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Constructor-based Logics

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Abstract: Many computer science applications concern properties that are true for a restricted class of models. In this paper, a couple of constructor-based *institutions* are presented. These institutions are defined on top of some base institutions, roughly speaking, by enhancing the syntax with *constructor symbols* and restricting the semantics to models with elements that are reachable by constructors. The proof rules for the constructor-based *Horn logics*, formalized as institutions, are defined in this paper, and a proof of completeness is provided in the abstract framework of institutions.

Key Words: institution, Horn logic, proof theory, completeness, induction, constructor.
Category: F.3, F.4

1 Introduction

Equational specification and programming constitute the basis of modern algebraic specification languages like CASL [Astesiano et al., 2002], Maude [Clavel et al., 2007] and CafeOBJ [Diaconescu and Futatsugi, 2002]. In 1935 Birkhoff provided the first proof of completeness for equational logic, in the unsorted case. Goguen and Meseguer extended this result to cover the many-sorted case, providing a full algebraisation of finitary many-sorted equational deduction. In [Codescu and Găină, 2008], the result is cast in the framework of institutions [Goguen and Burstall, 1992] capturing both the finitary and the infinitary cases. In this paper, an institution-independent completeness result for constructor-based Horn logics is presented. Preliminary results of this study were published in [Găină et al., 2009]. However, the full treatment of the topic, including the proofs, can be found in this paper.

The theory of institutions of Goguen and Burstall is a category-based formalization of the intuitive notion of logical system that has gained foundational status in algebraic specification theory. Horn institutions are obtained by restricting the sentences of a given institution, for example the institution of first-order logic, to the so-called Horn sentences of the form $(\forall X)\bigwedge H \Rightarrow C$, where H is a set of atoms, and C is an atom. Constructor-based institutions [Bidoit et al., 2003] are obtained from a base institution, basically by enhancing the syntax with *constructor symbols* and restricting the semantics to *reachable* models. The signatures of constructor-based institutions consist of a signature in the base institution and a distinguished family of sets of constructor operators. The result sorts of the constructors are called *constrained*, and a sort that is not constrained is called *loose*. The constructors determine the class of reachable models, which are of interest from the user point of view. Intuitively, the carrier sets of such models consist of constructor-generated elements. In other words, the elements "can be reached" through constructors and constants of loose sorts. The sentences and the satisfaction condition are preserved from the base institution. In order to obtain a constructor-based institution, the signature morphisms of the base institution are restricted such that the *reducts* along the signature morphisms of the models that are reachable (in the target signature) are again reachable (in the source signature). In the examples presented in this paper it is simply required that constructors are preserved by signature morphisms, and no "new" constructors are introduced for "old" constrained sorts (for sorts being in the image of some constrained sorts of the source signature).

The proof rules for the constructor-based Horn institutions are established in this research. Also, a proof of completeness using institution-independent techniques is provided in two steps:

1. We give an institution-independent definition of reachable model, and we apply it to obtain a new institution by enhancing the syntax of a base institution with constructors and restricting the semantics to reachable models. We prove an abstract layered completeness for *reachable institutions*, which are a particular case of constructor-based institutions, where all operators of constrained sorts are constructors.

2. We lift completeness from reachable institutions to constructor-based institutions such that the new result depends on sufficient completeness. Intuitively, a specification (Σ, Ax), where Ax is a set of conditional equations over the vocabulary given by the signature Σ, is sufficient complete when any term can be reduced to a term formed with constructors and variables of loose sorts using the equations from Ax.

The present work constitutes the logical foundation of the so-called OTS/CafeOBJ method. This is a method of modeling, specifying and verifying systems, which has been developed and refined through some case studies (see [Ogata and Futatsugi, 2008] and [Ogata and Futatsugi, 2003]). This paper fully justifies the practice of using constructors in connection with *induction schemes* and *case analysis*. One of the important advantages of our approach is the institution-independent status of our concepts and results. In [Codescu and Găină, 2008] and [Găină and Petria, 2010] the completeness is obtained directly, whereas in this research, an intermediate result is obtained for reachable institutions, and then it is extended to constructor-based institutions via sufficient completeness.

