Journal of Universal Computer Science, vol. 1, no. 10 (1995), 675-686
submitted: 31/5/95, accepted: 7/11/95, appeared: 28/10/95©Springer Pub. Co.

# An aperiodic set of Wang cubes [1]

Karel Culik II [2]
*Department of Computer Science*
*University of South Carolina*
*Columbia, S.C. 29208*
Jarkko Kari
Iterated Systems, Inc.
3525 Piedmont Road, Seven Piedmont Center, Suite 600
Atlanta, G.A. 30305-1530

**Abstract:** We introduce Wang cubes with colored faces that are a generalization of Wang tiles with colored edges. We show that there exists an aperiodic set of 21 Wang cubes, that is, a set for which there exists a tiling of the whole space with matching unit cubes but there exists no periodic tiling. We use the aperiodic set of 13 Wang tiles recently obtained by the first author using the new method developed by the second. Our method can be used to construct an aperiodic set of $n$-dimensional cubes for any $n \geq 3$.

**Key Words:** discrete mathematics, automata theory, aperiodic tilings, Wang tiles, Wang cubes, sequential machines

**Category:** G.2, F.1.1

## 1 Introduction

*Wang tiles* are unit square tiles with colored edges. The tile whose left, right, top and bottom edges have colors $l, r, t$ and $b$, respectively, is denoted by the 4-tuple $(l, r, t, b)$. A *tile set* is a finite set of Wang tiles. *Tilings* of the infinite Euclidean plane are considered using arbitrarily many copies of the tiles in the given tile set. The tiles are placed on the integer lattice points of the plane with their edges oriented horizontally and vertically. The tiles may not be rotated. A tiling is *valid* if everywhere the contiguous edges have the same color.

Let $T$ be a finite tile set, and $f : \mathbb{Z}^2 \to T$ a tiling. Tiling $f$ is *periodic* with period $(a, b) \in \mathbb{Z}^2 - \{(0, 0)\}$ iff $f(x, y) = f(x + a, y + b)$ for every $(x, y) \in \mathbb{Z}^2$. If there exists a periodic valid tiling with tiles of $T$, then there exists a *doubly periodic* valid tiling, i.e. a tiling $f$ such that, for some $a, b > 0$, $f(x, y) = f(x + a, y) = f(x, y + b)$ for all $(x, y) \in \mathbb{Z}^2$. A tile set $T$ is called *aperiodic* iff (i) there exists a valid tiling, and (ii) there does not exist any valid periodic tiling.

R. Berger in his well known proof of the undecidability of the tiling problem [2] refuted Wang's conjecture that no aperiodic set exists, and constructed the first aperiodic set containing 20426 tiles. He shortly reduced it to 104 tiles. Between 1966 and 1978 progressively smaller aperiodic sets were found by Knuth, Läuchli, Robinson, Penerose and finally a set of 16 tiles by R. Ammann. A discussion of these and related results is in chapters 10 and 11 of [6]. Recently, the second author developed a new method for constructing aperiodic sets that is not based on geometry, as are the earlier ones, but on sequential machines that multiply real numbers by rational constants. This approach makes short and precise correctness arguments possible. He used it to construct a new aperiodic set containing only 14 tiles over 6 colors in [8]. The first author added an additional trick in [3] and obtained an aperiodic set consisting of 13 tiles over 5 colors which we will use here to construct an aperiodic set of Wang cubes.

General 3-D tilings have been extensively studied as applied to crystallography and theoretical physics, see [7]. We introduce Wang cubes, the obvious generalization of Wang tiles to three dimensions. A *Wang cube* is a unit cube with colored faces. The cube with colors $l, r, f, g, t, b$ at the left, right, front, back, top, bottom faces, respectively, will be denoted by the six-tuple $(l, r, f, g, t, b)$. A cube set is a finite set of Wang cubes. We consider 3-D tilings of the infinite Euclidean 3-D space using arbitrarily many copies of the Wang cubes from the given cube set. The cubes are placed on the integer lattice points of the space with their sides oriented parallel to the $xy, xz$ and $yz$ coordinate planes. The cubes may not be rotated. A tiling is valid if everywhere the contiguous faces have the same color.

Let $S$ be a cube set and $f : \mathbb{Z}^3 \to S$ a 3-D tiling. Tiling $f$ is *periodic* with period $(a, b, c) \in \mathbb{Z}^3 - \{(0, 0, 0)\}$ if $f(x, y, z) = f(x + a, y + b, z + c)$ for every $(x, y, z) \in \mathbb{Z}^3$. A cube set $S$ is called *aperiodic* if

**(i)** there exists a valid 3-D tiling, and
**(ii)** there does not exist any valid periodic 3-D tiling.

Clearly, we can extend our definition to $n$-dimensional Wang cubes and $n$-dimensional tilings.

