Journal of Universal Computer Science, vol. 19, no. 6 (2013), 729-749
submitted: 16/5/11, accepted: 25/3/13, appeared: 28/3/13 © J.UCS

Investigations on a Pedagogical Calculus of Constructions

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Abstract: In the last few years appeared *pedagogical propositional natural deduction systems*. In these systems one must satisfy the *pedagogical constraint*: the user must give an *example* of any introduced notion. In formal terms, for instance in the propositional case, the main modification is that we replace the usual rule (hyp) by the rule (p-hyp)

$$\frac{F \in \Gamma}{\Gamma \vdash F}\,(\text{hyp}) \qquad \frac{F \in \Gamma \quad \vdash \sigma \cdot \Gamma}{\Gamma \vdash F}\,(\text{p-hyp})$$

where σ denotes a substitution which replaces variables of Γ with an example. This substitution σ is called the *motivation* of Γ.

First we expose the reasons of such a constraint and properties of these "pedagogical" calculi: the absence of negation at logical side, and the "usefulness" feature of terms at computational side (through the Curry-Howard correspondence). Then we construct a simple pedagogical restriction of the calculus of constructions (CC) called $\mathrm{CC_r}$. We establish logical limitations of this system, and compare its computational expressiveness to Gödel system T.

Finally, guided by the logical limitations of $\mathrm{CC_r}$, we give a formal and general definition of a "pedagogical calculus of constructions".

Key Words: mathematical logic, negationless mathematics, constructive mathematics, typed lambda-calculus, calculus of constructions, pedagogical system.
Category: F.1.1, F.4.1

1 Introduction and Motivations

1.1 The pedagogical constraint

Recently the articles [Colson and Michel(2007), Colson and Michel(2008), Colson and Michel(2009)] appeared in print, introducing *pedagogical natural deduction systems* and *pedagogical typed λ-calculi*. The main feature about these systems is that any proof (or any program) must satisfy the so named *pedagogical constraint*: in natural deduction systems (for instance) the rule (hyp) is replaced by (p-hyp)

$$\frac{F \in \Gamma}{\Gamma \vdash F}\,(\text{hyp}) \qquad \frac{F \in \Gamma \quad \vdash \sigma \cdot \Gamma}{\Gamma \vdash F}\,(\text{p-hyp})$$

where σ denotes a substitution which replaces propositional variables of Γ with an example, and $\vdash \sigma \cdot \Gamma$ stands for the derivations of those substituted formulas.

The idea of such a constraint is that, in order to assume a set Γ of hypotheses, one must first provide a "motivation" (the substitution σ under consideration) in which the set of hypotheses is fulfilled. In doing so, we can always exemplify introduced hypotheses: if $\Gamma \vdash F$ holds then there exists a substitution σ such that $\vdash \sigma \cdot \Gamma$ holds. This is the formal counterpart of the usual informal teaching practice, consisting in giving examples of objects satisfying the assumed properties. This last point is a justification of the terminology *pedagogical systems*, and the necessity of such a constraint was already observed by [Poincaré(1913)] [Sect. 3.1].

1.2 The pedagogical minimal propositional calculus

In [Colson and Michel(2007)], the minimal propositional calculus over $\rightarrow$, $\vee$ and $\wedge$ (MPC) has been constrained on the (hyp) rule as previously explained. It is shown in the article that the resulting calculus (P-MPC) is equivalent to the original one: a judgment $\Gamma \vdash F$ is derivable in the usual system (MPC) if and only if it is derivable in its pedagogical version (P-MPC).

1.3 The pedagogical second-order propositional calculi

The second-order propositional calculus (Prop^2) is considered in [Colson and Michel(2008)]. By constraining only the rule of hypothesis as above, one is led to a *weakly pedagogical second-order calculus* (P$_s$-Prop^2), where rules dealing with quantification are the usual ones:

$$\frac{\Gamma \vdash F \quad \alpha \notin \mathcal{V}(F)}{\Gamma \vdash \forall \alpha . F}\,(\forall_i) \qquad \frac{\Gamma \vdash \forall \alpha . F}{\Gamma \vdash F[\alpha \leftarrow U]}\,(\forall_e)$$

This calculus is *weakly* pedagogical since even if used sets of hypotheses can be exemplified, it is *not stable by normalization* of proofs. Indeed, it is shown that $\bot \rightarrow \bot$ is derivable in P$_s$-Prop^2 (where $\bot$ stands for $\forall \alpha . \alpha$):

$$\begin{array}{ll} 1.\, \beta \vdash \beta & (\beta \text{ is motivable}) \\ 2.\, \vdash \beta \rightarrow \beta & (\rightarrow_i \ 1) \\ 3.\, \vdash \forall \beta . \beta \rightarrow \beta & (\forall_i \ 2) \\ 4.\, \vdash \bot \rightarrow \bot & (\forall_e \ 3) \end{array}$$

But a normal form of this proof must end with a $(\rightarrow_i)$ rule of $\bot$ then assuming $\bot$ as hypothesis, which is impossible since $\bot$ is not motivable. Hence the normal form of this proof is not a proof of P$_s$-Prop^2.

This motivates the more constrained system P-Prop^2 where the $(\forall_e)$ rule has been replaced by

$$\frac{\Gamma \vdash \forall\alpha.F \quad \vdash \sigma \cdot U}{\Gamma \vdash F[\alpha \leftarrow U]}\ (\text{P-}\forall_e)$$

It is shown about this last system that the usual second-order encoding of connectives $\vee$ and $\wedge$ essentially works but it must be observed that the $\vee_i$ (at right for instance) becomes:

$$\frac{\Gamma \vdash A \quad \vdash \sigma \cdot B}{\Gamma \vdash A \vee B}\ (\vee_{ir})$$

The main result concerning P-Prop^2 is that there exists a translation $F \mapsto F^\gamma$ inspired by the A-translation of [Friedman(1978)] such that: $\Gamma \vdash F$ is derivable in Prop^2 if and only if $\Gamma^\gamma \vdash F^\gamma$ is derivable in P-Prop^2.

1.4 The pedagogical second-order λ-calculus

Through the Curry-Howard isomorphism, previous work about second-order propositional calculus is extended in [Colson and Michel(2009)] to the second-order λ-calculus. The system is shown to be stable by reduction (i.e. enjoys the so-called subject reduction property). Also, an important feature for a λ-calculus is observed: the *usefulness* of functions/programs. It means that every typable function in this pedagogical λ-calculus can be applied to a term: if $\vdash f : A \to B$ with A closed, then A is inhabited by a closed term. Indeed, pedagogical λ-calculi do not allow one to write useless programs, which are not needed.

1.5 The calculus of constructions

The calculus of constructions (CC) has been first introduced in [Coquand and Huet(1984), Coquand(1985)]: it is a λ-calculus which encompasses higher-order λ-calculi and calculi with dependent types. It is then natural to extend previous works on "pedagogization" to CC in the aim of obtaining a uniform treatment of pedagogical λ-calculi, and a first step toward a possible generalization to *pure type systems* (PTS).