Section 2 introduces the notions of institution and entailment system. These will constitute the base for expressing the soundness and completeness properties for a logic, i.e. that semantic deduction coincides with syntactic provability. Section 3 introduces the abstract concept of universal institution and reachable universal entailment system, which is proved sound and complete under conditions that are also investigated. Section 4 contains the main result. The completeness result of reachable institutions is extended to constructor-based institutions via sufficient completeness. Section 5 concludes the paper and discusses future work.

We assume that the reader is familiar with the basic notions of category theory. See [Mac Lane, 1998] for the standard definitions of category, functor, pushout, etc., which have been omitted here.

2 Institutions

Institutions were introduced in [Goguen and Burstall, 1992] with the original goal of providing an abstract framework for algebraic specification languages.

Definition 1. An *institution* $I = (\mathbb{S}ig^I, \mathbb{S}en^I, \mathbb{M}od^I, \models^I)$ consists of

1. a category $\mathbb{S}ig^I$, whose objects are called *signatures*,
2. a functor $\mathbb{S}en^I : \mathbb{S}ig^I \to \mathbb{S}et$, providing for each signature Σ a set whose elements are called *(Σ-)sentences*,
3. a functor $\mathbb{M}od^I : \mathbb{S}ig^I \to \mathbb{C}at^{op}$, providing for each signature Σ a category whose objects are called *(Σ-)models* and whose arrows are called *(Σ-)morphisms*,
4. a relation $\models^I_\Sigma \subseteq |\mathbb{M}od^I(\Sigma)| \times \mathbb{S}en^I(\Sigma)$ for each signature Σ, called *(Σ-)satisfaction*, such that for each signature morphism $\varphi : \Sigma \to \Sigma'$ the following *satisfaction condition* holds:
$$M' \models^I_{\Sigma'} \mathbb{S}en^I(\varphi)(e) \text{ iff } \mathbb{M}od^I(\varphi)(M') \models^I_\Sigma e$$
for all $M' \in |\mathbb{M}od^I(\Sigma')|$ and $e \in \mathbb{S}en^I(\Sigma)$.

We denote the *reduct* functor $\mathbb{M}od^I(\varphi)$ by $\_\!\upharpoonright_\varphi$ and the sentence translation $\mathbb{S}en^I(\varphi)$ by $\varphi(\_)$. When $M = M'\!\upharpoonright_\varphi$ we say that M is the φ-reduct of M' and M' is a φ-expansion of M. When there is no danger of confusion, we omit the superscript from the notations of the institution components; for example $\mathbb{S}ig^I$ may be simply denoted by $\mathbb{S}ig$.

*Example 1 First-order logic (***FOL***) [Goguen and Burstall, 1992].* The signatures are triplets (S, F, P), where S is the set of sorts, $F = (F_{w\to s})_{(w,s)\in S^*\times S}$ is the ($S^*\times S$ -indexed) set of operation symbols, and $P = (P_w)_{w\in S^*}$ is the (S^*-indexed) set of relation symbols. If $w = \lambda$, an element of $F_{w\to s}$ is called a *constant symbol*, or a *constant*. By a slight notational abuse, we let F and P also denote $\bigcup_{(w,s)\in S^*\times S} F_{w\to s}$ and $\bigcup_{w\in S^*} P_w$ respectively. A signature morphism between (S, F, P) and (S', F', P') is a triplet $\varphi = (\varphi^{st}, \varphi^{op}, \varphi^{rl})$, where $\varphi^{st} : S \to S'$, $\varphi^{op} : F \to F'$, $\varphi^{rl} : P \to P'$ such that $\varphi^{op}(F_{w\to s}) \subseteq F'_{\varphi^{st}(w)\to\varphi^{st}(s)}$ and $\varphi^{rl}(P_w) \subseteq P'_{\varphi^{st}(w)}$ for all $(w,s) \in S^* \times S$. When there is no danger of confusion, we may let φ denote each of φ^{st}, φ^{rl} and φ^{op}. Given a signature $\Sigma = (S,F,P)$, a Σ-model M is a triplet $M = ((M_s)_{s\in S}, (M_{w,s}(\sigma))_{(w,s)\in S^*\times S, \sigma\in F_{w\to s}}, (M_w(R))_{w\in S^*, R\in P_w})$ interpreting each sort s as a set M_s, each operation symbol $\sigma \in F_{w\to s}$ as a function $M_{w,s}(\sigma) : M_w \to M_s$ (where M_w stands for $M_{s_1} \times \ldots \times M_{s_n}$ if $w = s_1 \ldots s_n$), and each relation symbol $R \in P_w$ as a relation $M_w(R) \subseteq M^w$. When there is no danger of confusion we may let M_σ and M_R denote $M_{w,s}(\sigma)$ and $M_w(R)$ respectively. Morphisms between models are the usual Σ-morphisms, i.e., S-sorted functions that preserve the structure. The Σ-sentences are obtained from

