{(0)} = \omega,$$
$$r^{(j+1)} = P(h_1(r^{(j)}), \ldots, h_n(r^{(j)})), j \geq 0.$$

If $\lim r^{(j)}$ exists we denote

$$S(G) = \lim r^{(j)}$$

and say that $S(G)$ is the *series generated* by G. The sequence $(r^{(j)})$ is the *approximation sequence associated to* G. A series r is called an LS *series* if there exists an LS system G such that $r = S(G)$. A series r is an LS *series with the zero axiom* if there exists an LS system $G = (A \ll \Sigma^* \gg, \mathcal{D}, P, \varphi, 0)$ such that $r = S(G)$. ELS series are obtained from LS series by considering only terms over a terminal alphabet. Formally, an ELS system is a construct $G = (A \ll \Sigma^* \gg, \mathcal{D}, P, \varphi, \omega, \Delta)$ consisting of the LS system $U(G) = (A \ll \Sigma^* \gg, \mathcal{D}, P, \varphi, \omega)$ called the *underlying system* of G and a subset Δ of Σ. If $S(U(G))$ exists, G generates the series

$$S(G) = S(U(G)) \odot \mathrm{char}(\Delta^*).$$

A series r is called an ELS *series* if there exists an ELS system G such that $r = S(G)$. A series r is called an ELS *series with the zero axiom* if there exists an ELS system $G = (A \ll \Sigma^* \gg, \mathcal{D}, P, \varphi, 0, \Delta)$ such that $r = S(G)$.

If A and $\mathcal{D}$ are understood, the class of LS series with the zero axiom (resp. LS series, ELS series with the zero axiom, ELS series) is denoted by $\mathcal{S}(LS_0)$ (resp. $\mathcal{S}(LS)$, $\mathcal{S}(ELS_0)$, $\mathcal{S}(ELS)$).

In the sequel we will always use the convergence $\mathcal{D}_d$ obtained by transferring the discrete convergence in A to $A \ll \Sigma^* \gg$ as explained in [Kuich and Salomaa (86)].

3 Results

The purpose of this section is to prove the following result.

Theorem 1. *(i) If A is a commutative semiring and $r \in A \ll \Sigma^* \gg$ is quasiregular, then $r \in \mathcal{S}(LS)$ if and only if $r \in \mathcal{S}(ELS)$.*

(ii) If $A = N$, then $\mathcal{S}(LS_0)$ is properly included in $\mathcal{S}(ELS_0)$ and $\mathcal{S}(ELS_0)$ is properly included in $\mathcal{S}(ELS)$.

Lemma 2. *Suppose $G = (A \ll \Sigma^* \gg, \mathcal{D}_d, P(x_1, \ldots, x_n), \varphi, \omega)$ is an LS system such that $S(G)$ exists and is quasiregular. If $\Delta \subseteq \Sigma$, then there exists an LS system G_1 such that*

$$S(G_1) = S(G) \odot char(\Delta^*).$$

Proof. We assume without loss of generality that each term of the approximation sequence $(r^{(i)})$ of G is quasiregular.

Choose new letters $\#, \$ \notin \Sigma$ and new variables z_1 and z_2. Let $\Sigma^{(1)} = \{\sigma^{(1)} | \sigma \in \Sigma\}$ be an isomorphic copy of Σ and let $\mathrm{copy}_1 : \Sigma \longrightarrow \Sigma^{(1)}$ be the mapping defined by $\mathrm{copy}_1(\sigma) = \sigma^{(1)}$. Denote $P = P_0 + P_1$ where $P_0 \in A < \Sigma^* >$ and each term of P_1 contains a variable. Define $R = \#P_1 + \#z_1 + z_2$. If $x \in \{x_1, \ldots, x_n\}$, define $\varphi_1(x)$ by

$$\varphi_1(x)(\sigma) = \begin{cases} \varphi(x)(\sigma) & \text{if } \sigma \in \Sigma \\ \# & \text{if } \sigma = \# \\ 0 & \text{otherwise} \end{cases}$$

and $\varphi_1(z_1)$ by

$$\varphi_1(z_1)(\sigma) = \begin{cases} \$ + P_0 & \text{if } \sigma = \$ \\ \# & \text{if } \sigma = \# \\ 0 & \text{otherwise} \end{cases}$$

and $\varphi_1(z_2)$ by

$$\varphi_1(z_2)(\sigma) = \begin{cases} \mathrm{copy}_1(\sigma) & \text{if } \sigma \in \Delta \\ \lambda & \text{if } \sigma = \# \\ 0 & \text{otherwise} \end{cases} .$$

Define the LS system G_1 by $G_1 = (A \ll (\Sigma \cup \Sigma^{(1)} \cup \# \cup \$)^* \gg, \mathcal{D}_d, R(x_1, \ldots, x_n, z_1, z_2), \varphi_1, \omega + \$)$. Denote the approximation sequence associated to G_1 by $(s^{(i)})$. It follows inductively that there exists a sequence $(t^{(i)})$ such that

$$s^{(i)} = t^{(i)} + \mathrm{copy}_1(r^{(i-1)} \odot \mathrm{char}(\Delta^*))$$

and

$$\mathrm{proj}_{\Sigma \cup \$}(t^{(i)}) = r^{(i)} + \$$$

for $i \geq 1$. Furthermore, each word in $\mathrm{supp}(t^{(i)})$ contains at least i occurrences of $\#$. (Here the morphism $\mathrm{proj}_{\Sigma \cup \$}$ is defined by $\mathrm{proj}_{\Sigma \cup \$}(\sigma) = \sigma$ if $\sigma \in \Sigma \cup \$$, and $\mathrm{proj}_{\Sigma \cup \$}(\sigma) = \lambda$ if $\sigma \notin \Sigma \cup \$$.) This implies $\lim t^{(i)} = 0$. Therefore $\lim s^{(i)}$ exists and

$$\lim s^{(i)} = \mathrm{copy}_1(S(G) \odot \mathrm{char}(\Delta^*)).$$

Now the claim follows by renaming the letters. □

In the proof of Lemma 2 the letters of $\Sigma \cup \# \cup \$$ play the role of nonterminals. However, because $\lim s^{(i)}$ does not contain any letters of $\Sigma \cup \# \cup \$$, we do not need the Hadamard product.