1.6 Negationless mathematics

In the middle of the last century, Griss expressed the negationless mathematics as a step further of the intuitionistic philosophy of Brouwer. Indeed, in intuitionistic mathematics, a proof of a negative statement $\neg A$ impose to assume A in order to obtain a contradiction. But assuming A is no *intuitive* method for Griss since it will reveal to be an impossible construction.

Works of [Griss(1946), Griss(1950), Griss(1951a), Griss(1951b)] constitute an informal outline of a geometry, an arithmetic, a set theory and an analysis without negation. [Heyting(1955), Franchella(1994)] summarize differences of viewpoint about intuitionism of Brouwer and that of Griss.

Some formal developments of the Griss *desiderata* has been proposed, from which we can cite those of [Vredenduin(1953), Gilmore(1953), Valpola(1955)], [Nelson(1966), Nelson(1973), Minichiello(1969)], [López-Escobar(1972), López-Escobar(1974), Mezhlumbekova(1975)] and more recently of [Krivtsov(2000a), Krivtsov(2000b)], dealing with negationless predicate logic and arithmetic in natural deduction systems, or in sequent calculus. [Mints(2006)] provides an overview of those works.

The pedagogical constraint in an intuitionistic framework, by requiring formulas to be exemplified, leads naturally to negationless calculi.

1.7 Organization of the article

The paper is organized as follows: in section 2 we recall usual notations, definitions and results about the calculus of constructions (CC); in section 3 we introduce the main criterion for a subsystem of CC to be pedagogical, we discuss about the impossibility of a straightforward modification of CC, and we propose a better one ($\mathrm{CC_r}$); then in section 4 we show that this restriction $\mathrm{CC_r}$ meets this criterion; we present some limitations of it at logical and computational side in sections 5 and 6; finally we conclude by stating the first formal definition of a pedagogical subsystem of CC.

2 Background and Notations

In this section, we briefly recall usual notations, definitions and results about the calculus of constructions CC.

2.1 Definitions and notations

We try to use $x, y, \ldots$ as symbols for variables, $u, v, w, t, \ldots$ to denote terms, $A, B, \ldots$ for types and formulas, $\Gamma, \Gamma', \ldots$ for environments and $[\,]$ for the empty environment.

$\equiv$ is the syntactical equality of terms [1]. We note by $\leadsto_\beta$ the usual beta-reduction relation between terms; $\overset{*}{\leadsto}_\beta$ its reflexive and transitive closure; and $=_\beta$ its equivalence closure. A term u is in normal form if it is not reducible, i.e.

[1] As in [Coquand(1989)], we assume De Bruijn indexes for bound variables and identifiers for free variables. So there is no need for α-conversion notion.

$$\frac{}{[\,]\ \mathit{wf}}\,(\mathrm{env}_1) \qquad \frac{\Gamma \vdash A : \kappa \quad x \notin \mathcal{V}(\Gamma)}{\Gamma, x : A\ \mathit{wf}}\,(\mathrm{env}_2)$$

$$\frac{\Gamma\ \mathit{wf}}{\Gamma \vdash \mathrm{Prop} : \mathrm{Type}}\,(\mathrm{ax}) \qquad \frac{\Gamma, x : A, \Gamma'\ \mathit{wf}}{\Gamma, x : A, \Gamma' \vdash x : A}\,(\mathrm{var})$$

$$\frac{\Gamma, x : A \vdash u : B : \kappa}{\Gamma \vdash \lambda x^A.u : \forall x^A.B}\,(\mathrm{abs}) \qquad \frac{\Gamma, x : A \vdash B : \kappa}{\Gamma \vdash \forall x^A.B : \kappa}\,(\mathrm{prod})$$

$$\frac{\Gamma \vdash u : \forall x^A.B \quad \Gamma \vdash v : A}{\Gamma \vdash u\ v : B[x \leftarrow v\,]}\,(\mathrm{app}) \qquad \frac{\Gamma \vdash t : A \quad \Gamma \vdash A' : \kappa \quad A =_\beta A'}{\Gamma \vdash t : A'}\,(\mathrm{conv})$$

where κ stands for Prop or for Type.

Figure 1: Inference rules of CC.

there is no term t such that $u \rightsquigarrow_\beta t$. If all possible reductions from a term u lead to a normal form, then the term u is said to be strongly normalizing.

$\mathcal{V}(t)$ is the set of free variables of t. If $\mathcal{V}(t) = \emptyset$ then t is said to be closed. The usual capture avoiding substitution of u for x in t is noted $t[x \leftarrow u\,]$; and $t[x_1, \ldots, x_n \leftarrow u_1, \ldots, u_n\,]$ is the simultaneous substitution of u_1 for x_1, u_2 for x_2, etc. in t.

To shorten notations, we use a vector symbolism: $\vec{t}$ denotes the sequence of terms $t_1, \ldots, t_n$; and $\forall \vec{x}^{\vec{A}}.B$ denotes $\forall x_1^{A_1} \ldots \forall x_n^{A_n}.B$ (the notation x^A means that A is the type of the variable x). As usual, $A \to B$ is a shortcut for $\forall x^A.B$ when x does not appear in $\mathcal{V}(B)$.

In CC there are two kinds of judgments: $\Gamma\ \mathit{wf}$ means that the environment Γ is syntactically well-formed, and $\Gamma \vdash t : A$ expresses that the term t is of type A in the environment Γ.

Implicitly, $\Gamma \vdash A : \kappa$ signifies that there exists $\kappa \in \{\mathrm{Prop}, \mathrm{Type}\}$ such that this previous statement holds. $\Gamma \vdash A : B : C$ is the contraction of $\Gamma \vdash A : B$ and $\Gamma \vdash B : C$: appearing as premise of a rule it denotes two premises, and as a conclusion of a rule it expands to two possibles conclusions (i.e. two rules).

Rules of CC are presented in [Fig. 1]: close presentations can be found in [Coquand(1986)], with well formed judgments; in [Bunder and Seldin(2004)], avoiding weakening rule; or [Barendregt(1992)], presenting usual properties of CC.

2.2 Properties of CC

In the sequel we shall need the following well-known results (proofs can be found in [Barendregt(1992)]):

Property 1 (Church-Rosser).
If $u =_\beta v$ then there exists a term w such that $u \overset{*}{\rightsquigarrow}_\beta w$ and $v \overset{*}{\rightsquigarrow}_\beta w$.

Property 2 (Church numerals).
Let us define the following abbreviations for Church numbers:

$$\begin{aligned} \mathbb{N} &:= \forall A^{\mathrm{Prop}}.A \to (A \to A) \to A \\ 0 &:= \lambda A^{\mathrm{Prop}}.\lambda x^A.\lambda f^{A\to A}.x \\ S(n) &:= \lambda A^{\mathrm{Prop}}.\lambda x^A.\lambda f^{A\to A}.f\ (n\ A\ x\ f) \end{aligned}$$

Then the following rules are derivable:

$$\frac{\Gamma \text{ wf}}{\Gamma \vdash 0 : \mathbb{N} : \mathrm{Prop}} \qquad \frac{\Gamma \vdash n : \mathbb{N}}{\Gamma \vdash S(n) : \mathbb{N}}$$

Property 3. *The constant* Type *never appears in any well-formed environment: if Γ* wf *then* Type $\notin \Gamma$*; and if $\Gamma \vdash t : A$ then* Type $\notin \Gamma \cup \{t\}$.