- equality atoms $t_1 = t_2$, where $t_1, t_2 \in (T_{(S,F)})_s$ and $T_{(S,F)}$ is the (S,F)-algebra of ground terms, or
- relational atoms $R(t_1,\ldots,t_n)$, where $R \in P_{s_1\ldots s_n}$ and $t_i \in (T_{(S,F)})_{s_i}$ for all $i \in \{1,\ldots,n\}$,

by applying a finite number of times negation, conjunction, disjunction, and universal or existential quantification over finite sets of constants (variables). Satisfaction is the usual first-order satisfaction and is defined using the natural interpretations of ground terms t as elements M_t in models M. The definitions of functors $\mathbb{S}en$ and $\mathbb{M}od$ on morphisms are the natural ones: for any signature morphism $\varphi : \Sigma \to \Sigma'$, $\mathbb{S}en(\varphi) : \mathbb{S}en(\Sigma) \to \mathbb{S}en(\Sigma')$ translates sentences symbol-wise, and $\mathbb{M}od(\varphi) : \mathbb{M}od(\Sigma') \to \mathbb{M}od(\Sigma)$ is the forgetful functor.

*Example 2 Horn clause logic (***HCL***). A universal Horn sentence* for a **FOL** signature (S,F,P) is a (universal) sentence of the form $(\forall X)(\bigwedge H) \Rightarrow C$, where X is a finite set of variables, H is a finite set of (relational or equational) atoms, and C is a (relational or equational) atom. In the tradition of logic programming, universal Horn sentences are known as *Horn Clauses*. Thus **HCL** has the same signatures and models as **FOL** but only universal Horn sentences as sentences.

*Example 3 Constructor-based Horn clause logic (***CHCL***).* Its signatures (S,F,F^c,P) consist of a first-order signature (S,F,P), and a distinguished subfamily $F^c \subseteq F$ of sets of operation symbols called *constructors*. The constructors determine the set of *constrained* sorts $S^c \subseteq S$: $s \in S^c$ iff there exists a constructor $\sigma \in F^c_{w\to s}$. We call the sorts in $S^l = S - S^c$ *loose*. The (S,F,F^c,P)-sentences are the *universal Horn sentences* of the form $(\forall X)(\forall Y)\bigwedge H \Rightarrow C$, where X is a finite set of variables of constrained sorts, Y is a finite set of variables of loose sorts, H is a finite set of atoms and C is an atom.

The (S,F,F^c,P)-models are the usual first-order (S,F,P)-models M *reachable* by the constructors in F^c, i.e. there exists a set Y of variables of loose sorts, and a function $f : Y \to M$ such that for every constrained sort $s \in S^c$ the function $f^\#_s : (T_{(S,F^c)}(Y))_s \to M_s$ is a surjection, where $f^\# : T_{(S,F^c)}(Y) \to M\!\upharpoonright_{(S,F^c)}$ is the unique extension of f to a (S,F^c)-morphism.