A 3-D tiling $f : \mathbb{Z}^3 \to S$ is called *triply periodic* if for some $a, b, c > 0$, and for all $x, y, z \in \mathbb{Z}$ we have $f(x, y, z) = f(x + a, y, z) = f(x, y + b, z) = f(x, y, z + c)$.

Note that in a plane every set of tiles that admits a periodic tiling also admits a doubly periodic tiling. Similarly, in space the existence of a doubly periodic 3-D tiling implies the existence of a triply periodic tiling. However, there are sets of cubes that admit a periodic tiling but not any triply periodic tiling, for example, sets $W_1$ and $W_2$ in Section 4.

In Section 2 we review the relation between sets of Wang tiles and sequential machines, introduce the balanced representation of reals, and show how to construct a sequential machine that implements the multiplication of a number in balanced representation by a constant. These techniques are then used in Section 3 to construct tile set $T_{13}$ and prove its aperiodicity. We include Sections 2 and 3 not only to make this paper self-contained but because we need to use some properties of the computations of the sequential machine corresponding to $T_{13}$ in the proof of our main result in Section 4. In the last Section we construct three sets of cubes which are progressively less and less periodic. The first, $W_1$, admits tilings which are periodic in every non-horizontal direction. The second set, $W_2$, admits tilings that could be periodic in the vertical direction only. Finally, we present our main result, an aperiodic set of 21 cubes over 7 colors.

## 2 Balanced representation of numbers, sequential machines and tile sets

For an arbitrary real number $r$ we denote by $\lfloor r \rfloor$ the integer part of $r$, i.e. the largest integer that is not greater than $r$, and by $\{r\}$ the fractional part $r - \lfloor r \rfloor$. In proving that our tile set can be used to tile the plane we use *Beatty sequences*

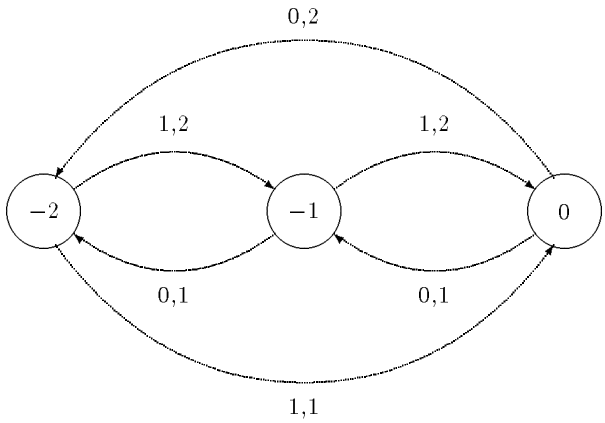



**Figure 1:** Sequential machine $M_3$.

of numbers. Given a real number $\alpha$ its bi-infinite Beatty sequence is the integer sequence $A(\alpha)$ consisting of the integral parts of the multiples of $\alpha$. In other words, for all $i \in \mathbb{Z}$,

$$A(\alpha)_i = \lfloor i \cdot \alpha \rfloor.$$

Beatty sequences were introduced by S.Beatty [1] in 1926.

We use sequences obtained by computing the differences of consecutive elements of Beatty sequences. Define, for every $i \in \mathbb{Z}$,

$$B(\alpha)_i = A(\alpha)_i - A(\alpha)_{i-1}.$$

The bi-infinite sequence $B(\alpha)_i$ will be called the *balanced* representation of $\alpha$. The balanced representations consist of at most two different numbers: If $k \leq \alpha \leq k+1$ then $B(\alpha)$ is a sequence of $k$'s and $(k+1)$'s. Moreover, the averages over finite subsequences approach $\alpha$ as the lengths of the subsequences increase. In fact, the averages are as close to $\alpha$ as they can be: The difference between $l \cdot \alpha$ and the sum of any $l$ consecutive elements of $B(\alpha)$ is always smaller than one.

For example,

$$B(1.5) = \ldots 121212 \ldots,\ B(\tfrac{1}{3}) = \ldots 001001 \ldots \text{ and } B(\tfrac{8}{3}) = \ldots 233233 \ldots.$$

Now, we introduce sequential machines which define mappings on bi-infinite strings. We will use them to implement multiplication of numbers in balanced representation and later shown that they are isomorphic to set of tiles.