Property 4. *If $x_1 : A_1, \dots, x_n : A_n$* wf *or $x_1 : A_1, \dots, x_n : A_n \vdash v : C$ then $x_1 : A_1, \dots, x_i : A_i$* wf *and $x_1 : A_1, \dots, x_i : A_i \vdash A_{i+1} : \kappa$ are sub-derivations.*

Property 5. *If $\Gamma \vdash t : A$ then $A \equiv$* Type *or $\Gamma \vdash A : \kappa$.*

Property 6 (weakening). *If $\Gamma \vdash u : A$ and Γ'* wf *with $\Gamma \subseteq \Gamma'$ then $\Gamma' \vdash u : A$.*

Property 7 (generation). *If $\Gamma \vdash \forall x^A.B : T$ then there exists κ such that $T =_\beta \kappa$ and $\Gamma, x : A \vdash B : \kappa$ is a sub-derivation.*

Property 8 (substitution lemma). *If $\Gamma \vdash u : A$ then:*

(i) if $\Gamma, x : A, \Gamma'$ wf *then $\Gamma, \Gamma'[x \leftarrow u]$* wf*;*

(ii) if $\Gamma, x : A, \Gamma' \vdash t : B$ then $\Gamma, \Gamma'[x \leftarrow u] \vdash t[x \leftarrow u] : B[x \leftarrow u]$.

Property 9 (subject reduction). *If $\Gamma \vdash u : A$ and $u \rightsquigarrow_\beta u'$ then $\Gamma \vdash u' : A$.*

Property 10 ($\forall$-telescope). *If $\Gamma \vdash A : B$ then: $A \equiv \forall \vec{y}^{\vec{C}}.$* Prop *if and only if $B \equiv$* Type.

Proof. By simple induction on the first derivation (using Church-Rosser and subject reduction).

Property 11 (strong normalization). *If $\Gamma \vdash u : A$ then u and A are strongly normalizing.*

3 Pedagogizing CC

3.1 The Poincaré criterion

Let us recall the origin of the pedagogical constraint —here in the case of definitions by postulate— by the following quotation:

> *A definition by postulate has value only when the existence of the object defined has been proved. In mathematical language, this means that the postulate does not imply a contradiction, we do not have the right to neglect this condition. Either it is necessary to admit the absence of contradiction as an intuitive truth, as an axiom, by a kind of act of faith —but then it is necessary to realize what we are doing and to remember that we have extended the list of indemonstrable axioms— or else it is necessary to construct a formal proof, either* by means of examples *or by the use of reasoning by recurrence. Not that this proof is less necessary when a direct definition is involved, but it is generally easier.*
>
> Henri Poincaré – Last thoughts [Poincaré(1913)]

In CC, a definition by postulate of an object x may be seen as an environment containing x followed by hypotheses about x. For instance,

Let x be a natural number verifying $P(x)$ and $Q(x)$.

is formally represented in CC by the following environment

$$x : \mathbb{N}, H_1 : P(x), H_2 : Q(x)$$

Poincaré pointed out that such a set of hypotheses is an admissible definition by postulate of x *only if* we are able to exhibit a natural satisfying both predicates P and Q. In other words, types $P(x)$ and $Q(x)$ must be inhabited for a given x (say n) in CC. Namely the following statements must hold:

$$\vdash n : \mathbb{N} \qquad \vdash t_1 : P(n) \qquad \vdash t_2 : Q(n)$$

If this is not possible (i.e. there is no such n, t_1 or t_2) then the definition is meaningless and should be avoided.

Since every environment can be seen as a set of definitions by postulate, let us generalize to any environment:

Definition 1 (Poincaré criterion). The environment $x_1 : A_1, .., x_n : A_n$ is respectful of the Poincaré criterion if there are terms $t_1, .., t_n$ such that:

$$\begin{gathered} \vdash t_1 : A_1 \\ \vdash t_2 : A_2[x_1 \leftarrow t_1\,] \\ \vdots \\ \vdash t_n : A_n[x_1, \dots, x_{n-1} \leftarrow t_1, \dots, t_{n-1}\,] \end{gathered}$$

$$\frac{\vdash_n \sigma \cdot \Gamma}{\Gamma \vdash_n o : \top : \mathrm{Prop} : \mathrm{Type}}\ (\mathrm{ax_n}) \qquad \frac{\vdash_n \sigma \cdot (\Gamma, x : A, \Gamma')}{\Gamma, x : A, \Gamma' \vdash_n x : A}\ (\mathrm{var_n})$$

$$\frac{\Gamma, x : A \vdash_n u : B : \kappa}{\Gamma \vdash_n \lambda x^A.u : \forall x^A.B}\ (\mathrm{abs_n}) \qquad \frac{\Gamma, x : A \vdash_n B : \kappa}{\Gamma \vdash_n \forall x^A.B : \kappa}\ (\mathrm{prod_n})$$

$$\frac{\Gamma \vdash_n u : \forall x^A.B \quad \Gamma \vdash_n v : A}{\Gamma \vdash_n u\ v : B[x \leftarrow v]}\ (\mathrm{app_n}) \qquad \frac{\Gamma \vdash_n t : A \quad \Gamma \vdash_n A' : \kappa \quad A =_\beta A'}{\Gamma \vdash_n t : A'}\ (\mathrm{conv_n})$$

where:

- κ stands for Prop or for Type;
- $\vdash_n \sigma \cdot (x_1 : A_1, \ldots, x_n : A_n)$ denotes the derivations $\vdash_n \sigma(x_i) : \sigma(A_i)$;
- o and $\top$ are two constants added to the calculus to start derivations.

Figure 2: Inference rules of $\mathrm{CC_n}$, a too naive adaptation of CC.

A formal system is said to meet the Poincaré criterion if every well-formed environments are respectful of the Poincaré criterion.

3.2 On the naive extension of previous work

In the previous works on pedagogization [Sect. 1], each environment is motivated by an example (the substitution σ) before being used:

$$\frac{F \in \Gamma \quad \vdash \sigma \cdot \Gamma}{\Gamma \vdash F}\ (\text{p-hyp})$$

It follows that each used environment can be exemplified, hence such a system trivially satisfies the Poincaré criterion. Unfortunately such a simple adjustment can not be performed into CC: take the naive system $\mathrm{CC_n}$ [Fig. 2] in which we have replaced the notion of well-formed environment by the simple fact that it is motivable.