A signature morphism $\varphi : (S,F,F^c,P) \to (S',F',F'^c,P')$ in **CHCL** is a first-order signature morphism $\varphi : (S,F,P) \to (S',F',P')$ such that the constructors are preserved along the signature morphisms (i.e. if $\sigma \in F^c$ then $\varphi(\sigma) \in F'^c$) and no "new" constructors are introduced for "old" constrained sorts (i.e. if $s \in S^c$ and $\sigma' \in (F'^c)_{w'\to\varphi(s)}$ then there exists $\sigma \in F^c_{w\to s}$ such that $\varphi(\sigma) = \sigma'$).

Lemma 2. *For every* **CHCL** *signature morphism* $\varphi : (S,F,F^c,P) \to (S',F',F'^c,P')$ *and any* (S',F',F'^c,P')*-model* M'*, we have* $M'\restriction_\varphi \in \mathbb{M}od(S,F,F^c,P)$*.*

Proof. Let $Y_{M'}$ be a renaming of the loose elements of M such that $Y_{M'}$ is disjoint from the constants in F'. This means that there exists a S'-sorted function $con_{M'} : Y_{M'} \to M'$ such that for all $s' \in S'^c$, $Y_{M'} = \emptyset$, and for all $s' \in S'^l$, $(con_{M'})_{s'} : (Y_{M'})_{s'} \to M'_{s'}$ is bijective. Since $M' \in \mathbb{M}od(S',F',F'^c,P')$, the unique extension $con^\#_{M'} : T_{(S',F'^c)}(Y_{M'}) \to M'\restriction_{(S',F'^c)}$ of $con_{M'}$ to a (S',F'^c)-morphism is surjective. We define $M = M'\restriction_\varphi$. Let Y_M be a renaming of the loose elements of M such that Y_M is disjoint from the constants in F. There exists a S-sorted function $con_M : Y_M \to M$ such that for all $s \in S^c$, $Y_M = \emptyset$, and for all $s \in S^l$, $(con_M)_s : (Y_M)_s \to M_s$ is bijective. We show that for all $s \in S$ and $t' \in T_{(S',F'^c)}(Y_{M'})$ there exists $t \in T_{(S,F^c)}(Y_M)$ such that $con^\#_{M'}(t') = con^\#_M(t)$, where $con^\# : T_{(S,F^c)}(Y_M) \to M\restriction_{(S,F^c)}$ is the unique extension of con_M to a (S,F^c)-morphism. We proceed by induction on the structure of the terms in $T_{(S',F'^c)}(Y_{M'})$.

1. For the base case, let $s \in S$ and $t' \in (Y_{M'})_{\varphi(s)}$. It follows that $\varphi(s) \in S'^l$, which implies $s \in S^l$. Take $t = con_M^{-1}(con_{M'}(t'))$, and we have $con_M(t) = con_{M'}(t')$.

2. For the induction step, let $s \in S$, $\sigma' \in F'^c_{w'\to\varphi(s)}$ and $t' \in (T_{(S',F'^c)}(Y_{M'}))_{w'}$. There exists $\sigma \in F^c_{w\to s}$ such that $\varphi(\sigma) = \sigma'$. By the induction hypothesis, there exists $t \in (T_{(S,F^c)}(Y_M))_w$ such that $con^\#_M(t) = con^\#_{M'}(t')$. Hence, $con^\#_M(\sigma(t)) = M_\sigma(con^\#_M(t)) = M'_{\sigma'}(con^\#_{M'}(t')) = con^\#_{M'}(\sigma'(t'))$.

Since $con_{M'}$ is surjective, con_M is surjective too. Therefore, $M \in \mathbb{M}od(S,F,F^c,P)$. □

Variants of constructor-based first-order logic were studied in [Bidoit et al., 2003] and [Bidoit and Hennicker, 2006].