A *sequential machine* is a 4-tuple $M = (K, \Sigma, \Delta, \gamma)$ where $K$ is the set of states, $\Sigma$ is the input alphabet, $\Delta$ is the output alphabet, and $\gamma \subseteq K \times \Sigma \times \Delta \times K$ is the transition set. Sequential machine $M$ can be represented by a labeled directed graph with nodes $K$ and an edge from node $q$ to node $p$ labeled $a, b$ for each transition $(q, a, b, p)$ in $\gamma$.

Machine $M$ computes a relation $\rho(M)$ between bi-infinite sequences of letters. A bi-infinite sequence $x$ over set $S$ is a function $x : \mathbb{Z} \to S$. We will abbreviate $x(i)$ by $x_i$. Bi-infinite sequences $x$ and $y$ over input and output alphabets, respectively, are in relation $\rho(M)$ if and only if there is a bi-infinite sequence $s$

of states of $M$ such that, for every $i \in \mathbb{Z}$, there is a transition from $s_{i-1}$ to $s_i$ labeled by $x_i, y_i$.

For a given positive rational number $q = \frac{n}{m}$, let us construct a sequential machine (nondeterministic Mealy machine) $M_q$ that multiplies a real number in balanced representation $B(\alpha)$ by $q$. The states of $M_q$ will represent all possible values of $q\lfloor r \rfloor - \lfloor qr \rfloor$ for $r \in \mathbb{R}$. Because

$$q\lfloor r \rfloor - 1 \leq qr - 1 < \lfloor qr \rfloor \leq qr < q(\lfloor r \rfloor + 1),$$

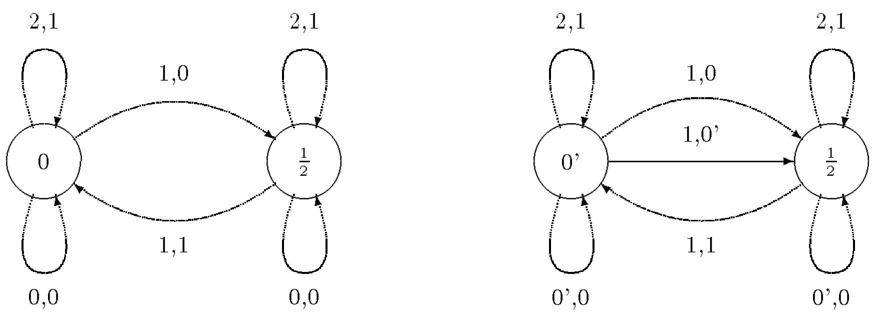
we have

$$-q < q\lfloor r \rfloor - \lfloor qr \rfloor < 1.$$

Because the possible values of $q\lfloor r \rfloor - \lfloor qr \rfloor$ are multiples of $\frac{1}{m}$, they are among the $n + m - 1$ elements of

$$S = \{-\frac{n-1}{m}, -\frac{n-2}{m}, \ldots, \frac{m-2}{m}, \frac{m-1}{m}\}.$$

$S$ is the state set of $M_q$.




(a) $M_{\frac{1}{2}}$ (b) $M'_{\frac{1}{2}}$

**Figure 2:** Sequential machines $M_{\frac{1}{2}}$ and $M'_{\frac{1}{2}}$.

The transitions of $M_q$ are constructed as follows: There is a transition from state $s \in S$ with input symbol $a$ and output symbol $b$ into state $s + qa - b$, if such a state exists. If there is no state $s + qa - b$ in $S$ then no transition from $s$ with label $a, b$ is needed. After reading input $\ldots B(\alpha)_{i-2}\ B(\alpha)_{i-1}$ and producing output $\ldots B(q\alpha)_{i-2}\ B(q\alpha)_{i-1}$, the machine is in state

$$s_{i-1} = qA(\alpha)_{i-1} - A(q\alpha)_{i-1} \in S.$$

On the next input symbol $B(\alpha)_i$ the machine outputs $B(q\alpha)_i$ and moves to state

$$\begin{aligned} s_{i-1} + qB(\alpha)_i - B(q\alpha)_i &= qA(\alpha)_{i-1} + qB(\alpha)_i - (A(q\alpha)_{i-1} + B(q\alpha)_i) \\ &= qA(\alpha)_i - A(q\alpha)_i \\ &= s_i \in S \end{aligned}$$

The sequential machine was constructed in such a way that the transition is possible. This shows that if the balanced representation $B(\alpha)$ is a sequence of input letters and $B(q\alpha)$ is over output letters, then $B(\alpha)$ and $B(q\alpha)$ are in relation $\rho(M_q)$.