Those modifications of CC are comparable in all respect to those made by [Colson and Michel(2009)] for their pedagogical second-order λ-calculus. Unfortunately, short-circuiting rules of well-formed environments and types, CC_{n} is not even a subsystem of CC:

Lemma 2. *The following derivations hold in CC_n but not in CC:*

(a) $x_1 : \mathrm{Type} \vdash_n \mathrm{Prop} : \mathrm{Type}$

(b) $x_1 : \mathrm{Prop}, x_2 : (\lambda H^{\top \to x_1}.\top)\ (\lambda y^{\top}.y) \vdash_n \mathrm{Prop} : \mathrm{Type}$

(c) $x_1 : \mathbb{N}, x_2 : (\lambda H^{x_1 = 0}.\top)\ (\lambda P^{\mathbb{N} \to \mathrm{Prop}}.\lambda H^{P\ 0}.H) \vdash_n \mathrm{Prop} : \mathrm{Type}$

Proof. Proofs are trivial as soon as one exhibits a motivation:

(a) $\sigma_1 := [x_1 \mapsto \mathrm{Prop}]$

(b) $\sigma_2 := [x_1 \mapsto \top; x_2 \mapsto o]$

(c) $\sigma_3 := [x_1 \mapsto 0; x_2 \mapsto o]$

And it is easy to see that they are not derivable in CC:

(a) Type appears into an environment, which is forbidden in CC [Prop. 3];

(b) $(\lambda H^{\top \to x_1}.\top)\ (\lambda y^{\top}.y)$ is ill-typed since the function is waiting for an element of type $\top \to x_1$, but an element of type $\top \to \top$ is given instead;

(c) same reason as for (b): the function is waiting for a proof of $x_1 = 0$, whereas a proof of $0 = 0$ is provided.

□

Remark. The first case can be avoided by enforcing the A_i to be of type Prop or Type in the definition of $\sigma \cdot \Gamma$.

CC has the advantage that well-formed types are built into the system. So we just need to find which rules need to be constrained and how in order to avoid non exemplifiable types (and especially empty types) as soon as possible.

3.3 A simple attempt: $\mathbf{CC_r}$

In CC, we are able to introduce $\bot := \forall A^{\mathrm{Prop}}.A$ as an hypothesis if we have been able to derive $\bot$ as a type, which is allowed by the (prod) rule. Actually, the (prod) rule is the only one able to create vacuity, since other rules construct type and an inhabitant of it simultaneously. We then impose products to always be inhabited by replacing the usual (prod) rule of CC by the following more restrictive ($\mathrm{prod_r}$) in $\mathrm{CC_r}$ [Fig. 3]:

$$\frac{\Gamma, x : A \vdash_r \mathbf{t} : B : \kappa}{\Gamma \vdash_r \forall x^A.B : \kappa}\ (\mathrm{prod_r})$$

Remark. This rule may be condensed together with ($\mathrm{abs_r}$) to obtain a rule with two conclusions. So $\mathrm{CC_r}$ can be viewed as CC without the (prod) rule.

$$\frac{}{[\,]\ wf_r}\ (\mathrm{env_{1r}}) \qquad \frac{\Gamma \vdash_r A : \kappa \quad x \notin \mathcal{V}(\Gamma)}{\Gamma, x : A\ wf_r}\ (\mathrm{env_{2r}})$$

$$\frac{\Gamma\ wf_r}{\Gamma \vdash_r \mathrm{Prop} : \mathrm{Type}}\ (\mathrm{ax_r}) \qquad \frac{\Gamma, x : A, \Gamma'\ wf_r}{\Gamma, x : A, \Gamma' \vdash_r x : A}\ (\mathrm{var_r})$$

$$\frac{\Gamma, x : A \vdash_r u : B : \kappa}{\Gamma \vdash_r \lambda x^A.u : \forall x^A.B}\ (\mathrm{abs_r}) \qquad \frac{\Gamma, x : A \vdash_r \mathbf{t} : B : \kappa}{\Gamma \vdash_r \forall x^A.B : \kappa}\ (\mathrm{prod_r})$$

$$\frac{\Gamma \vdash_r u : \forall x^A.B \quad \Gamma \vdash_r v : A}{\Gamma \vdash_r u\ v : B[x \leftarrow v]}\ (\mathrm{app_r}) \qquad \frac{\Gamma \vdash_r t : A \quad \Gamma \vdash_r A' : \kappa \quad A =_\beta A'}{\Gamma \vdash_r t : A'}\ (\mathrm{conv_r})$$

where κ stands for Prop or for Type.

Figure 3: Inference rules of $\mathrm{CC_r}$, a Poincarean calculus.

Lemma 3. *CC_r is a subsystem of CC: if $\Gamma \vdash_r u : A$ then $\Gamma \vdash u : A$.*

Proof. Immediate by induction on the derivation.

Usual properties of CC [Sect. 2.2] still hold for $\mathrm{CC_r}$, especially substitution lemma [Prop. 8], weakening [Prop. 6], subject reduction [Prop. 9] and strong normalization [Prop. 11]. They were formally checked in the Coq proof assistant [2] by straightforward adaptation of the work of [Barras(1996)].

3.4 Example of derivation in $\mathbf{CC_r}$

Lemma 4. *The following rule is derivable:*

$$\frac{\Gamma\ \mathrm{wf_r}}{\Gamma \vdash_r \mathrm{id} : \mathrm{True} : \mathrm{Prop}}$$

where $\mathrm{id} := \lambda A^{\mathrm{Prop}}.\lambda x^A.x$ *and* $\mathrm{True} := \forall A^{\mathrm{Prop}}.A \to A$.

[2] Sources can be found at `http://lita.sciences.univ-metz.fr/~demange/publications/sources/CoqR.tar.gz`

Proof.

1. $\Gamma\ \mathit{wf}_r$	$(\mathrm{hyp_r})$
2. $\Gamma \vDash_r \mathrm{Prop} : \mathrm{Type}$	$(\mathrm{ax_r}\ 1)$
3. $\Gamma, A : \mathrm{Prop}\ \ \mathit{wf}_r$	$(\mathrm{env_{2r}}\ 2)$
4. $\Gamma, A : \mathrm{Prop} \vDash_r A : \mathrm{Prop}$	$(\mathrm{var_r}\ 3)$
5. $\Gamma, A : \mathrm{Prop}, x : A\ \ \mathit{wf}_r$	$(\mathrm{env_{2r}}\ 4)$
6. $\Gamma, A : \mathrm{Prop}, x : A \vDash_r x : A : \mathrm{Prop}$	$(\mathrm{var_r}\ 5)$
7. $\Gamma, A : \mathrm{Prop} \vDash_r \lambda x^A.x : A \to A : \mathrm{Prop}$	$(\mathbf{abs_r{+}prod_r}\ 6)$
8. $\Gamma \vDash_r \lambda A^{\mathrm{Prop}}.\lambda x^A.x : \forall A^{\mathrm{Prop}}.A \to A : \mathrm{Prop}$	$(\mathbf{abs_r{+}prod_r}\ 7)$

Remark. id and True play the role of the constants o and $\top$ that needed to be added in the pedagogical second-order λ-calculus of [Colson and Michel(2009)].