*Example 4 Order-sorted algebra (***OSA***) [Goguen and Meseguer, 1992].* A **OSA** signature $(S,\leq,F)$ consists of an algebraic signature (S,F), with a partial ordering $(S,\leq)$ such that the following *monotonicity* condition is satisfied: $\sigma \in F_{w_1\to s_1} \cap F_{w_2\to s_2}$ and $w_1 \leq w_2$ imply $s_1 \leq s_2$. A morphism of **OSA**-signatures $\varphi : (S,\leq,F) \to (S',\leq',F')$ is just a morphism of algebraic signatures $(S,F) \to (S',F')$ such that the ordering is preserved, i.e. $\varphi(s_1) \leq' \varphi(s_2)$ whenever $s_1 \leq s_2$.

Given an order-sorted signature $(S,\leq,F)$, an order-sorted $(S,\leq,F)$-algebra is a (S,F)-algebra M such that $s_1 \leq s_2$ implies $M_{s_1} \subseteq M_{s_2}$, and $\sigma \in F_{w_1\to s_1} \cap F_{w_2\to s_2}$ and $w_1 \leq w_2$ implies $M^{w_1,s_1}_\sigma = M^{w_2,s_2}_\sigma|_{M_{w_1}}$. Given order-sorted $(S,\leq,F)$-algebras M and N, an order-sorted $(S,\leq,F)$-morphism $h : M \to N$ is a (S,F)-morphism such that $s_1 \leq s_2$ implies $h_{s_1} = h_{s_2}|_{M_{s_1}}$.

An **OSA** signature $(S,\leq,F)$ is *regular* iff for each $\sigma \in F_{w_1\to s_1}$ and each $w_0 \leq w_1$ there is a unique least element in the set $\{(w,s) \mid \sigma \in F_{w\to s} \text{ and } w_0 \leq w\}$. For regular signatures $(S,\leq,F)$, any $(S,\leq,F)$-term has a least sort and the initial $(S,\leq,F)$-algebra can be defined as a term algebra, cf. [Goguen and Meseguer, 1992]. Let $(S,\leq,F)$ be an order-sorted signature. We say that the sorts s_1 and s_2 are in the same *connected component* of S iff $s_1 \equiv s_2$, where $\equiv$ is the least equivalence on S that contains $\leq$. A partial ordering $(S,\leq)$ is *filtered* iff for all $s_1,s_2 \in S$, there is some $s \in S$ such that $s_1 \leq s$ and $s_2 \leq s$. A partial ordering is *locally filtered* iff every connected component of it is filtered. An order-sorted signature $(S,\leq,F)$ is *locally filtered* iff $(S,\leq)$ is locally filtered, and it is *coherent* iff it is both locally filtered and regular. Hereafter we assume that all **OSA** signatures are coherent. The atoms of the signature $(S,\leq,F)$ are equations of the form $t_1 = t_2$ such that the least sort of the terms t_1 and t_2 are in the same connected component. The sentences are closed formulas built by application of Boolean connectives and quantification to the equational atoms. Order-sorted algebras were extensively studied in [Goguen and Diaconescu, 1994] and [Goguen and Meseguer, 1992]. **HOSA** is obtained by restricting **OSA** to universal Horn sentences.

*Example 5 Constructor-based Horn order-sorted algebra (***CHOSA***).* This institution is defined on top of **HOSA** similarly as **CHCL** is defined on top of **HCL**. The constructor-based order-sorted signatures $(S,\leq,F,F^c)$ consist of an order-sorted signature $(S,\leq,F)$, and a distinguished subfamily $F^c \subseteq F$ of sets of operational symbols, called *constructors*, such that $(S,\leq,F^c)$ is an order-sorted signature (the monotonicity and coherence conditions are satisfied). The sort $s \in S$ is *constrained* if there exists a constructor $\sigma \in F^c_{w\to s}$ with the result sort s. As in the first case, we let S^c to denote the set of all constrained sorts. We call the sorts in $S^l = S - S^c$ *loose*. The $(S,\leq,F,F^c)$-sentences are the *universal Horn sentences* of the form $(\forall X)(\forall Y) \bigwedge H \Rightarrow C$, where X is a finite set of variables of constrained sorts, Y is a finite set of variables of loose sorts, H is a finite set of equational atoms and C is an equational atom.