Sequential machine $M_3$ in Fig. 1 is constructed in this fashion for multiplying by 3, using input symbols $\{0,1\}$ and output symbols $\{1,2\}$. This means that $B(\alpha)$ and $B(3\alpha)$ are in relation $\rho(M_3)$ for all real numbers $\alpha$ satisfying $0 \leq \alpha \leq 1$ and $1 \leq 3\alpha \leq 2$, that is, for all $\alpha \in \left[\frac{1}{3}, \frac{2}{3}\right]$. Similarly, $M_{1/2}$, shown in Fig. 2(a), is constructed for input symbols $\{0,1,2\}$ and output symbols $\{0,1\}$, so that $B(\alpha)$ and $B(\frac{1}{2}\alpha)$ are in relation $\rho(M_{1/2})$ for all $\alpha \in [0,2]$.

Our intention is to iterate sequential machines $M_3$ and $M_{\frac{1}{2}}$ without allowing $M_{\frac{1}{2}}$ to be used more than twice in a row. To assure that we modify $M_{\frac{1}{2}}$ by introducing new input/output symbol $0'$ and changing its diagram to $M'_{\frac{1}{2}}$ as shown in Fig. 2(b). We also change the state 0 to $0'$ to make the sets of states of $M_3$ and $M'_{\frac{1}{2}}$ disjoint. That allows us to view the union of $M_3$ and $M'_{\frac{1}{2}}$ as one sequential machine $M$.

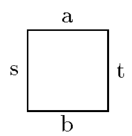



**Figure 3:** The tile $(s,a,b,t)$ corresponding to the transition $s \xrightarrow{a,b} t$

There is a one-to-one correspondence between the tile sets and sequential machines which translates the properties of tile sets to properties of computations of sequential machines.

A finite tile set $T$ over set of colors $C_{EW}$ on east-west edges and set of colors $C_{NS}$ on north-south edges is represented by sequential machine $M = (C_{EW}, C_{NS}, C_{NS}, \gamma)$ where $(s,a,b,t) \in \gamma$ iff there is a tile $(s,a,b,t)$ in $T$ , see Fig. 3.

Obviously, bi-infinite sequences $x$ and $y$ are in the relation $\rho(M)$ iff there exists a row of tiles, with matching vertical edges, whose upper edges form sequence $x$ and lower edges sequence $y$. So there is a one-to-one correspondence between valid tilings of the plane, and bi-infinite iterations of the sequential machine on bi-infinite sequences.

Clearly, the two conditions for $T$ being aperiodic can be translated to conditions on computations of $M$. Clearly, set $T$ is aperiodic if (i) there exists a bi-infinite computation of $M$, and (ii) there is no bi-infinite word $w$ over $C_{NS}$ such that $(w,w) \in [\rho(M)]^+$, where $\rho^+$ denotes the transitive closure of $\rho$.

## 3 An aperiodic set of tiles

We say that the tile in Fig. 3 multiplies by $q$ if $aq + s = b + t$.

Let denote by $T_3$ and by $T_{\frac{1}{2}}$ the tile sets representing the sequential machines $M_3$ and $M'_{\frac{1}{2}}$, respectively. Therefore, $T_3$ and $T_{\frac{1}{2}}$ multiply by 3 or by 1/2, respectively. The tile set $T_{13} = T_3 \cup T_{\frac{1}{2}}$, consisting of 13 tiles, is shown in Fig. 4.

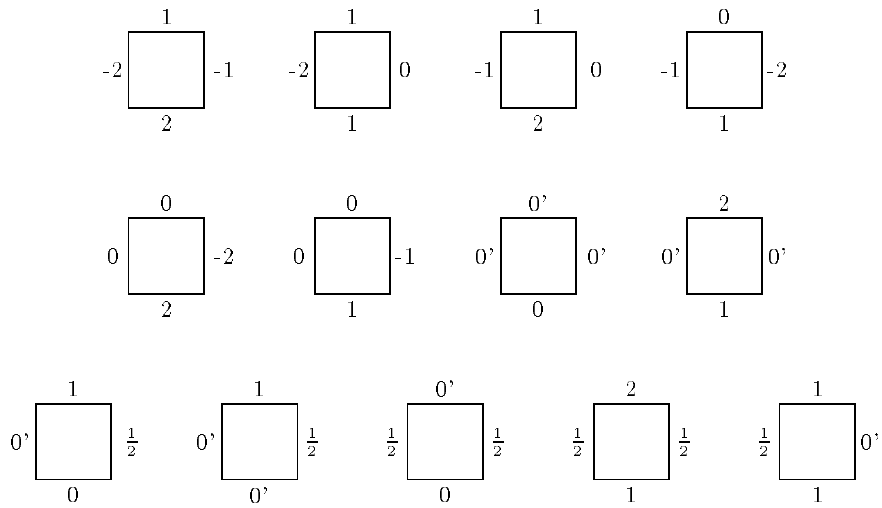



**Figure 4:** $T_{13}$ — an aperiodic set of 13 Wang tiles

Now, we proceed to prove that $T_{13}$ is an aperiodic set of tiles.