4 $\mathbf{CC_r}$ meets the Poincaré criterion

In this section we show that every type (term of sort Prop or Type) in a well-formed environment of $\mathrm{CC_r}$ is inhabited. The sketch of the proof is as follows: we first notice that in $\mathrm{CC_r}$ every product is inhabited, then because each closed type reduces to a product we can inhabit every type of a well-formed environment, beginning by its leftmost type which is closed and continuing for the whole environment using substitution lemma.

Lemma 5 (generation). *If $\Gamma \vDash_r \forall x^A.B : T$ then there exist κ and a term t such that $T =_\beta \kappa$ and $\Gamma, x : A \vDash_r t : B : \kappa$.*

Proof. Immediate by induction on the derivation. □

Lemma 6. *If $\Gamma \vDash_r C : \mathrm{Type}$ then there is a term t such that $\Gamma \vDash_r t : C$.*

Proof. By cases on the last applied rule; $(\mathrm{ax_r})$ case is dealt with [Lem. 4]; $(\mathrm{var_r})$, $(\mathrm{app_r})$ and $(\mathrm{conv_r})$ cases are eliminated using [Prop. 3] and [Prop. 5]; $(\mathrm{prod_r})$ case is trivial using $(\mathrm{abs_r})$ rule. □

Remark. Indeed every element of type Type is a $\forall$-telescope [Prop. 10], i.e. syntactically of the form $\forall \vec{x}^{\vec{A}}.\,\mathrm{Prop}$, and then trivially inhabited by $\lambda \vec{x}^{\vec{A}}.\,\mathrm{True}$.

Lemma 7. *If $\Gamma \vdash C : \forall \vec{y}^{\vec{D}}.\,\mathrm{Prop}$ with C closed, then for all closed terms $w_1, \ldots, w_n$ verifying*

$$\begin{gathered}\Gamma \vdash w_1 : D_1 \\ \Gamma \vdash w_2 : D_2[y_1 \leftarrow w_1\,] \\ \vdots \\ \Gamma \vdash w_n : D_n[y_1, \ldots, y_{n-1} \leftarrow w_1, \ldots, w_{n-1}\,]\end{gathered}$$

there are terms E and F such that

$$C\ \vec{w} \overset{*}{\rightsquigarrow}_\beta \forall z^E.F$$

Proof. Let us define by $\|t\|$ the length of the longest path of reduction from the term t to its normal form (which exists because terms of CC are strongly normalizing [Prop. 11]).

We proceed by induction on the lexicographical order of $\|C\ \vec{w}\|$ and the height of the derivation of $\Gamma \vdash C : \forall \vec{y}^{\vec{D}}.\,\mathrm{Prop}$.

Let us deal with non-trivial cases (others being mostly eliminated by [Prop. 3] and [Prop. 5]):

(abs) If the last rule of the derivation is

$$\frac{\Gamma, y_1 : D_1 \vdash u : \forall y_2^{D_2} \ldots \forall y_n^{D_n}.\,\mathrm{Prop} : \mathrm{Type}}{\Gamma \vdash \lambda y_1^{D_1}.u : \forall y_1^{D_1}.\forall y_2^{D_2} \ldots \forall y_n^{D_n}.\,\mathrm{Prop}}$$

Substituting w_1 for y_1 in the premise [Prop. 8], we obtain

$$\Gamma \vdash u[y_1 \leftarrow w_1\,] : \forall y_2^{D_2[y_1 \leftarrow w_1\,]} \ldots \forall y_n^{D_n[y_1 \leftarrow w_1\,]}.\,\mathrm{Prop}$$

As we have $\|u[y_1 \leftarrow w_1\,]\ w_2\ \ldots\ w_n\| < \|(\lambda y_1^{D_1}.u)\ w_1\ w_2\ \ldots\ w_n\|$ and also $u[y_1 \leftarrow w_1\,]$ is closed (since $\lambda {y_1}^{D_1}.u$ and w_1 are), we can apply induction hypothesis to $w_2, \ldots, w_n$ and get

$$(\lambda y_1^{D_1}.u)\ w_1\ w_2\ \ldots\ w_n \rightsquigarrow_\beta u[y_1 \leftarrow w_1\,]\ w_2\ \ldots\ w_n \overset{*}{\rightsquigarrow}_\beta \forall z^E.F$$

(app) If the last rule of the derivation looks like

$$\frac{\Gamma \vdash u : \forall x^A.\forall {y_1}^{G_1} \ldots \forall {y_n}^{G_n}.\,\mathrm{Prop} \quad \Gamma \vdash v : A}{\Gamma \vdash u\ v : \forall {y_1}^{G_1[x \leftarrow v\,]} \ldots \forall {y_n}^{G_n[x \leftarrow v\,]}.\,\mathrm{Prop}}$$

where $D_i \equiv G_i[x \leftarrow v\,]$ and $C \equiv u\ v$.

Since for every i $y_i \notin \mathcal{V}(v)$ then

$$G_i[x \leftarrow v\,][y_1, \ldots, y_{i-1} \leftarrow w_1, \ldots, w_{i-1}\,] \equiv G_i[x, y_1, \ldots, y_{i-1} \leftarrow v, w_1, \ldots, w_{i-1}\,]$$

Noticing we have $\|u\ v\ \vec{w}\| = \|(u\ v)\ \vec{w}\|$, we can then apply induction hypothesis on the first premise and terms $v, \vec{w}$ (v is closed since $C \equiv u\ v$ is) to finally obtain

$$(u\ v)\ \vec{w} \overset{*}{\rightsquigarrow}_\beta \forall z^E.F$$

(conv)

$$\frac{\Gamma \vdash u : A \quad \Gamma \vdash \forall \vec{y}^{\vec{D}}.\,\mathrm{Prop} : \mathrm{Type} \quad A =_\beta \forall \vec{y}^{\vec{D}}.\,\mathrm{Prop}}{\Gamma \vdash u : \forall \vec{y}^{\vec{D}}.\,\mathrm{Prop}}$$

First, by Church-Rosser [Prop. 1] on the third premise and the definition of beta-reduction, we have $A \overset{*}{\rightsquigarrow}_\beta \forall \vec{y}^{\vec{G}}.\,\mathrm{Prop}$ and $\forall \vec{y}^{\vec{D}}.\,\mathrm{Prop} \overset{*}{\rightsquigarrow}_\beta \forall \vec{y}^{\vec{G}}.\,\mathrm{Prop}$.

We have three possible cases [Prop. 5]:

- $A \equiv \text{Type}$ which is impossible by definition of $\rightsquigarrow_\beta$;
- $\Gamma \vdash A : \text{Prop}$ which is also impossible by subject reduction [Prop. 9] since A reduces to a $\forall$-telescope [Prop. 10];
- $\Gamma \vdash A : \text{Type}$ implying that $A \equiv \forall \vec{y}^{\vec{H}}.\,\text{Prop}$ and then also $H_i =_\beta D_i$.