The $(S,\leq,F,F^c)$-models are the usual order-sorted algebras with the carrier sets for the constrained sorts consisting of interpretations of terms formed with constructors and elements of loose sorts, i.e. there exists a set of loose variables Y, and a function $f : Y \to M$ such that for every constrained sort $s \in S^c$ the function $f^{\#}_s : (T_{(S,\leq,F^c)}(Y))_s \to M_s$ is a surjection, where $f^{\#} : T_{(S,\leq,F^c)}(Y) \to M\!\upharpoonright_{(S,\leq,F^c)}$ is the unique extension of f to a $(S,\leq,F^c)$-morphism.

A **CHOSA** signature morphism $\varphi : (S,\leq,F,F^c) \to (S',\leq',F',F'^c)$ is an order-sorted signature morphism such that constructors are preserved along the signature morphisms (i.e. if $\sigma \in F^c$ then $\varphi(\sigma) \in F'^c$), and no "new" constructors are introduced for "old" constrained sorts (i.e. if $s \in S^c$ and $\sigma' \in (F'^c)_{w'\to\varphi(s)}$ then there exists $\sigma \in F^c_{w\to s}$ such that $\varphi(\sigma) = \sigma'$, and if $s'_0 \in S'$ and $s \in S^c$ such that $s'_0 \leq' \varphi(s)$ then there exists $s_0 \in S$ such that $s_0 \leq s$ and $\varphi(s_0) = s'_0$).

*Example 6 Preorder algebra (***POA***) [Diaconescu and Futatsugi, 1998].* The **POA** signatures are just the ordinary algebraic signatures. The **POA** models are *preordered al-*

gebras which are interpretations of the signatures into the category of preorders $\mathbb{P}re$ rather than the category of sets $\mathbb{S}et$. This means that each sort gets interpreted as a preorder, and each operation as a preorder functor, which means a preorder-preserving (i.e. monotonic) function. A *preordered algebra morphism* is just a family of preorder functors (preorder-preserving functions) which is also an algebra morphism. **POA** constitutes an unlabeled form of Meseguer's rewriting logic [Meseguer, 1992], but the later is not an institution.

The sentences have two kinds of atoms: equations and *preorder atoms*. A preorder atom $t \leq t'$ is satisfied by a preorder algebra M when the interpretations of the terms are in the preorder relation of the carrier, i.e. $M_t \leq M_{t'}$. Full sentences are constructed from equational and preorder atoms by using Boolean connectives and first-order quantification.

As in the case of **FOL** we define Horn preorder algebra (**HPOA**) by restricting the sentences to universal Horn sentences. The institution of constructor-based Horn preorder algebra (**CHPOA**) is obtained in the same way as in the first-order case.

*Example 7 Partial algebra (***PA***) [Reichel, 1984, Burmeister, 1986].* A *partial algebraic signature* (S,F) consists of a set S of sorts and a set F of *partial* operations. We assume that there is a distinguished constant on each sort $\perp_s : s$. Signature morphisms map the sorts and operations in a compatible way, preserving $\perp_s$.

A partial algebra is just like an ordinary algebra but interpreting the operations of F as partial rather than total functions; $\perp_s$ is always interpreted as undefined. A *partial algebra* (S,F)*-morphism* $h : A \to B$ is a family of (total) functions $(A_s \xrightarrow{h_s} B_s)_{s\in S}$ such that $h_s(A_\sigma(a)) = B_\sigma(h_w(a))$ for each operation $\sigma : w \to s$ and each string of arguments $a \in A_w$ for which $A_\sigma(a)$ is defined.