**Theorem 1.** *The tile set $T_{13}$ is aperiodic.*

1. We show that set $T$ admits uncountably many valid tilings of the plane. For any $\alpha \in [\frac{1}{3}, 2]$, from the input sequence $B(\alpha)$ the sequential machine $M$ computes output $B(3\alpha)$ if $\alpha \in [\frac{1}{3}, \frac{2}{3}]$ and output $B[\frac{\alpha}{2}]$ if $\alpha \in [\frac{2}{3}, 2]$. In the latter case, if $\alpha \in [1, 2]$ then output $\frac{\alpha}{2} \in [\frac{1}{2}, 1]$ can be encoded in alphabet $\{0, 1'\}$ and if $B\frac{\alpha}{2} \geq \frac{2}{3}$ the second application of $M$ computes $\frac{\alpha}{4} \in [\frac{1}{3}, \frac{1}{2}]$ represented in alphabet $\{0, 1\}$. In any case, the machine $M$ can be applied again using the previous output as input, and this may be repeated arbitrary many times.

On the other hand, if $\alpha \in [\frac{1}{3}, 2]$ there is input $B(\frac{\alpha}{3})$ or $B(2\alpha)$, that is in relation $\rho(M)$ with $B(\alpha)$. Input sequence $B(\frac{\alpha}{3})$ is used for $\alpha \geq 1$, and $B(2\alpha)$ for $\alpha \leq 1$. This can be repeated many times so $M$ can be iterated also backwards. Hence, for every bi-infinite $B(\alpha), \alpha \in [\frac{1}{3}, 2]$, there is a bi-infinite iteration yielding a tiling of the plane.

2. Now, we show that the tile set $T$ does not admit a periodic tiling. Assume that $f : \mathbb{Z}^2 \to T$ is a doubly periodic tiling with horizontal period $a$ and vertical period $b$. We can inspect that there is no tiling for $b = 1$ or 2 so we can assume that $b \geq 3$. Since no more than two consecutive rows of tiles can consist of tiles from subset $T_{1/2}$, we can assume without loss of generality that in row zero the

tiles are from $T_3$. Let $n_i$ denote the sum of colors on the upper edges of tiles $f(1,i), f(2,i), \ldots, f(a,i)$. Because the tiling is horizontally periodic with period $a$, the "carries" on the left edge of $f(1,i)$ and the right edge of $f(a,i)$ are equal.

Therefore $n_{i+1} = q_i n_i$, where $q_i = 3$ if tiles from $T_3$ are used in row $i$ and $q_i = \frac{1}{2}$ if tiles from $T_{1/2}$ are used. Because the vertical period of tiling is $b$,

$$n_1 = n_{b+1} = q_1 q_2 \ldots q_b . n_1 \ .$$

Since tiles from $T_3$ are used for $i = 0$, there are no 0's on the upper edges of the first row and thus $n_1 \neq 0$. Hence, $q_1 q_2 \ldots q_b = 1$. This contradicts the fact that no nonempty product of 3's and $\frac{1}{2}$'s can be 1.

We now show a property of valid tilings by $T_1 3$ which we will need in the next section.

**Lemma 2.** *On every valid tiling of the plane by $T_{13}$ there exists arbitrarily long horizontal sequences of tiles with 2 on the upper edge.*

*Proof.* Let $f : \mathbb{Z}^2 \longrightarrow T_{13}$ be a valid tiling. For every $i \in \mathbb{Z}$ let $q_i = 3$ ($q_i = \frac{1}{2}$) iff $T_3$ ($T_{\frac{1}{2}}$, respectively) is used on row $i$. Let $n_{N,i}$ denote the sum of the upper edges of tiles $f(1,i), f(2,i), \ldots f(N,i)$, for every $N > 0$ and $i \in \mathbb{Z}$. Because the tiles on the $i$'th row multiply by $q_i$, and because the difference between available carries is at most 2, we have for every $i \in \mathbb{Z}$ and $N > 0$,

$$|n_{N,i+1} - q_i \cdot n_{N,i}| \leq 2.$$

Therefore, for every $N, m > 0$,

$$|n_{N,m} - q_0 q_1 \ldots q_{m-1} \cdot n_{N,0}| \leq 2(1 + q_0 + q_0 q_1 + \ldots + q_0 \cdot \ldots \cdot q_{m-2}) \leq 12m.$$

The last inequality follows from the fact that $q_i q_{i+1} \cdot \ldots \cdot q_{i+j}$ is always at most 6.