In order to apply induction hypothesis on the first premise, it is necessary to first show that

$$\begin{gathered} \Gamma \vdash w_1 : H_1 \\ \Gamma \vdash w_2 : H_2[y_1 \leftarrow w_1\,] \\ \vdots \\ \Gamma \vdash w_n : H_n[y_1, \ldots, y_{n-1} \leftarrow w_1, \ldots, w_{n-1}\,] \end{gathered}$$

which can be proved by (strong) induction on n:

For all i, since $D_i =_\beta H_i$ then

$$D_i[y_1, \ldots, y_{i-1} \leftarrow w_1, \ldots, w_{i-1}\,] =_\beta H_i[y_1, \ldots, y_{i-1} \leftarrow w_1, \ldots, w_{i-1}\,]$$

Also from $\Gamma \vdash \forall {y_1}^{H_1}.\ldots.\forall y_n^{H_n}.\,\text{Prop} : \text{Type}$ we have by generation [Prop. 7] $\Gamma, y_1 : H_1, \ldots, y_n : H_n \vdash \text{Prop} : \text{Type}$ and then it follows [Prop. 4] $\Gamma, y_1 : H_1, \ldots, y_{i-1} : H_{i-1} \vdash H_i : \kappa_i$.
By induction hypothesis on the $(w_k)_{k<i}$ and after substitutions [Prop. 8] we get

$$\Gamma \vdash H_i[y_1, \ldots, y_{i-1} \leftarrow w_1, \ldots, w_{i-1}\,] : \kappa_i$$

and then by (conv) on $\Gamma \vdash w_i : D_i[y_1, \ldots, y_{i-1} \leftarrow w_1, \ldots, w_{i-1}\,]$ we obtain

$$\Gamma \vdash w_i : H_i[y_1, \ldots, y_{i-1} \leftarrow w_1, \ldots, w_{i-1}\,]$$

The w_i being well-typed, we can apply induction hypothesis on the first premise to conclude:

$$u\ \vec{w} \overset{*}{\rightsquigarrow}_\beta \forall z^E.F$$

□

Lemma 8.
If $\Gamma \vdash_r C : \text{Prop}$ with C closed, then there is a term t such that $\Gamma \vdash_r t : C$.

Proof. Since $\Gamma \vdash_r C : \text{Prop}$ then $\Gamma \vdash C : \text{Prop}$ [Lem. 3], and C closed implies that it reduces to a product $\forall z^E.F$ [Lem. 7]. Hence by subject reduction [Prop. 9] in CC_r: $\Gamma \vdash_r \forall z^E.F : \text{Prop}$.

By generation [Lem. 5] there is a term u such that $\Gamma, z : E \vdash_r u : F : \mathrm{Prop}$ and by ($\mathrm{abs_r}$) $\Gamma \vdash_r \lambda z^E.u : \forall z^E.F$. Finally ($\mathrm{conv_r}$) gives that $\Gamma \vdash_r t : C$ with $t := \lambda z^E.u$.

The two previous lemmas about $\mathrm{CC_r}$ can be summed up by the following statement:

Corollary 9.
If $\Gamma \vdash_r C : \kappa$ with C closed, then there is a term t such that $\Gamma \vdash_r t : C$.

So the pedagogical character of the calculus follows, every type of a well-formed environment is inhabited:

Theorem 10 (Poincaré criterion). *If $x_1 : A_1, \ldots, x_n : A_n$ $\mathrm{wf_r}$ then there are terms $t_1, \ldots, t_n$ such that*

$$\begin{gathered} \vdash_r t_1 : A_1 \\ \vdash_r t_2 : A_2[x_1 \leftarrow t_1] \\ \vdots \\ \vdash_r t_n : A_n[x_1, \ldots, x_{n-1} \leftarrow t_1, \ldots, t_{n-1}] \end{gathered}$$

Proof. By induction on the size of the environment n.

From the derivation $x_1 : A_1, \ldots, x_n : A_n$ wf_r, we have $\vdash_r A_1 : \kappa$ as a sub-derivation [Prop. 4] where A_1 is closed. We then have t_1 [Cor. 9] such that

$$\vdash_r t_1 : A_1$$

then by substitution [Prop. 8] we have

$$x_2 : A_2[x_1 \leftarrow t_1], \ldots, x_n : A_n[x_1 \leftarrow t_1] \ \mathit{wf_r}$$

Following the same pattern, we build terms $t_2, \ldots, t_n$ verifying

$$\begin{gathered} \vdash_r t_2 : A_2[x_1 \leftarrow t_1] \\ \vdots \\ \vdash_r t_n : A_n[x_1, \ldots, x_{n-1} \leftarrow t_1, \ldots, t_{n-1}] \end{gathered}$$

□

This exemplification can be transmitted to the conclusion of judgments:

Corollary 11. *If $x_1 : A_1, \ldots, x_n : A_n \vdash_r u : B$ then there are terms $t_1, \ldots, t_n$ such that*

$$\begin{gathered} \vdash_r t_1 : A_1 \\ \vdash_r t_2 : A_2[x_1 \leftarrow t_1] \\ \vdots \\ \vdash_r t_n : A_n[x_1, \ldots, x_{n-1} \leftarrow t_1, \ldots, t_{n-1}] \end{gathered}$$

and

$$\vdash_r u[\vec{x} \leftarrow \vec{t}] : B[\vec{x} \leftarrow \vec{t}]$$

Proof. Immediate by applying n times the substitution lemma [Prop. 8] to the terms obtained from the theorem. □

Theorem 12 (usefulness).
If $\vdash_r f : \forall x^A.B$ *then there is a term* u *such that* $\vdash_r u : A$.

Proof. From $\vdash_r f : \forall x^A.B$ we have $\vdash_r \forall x^A.B : \kappa$ [Prop. 5], then by generation [Lem. 5] $x : A \vdash_r B : \kappa$ which implies that $x : A$ *wf* [Prop. 4], and finally by Poincaré criterion [Thm. 10] we construct u. □

5 Limitations of the logical power of $\mathbf{CC_r}$

To introduce an hypothesis (which is not a variable) in an environment, it is necessary to first inhabit it. For instance, defining Leibniz equality over a type A by

$$x =_A y := \forall Q^{A \to \mathrm{Prop}}.Q\ x \to Q\ y$$

it is not possible to prove nor symmetry nor transitivity of this relation over A (whatever this type is). Indeed, we are not permitted to add $x =_A y$ as an hypothesis because we can not derive $A : \mathrm{Prop}, x : A, y : A \vdash_r x =_A y : \mathrm{Prop}$ since $x =_A y$ is not inhabited in this environment.

Theorem 13.
There is no term u *such that* $\vdash_r u : \forall A^{\mathrm{Prop}}.\forall x^A.\forall y^A.x =_A y \to y =_A x$.