Next we recall some earlier results.

Lemma 3. *Let $A = N$ and denote $r = \sum_{n\geq 1}(a^n b^n + b^n a^n) \in N \ll \{a,b\}^* \gg$. Then $r \notin \mathcal{S}(LS_0)$ and $r \in \mathcal{S}(ELS_0)$.*

Proof. The claim is shown in Examples 3.6 and 4.3 of [Honkala 95]. □

Lemma 4. *Let $A = N$. Then the series $\sum_{n\geq 1}(a^n b)^n$ does not belong to $\mathcal{S}(ELS_0)$.*

Proof. See [Honkala 94b]. □

For the next lemma we need a definition. A *vector of LS systems* of dimension $k \geq 1$ is a k-tuple $\overline{G} = ((A \ll \Sigma^* \gg, \mathcal{D}, P_i(x_{11}, \ldots, x_{1k}, \ldots, x_{n1}, \ldots, x_{nk}), \varphi_i, \omega_i))_{1\leq i\leq k}$ of LS systems. The *approximation sequence* $((r_{j,1}, \ldots, r_{j,k}))_{j\geq 0}$ *associated to* G is defined recursively by

$$r_{0,s} = \omega_s,$$

$$r_{j+1,s} = P_s(\varphi_s(x_{11})(r_{j,1}), \ldots, \varphi_s(x_{1k})(r_{j,k}), \ldots, \varphi_s(x_{n1})(r_{j,1}), \ldots, \varphi_s(x_{nk})(r_{j,k})), 1 \leq s \leq k.$$

If $\lim_{j\to\infty} r_{j,s}$ exists for every $1 \leq s \leq k$, then we denote

$$S(\overline{G}) = (\lim r_{j,1}, \ldots, \lim r_{j,k})$$

and say that $S(\overline{G})$ is the *(vector of) series generated by* $\overline{G}$.

The next lemma is stated and proved as Theorem 4.5 in [Honkala 95].

Lemma 5. *Suppose $\overline{G} = ((A \ll \Sigma^* \gg, \mathcal{D}_d, P_i(x_{11}, \ldots, x_{1k}, \ldots, x_{n1}, \ldots, x_{nk}), \varphi_i, \omega_i))_{1\leq i\leq k}$ is a vector of LS systems such that $S(\overline{G}) = (r^{(1)}, \ldots, r^{(k)})$ exists and $r^{(1)}, \ldots, r^{(k)}$ are quasiregular. Then $r^{(s)}$ is an ELS series for any s.*

Lemma 6. *Let $A = N$. Then the series $\sum_{n\geq 1}(a^n b)^n$ belongs to $\mathcal{S}(ELS)$.*

Proof. Denote $\Sigma = \{a, b\}$ and define the LS systems G_1, G_2, G_3 by

$$G_1 = (N \ll \Sigma^* \gg, \mathcal{D}_d, x_{11}x_{12}, \varphi_1, ab),$$

$$G_2 = (N \ll \Sigma^* \gg, \mathcal{D}_d, x_{22}, \varphi_2, ab),$$

$$G_3 = (N \ll \Sigma^* \gg, \mathcal{D}_d, x_{31} + x_{33}, \varphi_3, 0)$$

where $\varphi_3(x_{31}) = \varphi_3(x_{33})$ is the identity morphism and $\varphi_1(x_{11}) = \varphi_1(x_{12}) = \varphi_2(x_{22}) = h$ is defined by $h(a) = a, h(b) = ab$. Furthermore, define the 3-dimensional vector $\overline{G}$ of LS systems by $\overline{G} = (G_1, G_2, G_3)$. Denote by $((r_{j,1}, r_{j,2}, r_{j,3}))_{j\geq 0}$ the approximation sequence of $\overline{G}$. Then

$$r_{0,1} = ab, r_{0,2} = ab, r_{0,3} = 0,$$

$$r_{j+1,1} = h(r_{j,1})h(r_{j,2}),$$
$$r_{j+1,2} = h(r_{j,2}),$$
$$r_{j+1,3} = r_{j,1} + r_{j,3}$$

for $j \geq 0$. It follows inductively that

$$r_{j,1} = (a^{j+1}b)^{j+1},$$
$$r_{j,2} = a^{j+1}b,$$
$$r_{j,3} = \sum_{1 \leq n \leq j} (a^n b)^n$$

for $j \geq 0$. Therefore $\lim r_{j,1} = \lim r_{j,2} = 0$ and $\lim r_{j,3} = \sum_{n \geq 1}(a^n b)^n$. Hence $S(\overline{G})$ exists and each component of $S(\overline{G})$ is quasiregular. Therefore the claim follows by Lemma 5. □

Proof of Theorem 1. Claim (i) follows by Lemma 2. Claim (ii) is a consequence of Lemmas 3,4 and 6. □

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