Let $\varepsilon$ be an arbitrarily small positive number. In the following we show that for every large enough $N$ there exists $m$ such that $\frac{n_{N,m}}{N} > 2 - 14\varepsilon$. Consequently the upper edges of tiles $f(1,m), \ldots, f(N,m)$ must contain a long sequence of 2's. (On the average, only every $1/(14\varepsilon)$'th symbol may be 1.)

Because $\log_2 3$ is an irrational number, the set $\{m \log_2 3 \bmod 1 | m \in \mathbb{Z}_+\}$ is dense in $[0,1]$. This means that for any subinterval $I \subset [0,1]$ there exists a number $M$ such that $\forall x\ \exists m < M$, $m$ positive, such that $x + m\log_2 3 \bmod 1 \in I$. Let us choose $I = [\log_2(2 - 13\varepsilon), \log_2(2 - 12\varepsilon)]$, and let $M$ be as above.

Let $N > 12M(1 + \log_2 3)/\varepsilon$ be so large that for every $i \in \mathbb{Z}$, $\frac{n_{N,i}}{N}$ is at least $1/3 - \varepsilon$. This is possible because there cannot be three consecutive 0's on the upper edges of the tiles on the valid tiling $f$. Now, choose $x = \log_2(\frac{n_{N,0}}{N})$. There exists a positive integer $m < M$ such that $x + m\log_2 3 - k \in I$, where $k$ is a positive integer not greater than $M \log_2 3$. This means that

$$2 - 13\varepsilon \leq \frac{n_{N,0}}{N} \frac{3^m}{2^k} \leq 2 - 12\varepsilon.$$

Necessarily $q_0 q_1 \ldots q_{m+k-1} = \frac{3^m}{2^k}$. Otherwise the product would be either at most $\frac{3^{m-1}}{2^{k+1}}$ or at least $\frac{3^{m+1}}{2^{k-1}}$. This would mean that $\frac{n_{N,0}}{N} q_0 q_1 \ldots q_{m+k-1} \leq 1/3 - 2\varepsilon$ or $\geq 12 - 78\varepsilon$. Because $m + k < M(1 + \log_2 3)$ we know that

$$\left| \frac{n_{N,m+k}}{N} - q_0 q_1 \ldots q_{m+k-1} \cdot \frac{n_{N,0}}{N} \right| \leq 12(m+k)/N < \varepsilon.$$

So we would have either $\frac{n_{N,m+k}}{N} < 1/3 - \varepsilon$ (a contradiction), or $\frac{n_{N,m+k}}{N} > 12 - 79\varepsilon$ (a contradiction, if $\varepsilon$ is small). We conclude that $q_0 q_1 \ldots q_{m+k-1} = \frac{3^m}{2^k}$, and consequently

$$2 - 14\varepsilon \leq \frac{n_{N,m+k}}{N} \leq 2 - 11\varepsilon.$$

## 4 Wang cube sets

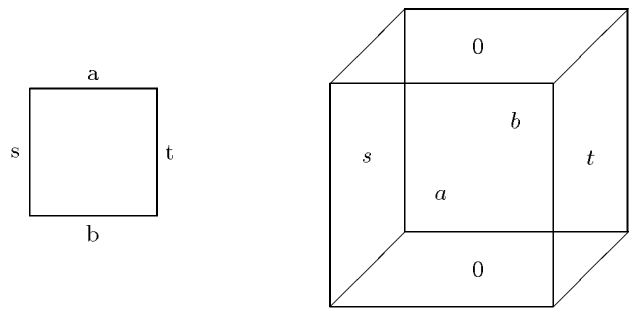



**Figure 5:** Corresponding Wang tile and Wang cube in Example 1.

*Example 1.* We convert each tile from tile set $T_{13}$ into a Wang cube by using the edge colors at the corresponding vertical faces, and a uniform color on all horizontal faces, see Fig. 5. Formally,

$$W_1 = \{(s, a, b, t, 0, 0) \mid (s, a, b, t) \in T_{13}\}.$$

Given any period $(a, b, c) \in \mathbb{Z}^3$, $c \neq 0$, set $W_1$ admits a 3-D tiling with period $(a, b, c)$. Indeed, we can choose any 2-D tiling $f : \mathbb{Z}^2 \to T_{13}$ and define 3-D tiling $g : \mathbb{Z}^3 \to W_1$ by $g(x, y, z) = f(x - \lfloor \frac{az}{c} \rfloor, y - \lfloor \frac{bz}{c} \rfloor)$. Clearly, $g(x + a, y + b, z + c) = g(x, y, c)$ for all $x, y, z \in \mathbb{Z}$.