Proof. Let us suppose such a term u exists. We then have a sort κ such that $A : \mathrm{Prop}, x : A, y : A \vdash_r x =_A y : \kappa$ [Lem. 5], [Prop. 4] and [Prop. 5]. And because $x =_A y$ is a product, it is inhabited [Lem. 5], say by t. But since $\mathrm{CC_r}$ is a subsystem of CC [Lem. 3], $A : \mathrm{Prop}, x : A, y : A \vdash t : x =_A y$ also holds in CC. Then applying it to $\mathbb{N}$ and 0 and 1 by substitution lemma [Prop. 8], we get a proof of $0 =_{\mathbb{N}} 1$ in the empty environment in CC, which is known to be impossible (by a simple combinatoric discussion about the normal form of such a proof). □

In fact, this calculus does not even natively contain simply typed λ-calculus:

Theorem 14. *There is no term* u *such that*

$$A\ B\ C : \mathrm{Prop} \vdash_r u : (A \to B) \to (B \to C) \to (A \to C)$$

Proof. Similarly as above, assuming such a u then [Lem. 5], [Prop. 4], [Prop. 5]

$$A : \mathrm{Prop}, B : \mathrm{Prop}, C : \mathrm{Prop} \vdash_r A \to B : \mathrm{Prop}$$

so there is an inhabitant t of the product type $A \to B$ in $\mathrm{CC_r}$ [Lem. 5] and hence in CC, implying by ($\mathrm{abs_r}$) rule that

$$\vdash \lambda ABC^{\mathrm{Prop}}.t : \forall ABC^{\mathrm{Prop}}.A \to B$$

which can be specialized to True and $\bot$ to obtain a proof of True $\to \bot$ and finally a proof of $\bot$ in the empty environment, which is impossible since CC is consistent. ◻

Actually, every instances of the types in $\mathrm{CC_r}$ must be inhabited:

Theorem 15.
If $x_1 : A_1, .., x_n : A_n \vdash_r B : \kappa$ then for all terms $w_1, \ldots, w_n$ verifying

$$\begin{array}{c} \vdash_r w_1 : A_1 \\ \vdash_r w_2 : A_2[x_1 \leftarrow w_1\,] \\ \vdots \\ \vdash_r w_n : A_n[x_1, \ldots, x_{n-1} \leftarrow w_1, \ldots, w_{n-1}\,] \end{array}$$

there is a term t such that

$$\vdash_r t : B[\vec{x} \leftarrow \vec{w}\,]$$

Proof. The proof is trivial by applying n times the substitution lemma [Prop. 8] to obtain $\vdash_r B[\vec{x} \leftarrow \vec{w}\,] : \kappa$ which is closed and hence inhabited [Cor. 9]. ◻

It is hard to precisely determine the logical expressiveness of $\mathrm{CC_r}$. We have at least natively simply typed λ-calculus on closed (and then inhabited) types of $\mathrm{CC_r}$ (e.g. $\top$, $\mathbb{N}$, etc.). The proof is the same as the one of [Lem. 20] below.

6 Computational expressivity of $\mathrm{CC_r}$

Although the logical strength of $\mathrm{CC_r}$ seems quite poor, its computational power is at least that of the system T of [Gödel(1958)]. We use the usual well-known way to define terms, types (except cartesian product), and recursor (from iterator) of system T in lambda-calculus (see[Girard et al.(1990)]).

Definition 16 (Church numerals and iterator).

$$\begin{aligned} \mathbb{N} &:= \forall A^{\mathrm{Prop}}.A \to (A \to A) \to A \\ 0 &:= \lambda A^{\mathrm{Prop}}.\lambda x^A.\lambda f^{A\to A}.x \\ S(n) &:= \lambda A^{\mathrm{Prop}}.\lambda x^A.\lambda f^{A\to A}.f\ (n\ A\ x\ f) \\ \mathrm{it}_T(n, b, (y^T)step) &:= n\ T\ b\ (\lambda y^T.step) \end{aligned}$$

Lemma 17. *The following rules are derivable:*

$$\frac{\Gamma \text{ wf}_{\mathrm{r}}}{\Gamma \vdash_r 0 : \mathbb{N} : \mathrm{Prop}} \qquad \frac{\Gamma \vdash_r n : \mathbb{N}}{\Gamma \vdash_r S(n) : \mathbb{N}}$$

$$\frac{\Gamma \vdash_r T : \mathrm{Prop} \quad \Gamma \vdash_r n : \mathbb{N} \quad \Gamma \vdash_r b : T \quad \Gamma, y : T \vdash_r \mathit{step} : T}{\Gamma \vdash_r \mathrm{it}_T(n, b, (y^T)\mathit{step}) : T}$$

Lemma 18. *The following reductions hold:*

$$\begin{aligned} \mathrm{it}_T(0, b, (y^T)\mathit{step}) &\overset{*}{\leadsto}_\beta b \\ \mathrm{it}_T(S(n), b, (y^T)\mathit{step}) &\overset{*}{\leadsto}_\beta \mathit{step}[y \leftarrow \mathrm{it}_T(n, b, (y^T)\mathit{step})\,] \end{aligned}$$

Definition 19 (simple types on $\mathbb{N}$).
Simple types on $\mathbb{N}$ are those obtained from $\mathbb{N}$ and $\rightarrow$.

Lemma 20. *If Γ wf_{r} and T is a simple type on $\mathbb{N}$, then there is a term t such that $\Gamma \vdash_r t : T : \mathrm{Prop}$.*

Proof. By induction on T (as a simple type on $\mathbb{N}$):

- If T is $\mathbb{N}$, then 0 fits.
- If T is $A \rightarrow B$ where A and B are simple types on $\mathbb{N}$ then by induction hypothesis on A we get $\Gamma \vdash_r A : \mathrm{Prop}$, hence $\Gamma, x : A$ wf_r by ($\mathrm{env}_{2\mathrm{r}}$) rule. By induction hypothesis on B, we get $\Gamma \vdash_r b : B : \mathrm{Prop}$, and weakening it [Prop. 6] we have $\Gamma, x : A \vdash_r b : B : \mathrm{Prop}$ and finally ($\mathrm{abs}_{\mathrm{r}}$) and ($\mathrm{prod}_{\mathrm{r}}$) rules give $\Gamma \vdash_r \lambda x^A.b : A \rightarrow B : \mathrm{Prop}$.

$\square$

CC_{r} does not allow us to derive the usual cartesian product defined by

$$A \times B := \forall C^{\mathrm{Prop}}.(A \rightarrow B \rightarrow C) \rightarrow C$$

To simulate recursor from iterator we define a restricted cartesian product $\mathbb{N} \times T$ for each T, simple type on $\mathbb{N}$, by encoding a natural into T.

Lemma 21. *If Γ wf_{r} and T is a simple type on $\mathbb{N}$ then there are two terms enc_T and dec_T such that $\Gamma \vdash_r \mathrm{enc}_T : \mathbb{N} \rightarrow T$ and $\Gamma \vdash_r \mathrm{dec}_T : T \rightarrow \mathbb{N}$ and for every term n we have $\mathrm{dec}_T(\mathrm{enc}_T\ \ n) \overset{*}{\leadsto}_\beta n$.*

Proof. By induction on T (as a simple type on $\mathbb{N}$):

- If T is $\mathbb{N}$, then we take the identity on $\mathbb{N}$ for enc_T and dec_T.