*Example 2.* Now we construct Wang cube set $W_2$ that admits tilings periodic at most in one direction. We modify the construction from Example 1. to force identical tilings at all horizontal levels, see Fig. 6. Formally, we define

$$W_2 = \{(s, a, b, t, s, s) \mid (s, s, b, t) \in T_{13}\}$$

In every bi-infinite computation of sequential machine $M$ (corresponding to set $T_{13}$) a bi-infinite sequence of bi-infinite strings of states uniquely determines the bi-infinite sequence of bi-infinite strings of inputs. Therefore in every valid 3-D tiling $g : \mathbb{Z}^3 \to W_2$ all horizontal levels are identical, i.e. $g(x, y, z) = g(x, y, 0)$ for all $x, y, z \in \mathbb{Z}$. Since every horizontal level simulates a 2-D tiling valid for $T_{13}$ and therefore is aperiodic, clearly, $g$ is periodic with period $(0, 0, 1)$, and its multiples, but with no other period.

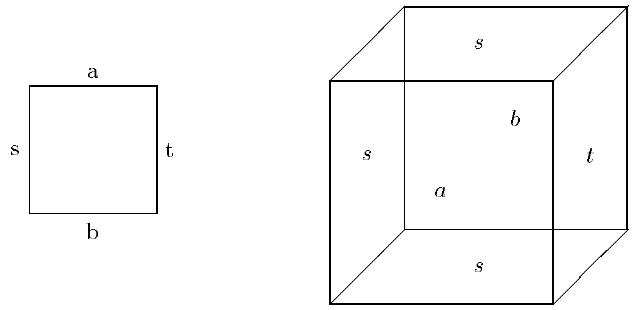



**Figure 6:** Corresponding Wang tile and Wang cube in Example 2.

Finally, we construct a cube set that does not admit any aperiodic 3-D tiling.

**Theorem 3.** *There exists an aperiodic set of 21 Wang cubes over 7 colors.*

*Proof.* We will modify a simplified version of the cube set $W_2$ from the previous example to prevent the periodicity of its tilings in the vertical direction. According to Lemma 2, on any valid tiling by $T_{13}$ there are arbitrarily long horizontal blocks of tiles $(0', 2, 1, 0')$ or $(\frac{1}{2}, 2, 1, \frac{1}{2})$. We will prevent the periodicity by simulating a simple cellular automaton (trellis automaton, see [4]) on blocks of cubes corresponding to blocks of $(0', 2, 1, 0')$ -tiles or blocks of $(\frac{1}{2}, 2, 1, \frac{1}{2})$ -tiles. We choose this cellular automaton so that its computations will have a period at least as big as the size of its input, i.e. the size of the block of identical tiles. Now, we proceed with technical details, $\oplus$ denotes the operation exclusive-or (addition mod 2).

Let $T_9 = T_{13} - \{(0', 2, 1, 0'), (\frac{1}{2}, 1, 1, 0'), (\frac{1}{2}, 2, 1, \frac{1}{2}), (0', 1, 1, \frac{1}{2})\}$, and define $W_{21} = A \cup B \cup C$, where

$$A = \{((s,1), a, b, (t,1), (1,1), (1,1)) \mid (s,a,b,t) \in T_9\},$$

$$B = \{((s,x), 2, 1, (s,y), (1,x), (1, x \oplus y)) \mid s \in \{0', \frac{1}{2}\}, x, y \in \{0,1\}\},$$

$$C = \{((\frac{1}{2},1),1,1,(0',x),(0,1),(0,1)), ((0',1)1,1,(\frac{1}{2},x),(0,1),(0,1)) \mid x \in \{0,1\}\}.$$

First, we have simplified the colors on the horizontal faces of cubes from $W_2$ by coloring by 0 the faces of the cube that simulates the tiles $(0', 2, 1, 0')$ and $(\frac{1}{2}, 2, 1, \frac{1}{2})$, and coloring by 1 all the other horizontal faces. Then we added a second bit to the colors of all the faces except front and back. For 9 of the cubes, simulating the tiles of $T_9$, we have made the second bit always 1 so there are no new cubes. The four cubes corresponding to the tiles $(\frac{1}{2}, 1, 1, 0')$ and $(0', 1, 1, \frac{1}{2})$ differ in the second bit in the right face which is the only one that can be either 0 or 1. Finally, for the tiles $(0', 2, 1, 0')$ and $(\frac{1}{2}, 2, 1, \frac{1}{2})$ we created 8 new cubes (4 for each tile) so that the second bits at the top and at the right face are arbitrary, the second bit at the left face is the same as at the top, and the second bit at the bottom is equal to the exclusive-or of the second bits at the top and at the right face. The values of the second bits in the four new cubes (vertical cuts) from $B$, corresponding to each of the tiles, are shown in Fig. 7.

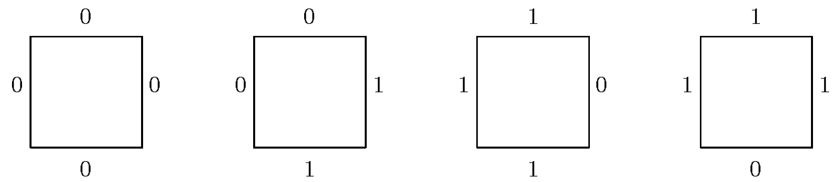



**Figure 7:** The second bits in each four-tuple of the cubes from $B$

Since there must be large blocks of either $(0', 2, 1, 0')$ or $(\frac{1}{2}, 2, 1, \frac{1}{2})$ in each 2-D tiling valid for $T_{13}$, which must match at all horizontal levels, there must be arbitrary wide vertical bi-infinite strips consisting entirely of cubes from $B$ bordered on left in each row by one cube from $C$. Our choice of bit values guarantees that each row of the cubes in these strips simulates one step of the computation of the trellis type cellular automaton each cell of which computes the logical operation exclusive-or (or equivalently addition mod 2). Indeed, the values of the "second bits" at each cube are first copied, half step to the left (identity) and then the bottom bit is set to the exclusive-or of the left and right bit (same as the left bit of the right neighbor). The whole cellular automaton is deterministic (in the top-down direction) since the first right neighbor out of the block has always new bit 1 in the left face. The neighbor on the left side is always a cube from $C$ and the value of the second bit in its right face is uniquely determined.

Since it is well known (see e.g. [4]) that the exclusive-or cellular automaton always repeats its initial value (string from $\{0,1\}^\star$) in no less steps than the size of input we conclude that every strip itself is periodic with a period not shorter than its width. Since there are arbitrary wide strips no tiling admitted by $W_{21}$ is periodic.

Clearly, every 3-D tiling admitted by $W_2$ can be converted into a tiling admitted by $W_{21}$, hence $W_{21}$ is an aperiodic set of Wang cubes.

Set $T_{13}$ requires 4 different colors on the horizontal edges, and 5 different colors (states) on the vertical edges. Since we "split" two of the states (colors of vertical edges) and use two bits for the colors of the horizontal faces the set $W_{21}$, requires 7, 4, and 4 distinct colors at faces parallel to yz, xz, and xy, respectively. Since no rotation of the cubes is allowed max(7,4,4) = 7 distinct colors is sufficient.

## References

1. Beatty,S.: "Problem 3173"; Am. Math. Monthly 33 (1926) 159; solutions in 34 (1927) 159.
2. Berger,R.: "The Undecidability of the Domino Problem"; Mem. Amer. Math. Soc. 66 (1966).
3. Culik II,K.: "An aperiodic set of 13 Wang tiles"; Discrete Math., to appear.
4. Culik II,K. and Dube,S.: " Fractal and Recurrent Behavior of Cellular Automata"; Complex Systems 3, 253-267 (1989).
5. Culik II,K. and Kari,J.: "Sequential Machines and aperiodic sets of Wang tiles"; manuscript.

6. Grünbaum,B. and Shephard,G.C.: "Tilings and Patterns"; W.H.Freeman and Company, New York (1987).
7. Jaric,M.V.: "Introduction to the Mathematics of Quasicrystals"; Academic Press, Inc., San Diego (1989).
8. Kari,J.: "A small aperiodic set of Wang tiles"; Discrete Math., to appear.

Acknowledgements

The authors are grateful to V. Valenta for implementing several algorithms for testing 2-D tile sets and composing 3-D tile sets, and for comments to the draft of this paper.