– If T is $A \to B$, we take

$$\begin{aligned}\mathrm{enc}_{A\to B} &:= \lambda x^{\mathbb{N}}.\lambda z^{A}.\,\mathrm{enc}_B\ \ x\\ \mathrm{dec}_{A\to B} &:= \lambda f^{A\to B}.\,\mathrm{dec}_B\ (f\ \ a)\end{aligned}$$

where a is a term of type A obtained from [Lem. 20].

□

Definition 22. We define the following abbreviations for couples

$$\begin{aligned}\mathbb{N}\times T &:= (T\to T\to T)\to T\\ \langle n,t\rangle^{T} &:= \lambda f^{T\to T\to T}.f\ (\mathrm{enc}_T\ \ n)\ t\\ \pi_1(c) &:= \mathrm{dec}_T\ (c\ (\lambda x^{T}.\lambda y^{T}.x))\\ \pi_2(c) &:= c\ (\lambda x^{T}.\lambda y^{T}.y)\end{aligned}$$

Lemma 23. *The following rules are derivable:*

$$\frac{\mathit{\Gamma}\ \mathrm{wf_r}}{\mathit{\Gamma}\vdash_r \mathbb{N}\times T:\mathrm{Prop}}\qquad \frac{\mathit{\Gamma}\vdash_r n:\mathbb{N}\quad \mathit{\Gamma}\vdash_r t:T}{\mathit{\Gamma}\vdash_r \langle n,t\rangle^{T}:\mathbb{N}\times T}$$

$$\frac{\mathit{\Gamma}\vdash_r c:\mathbb{N}\times T}{\mathit{\Gamma}\vdash_r \pi_1(c):\mathbb{N}}\qquad \frac{\mathit{\Gamma}\vdash_r c:\mathbb{N}\times T}{\mathit{\Gamma}\vdash_r \pi_2(c):T}$$

Lemma 24. *The following reductions hold:*

$$\begin{aligned}\pi_1(\langle n,t\rangle^{T}) &\overset{*}{\rightsquigarrow}_\beta n\\ \pi_2(\langle n,t\rangle^{T}) &\overset{*}{\rightsquigarrow}_\beta t\end{aligned}$$

Definition 25. We define recursor from iterator by

$$\mathrm{rec}_T(n,b,(x^{\mathbb{N}},y^{T})\mathit{step}) := \pi_2\left[\mathrm{it}_{T\times T}(n,\langle 0,b\rangle^{T},(z^{T\times T})\mathit{step}')\right]$$

where

$$\mathit{step}' := \langle S(\pi_1(z)),\mathit{step}[x,y\leftarrow \pi_1(z),\pi_2(z)\,]\rangle^{T\times T}$$

Lemma 26. *The following rule is derivable:*

$$\frac{\mathit{\Gamma}\vdash_r T:\mathrm{Prop}\quad \mathit{\Gamma}\vdash_r n:\mathbb{N}\quad \mathit{\Gamma}\vdash_r b:T\quad \mathit{\Gamma},x:\mathbb{N},y:T\vdash_r \mathit{step}:T}{\mathit{\Gamma}\vdash_r \mathrm{rec}_T(n,b,(x^{\mathbb{N}},y^{T})\mathit{step}):T}$$

Lemma 27. *The following reductions hold:*

$$\begin{aligned}\mathrm{rec}_T(0,b,(x^{\mathbb{N}},y^{T})\mathit{step}) &\overset{*}{\rightsquigarrow}_\beta b\\ \mathrm{rec}_T(S(n),b,(x^{\mathbb{N}},y^{T})\mathit{step}) &\overset{*}{\rightsquigarrow}_\beta \mathit{step}[x,y\leftarrow n,\mathrm{rec}_T(n,b,(x^{\mathbb{N}},y^{T})\mathit{step})\,]\end{aligned}$$

7 Conclusions and direction for further work

We have seen a simple attempt to pedagogize the calculus of constructions. It has a good computational power —at least Gödel system T— but lacks of logical expressivity —does not even natively contain simply typed λ-calculus. A pleasant aspect is the simplicity of the added constraint, which also emphasizes that the (prod) rule is responsible for vacuity in CC.

Logical limitations of our calculus $\mathrm{CC_r}$ suggest a more precise definition for a calculus of constructions to be pedagogical: in a pedagogical calculus, we should be able to prove the symmetry of the Leibniz equality, because the non-emptiness of $x =_A y$ can be justified by substituting $\mathbb{N}$ to A and 0 to x and y. It means that we not only need that a well-formed environment can be exemplified (i.e. meets the Poincaré criterion), but the converse should hold too.

However we must be careful in stating the converse of the Poincaré criterion: we already noticed that "all exemplifiable environments are well-formed" does not hold [Sect. 3.2]. We then propose the following definition:

Definition (converse of the Poincaré criterion). *A subsystem CC_p of CC meets the converse of the Poincaré criterion if whenever there are terms $t_1, \ldots, t_n$ verifying*

$$\begin{gathered} \vdash_p t_1 : A_1 \\ \vdash_p t_2 : A_2[x_1 \leftarrow t_1\,] \\ \vdots \\ \vdash_p t_n : A_n[x_1, \ldots, x_{n-1} \leftarrow t_1, \ldots, t_{n-1}\,] \end{gathered}$$

and

$$x_1 : A_1, \ldots, x_n : A_n \text{ wf}$$

then

$$x_1 : A_1, \ldots, x_n : A_n \text{ wf}_\mathrm{p}$$

In this definition we refer explicitly to CC as a base system that we need to constrain: it then prevents from escaping of CC, as it was unfortunately the case in our naive attempt $\mathrm{CC_n}$ [Sect. 3.2].

We can then express what we wait exactly for a subsystem of CC to be called pedagogical:

Definition (pedagogical subsystem of CC).
CC_p is a pedagogical subsystem of CC if:

1. *CC_p is a subsystem of CC;*
2. *CC_p verifies the subject reduction property;*

3. CC_p meets the Poincaré criterion and its converse.

The subject reduction property must be explicitly stated here: [Colson and Michel(2008)] defined a *weakly pedagogical second-order calculus* (P_s-$Prop^2$) satisfying 1 and 3 but not 2. While in this paper we give a system satisfying 1, 2 and only one direction of 3.

There are strong indications on the possibility to construct a second-order pedagogical λ-calculus in the new sense just defined: indeed the work has been initiated by [Colson and Michel(2009)] in a slightly different formalism with their P_s-$Prop^2$ system [Sect. 1]. It would be a great advance to express it in the formalism of CC since it would lead to a first step toward a "full" pedagogical Calculus of Constructions. Moreover it would raise the question of formally characterizing a maximally expressive pedagogical restriction of CC.

Acknowledgements

The authors wish to thank David Michel who initiated formal work about pedagogical systems and for his constant support to this work. We also would like to thank the referees of this paper for their useful suggestions.

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