Remark. For every inclusion $\Sigma \hookrightarrow \Sigma(Z)$, where $\Sigma = (S,F)$ and $\Sigma(Z) = (S,F \cup Z)$, the $\Sigma(Z)$-models can be represented as pairs (M,h) where M is a Σ-model and $h : Z' \to A$ is a function such that $Z' \subseteq Z$ is the set of constants which are defined in M.

We consider one kind of "base" sentences: *existence equality* $t \stackrel{e}{=} t'$. The existence equality $t \stackrel{e}{=} t'$ holds when both terms are defined and equal. The definedness predicate and strong equality can be introduced as notations: $def(t)$ stands for $t \stackrel{e}{=} t$ and $t \stackrel{s}{=} t'$ stands for $(t \stackrel{e}{=} t') \vee (\neg def(t) \bigwedge \neg def(t'))$. We consider the atomic sentences in $\mathbb{S}en(S,F)$ to be the atomic existential equalities. The sentences are formed from these "base" sentences by Boolean connectives and quantification over variables. The definition of **PA** given here is slightly different from the one in [Mosses, 2004, Astesiano et al., 2002, Mossakowski, 2002] since it does not consider total operation symbols.

*Example 8 Constructor-based partial algebra (***CHPA***).* The signatures of constructor-based partial algebra (S,F,F^c) consist of a signature (S,F) in the base institution, and a distinguished subfamily $F^c \subseteq F$ of sets of *constructors*. The constructors determine the set of *constrained* sorts $S^c \subseteq S$: $s \in S^c$ iff there exists a constructor $\sigma \in F^c_{w\to s}$ with the result sort s, and the set of *loose* sorts $S^l = S - S^c$.

The (S,F,F^c)-sentences are the universal Horn sentences formed over existential equations. The (S,F,F^c)-models are the usual partial algebras M with the carrier sets for the constrained sorts consisting of interpretations of terms formed with constructors and variables of loose sorts, i.e. there exists a set Y of variables of loose sorts, and a function $f : Y \to M$ such that for every constrained sort $s \in S^c$ the function $f^\#_s : (T_{(M,f)})_s \to M_s$ is a surjection, where $T_{(M,f)} \subseteq T_{(S,F^c\cup Y)}$ is the maximal partial $(S,F^c\cup Y)$-algebra of terms such that $(M,f) \models def(t)$ for all $t \in T_{(M,f)}$, and $f^\# : T_{(M,f)} \to (M,f)\restriction_{(S,F^c\cup Y)}$ is the unique $(S,F^c\cup Y)$-morphism extending f.

A constructor-based partial signature morphisms $\varphi : (S,F,F^c) \to (S_1,F_1,F^c_1)$ is a **PA** signature morphism $\varphi : (S,F) \to (S_1,F_1)$ such that the constructors are preserved along the signature morphisms (i.e. if $\sigma \in F^c$ then $\varphi(\sigma) \in F^c_1$), and no "new" constructors are introduced for "old" constrained sorts (i.e. if $s \in S^c$ and $\sigma_1 \in (F^c_1)_{w1\to\varphi(s)}$ then there exists $\sigma \in F^c_{w\to s}$ such that $\varphi(\sigma) = \sigma_1$).

*Example 9 Reachable Horn clause logic (***RHCL***).* **RHCL** is obtained from **CHCL** by restricting the signatures such that all operation symbols of constrained sorts are constructors, i.e. for each (S,F,F^c,P) we have $F^c = F^{S^c}$ where

$$F^{S^c}_{w\to s} = \begin{cases} F_{w\to s} & : \quad s \in S^c \\ \emptyset & : \quad s \notin S^c \end{cases}$$

In this case, we choose to denote each signature (S,F,F^c,P) by (S,S^c,F,P) where S^c is the set of constrained sorts.

Similarly, one may define **RHOSA**, **RHPOA** and **RHPA**.

2.1 Substitutions

In **CHCL**, consider $\Sigma \overset{\chi_1}{\hookrightarrow} \Sigma(X_1)$ and $\Sigma \overset{\chi_2}{\hookrightarrow} \Sigma(X_2)$ two inclusion signature morphisms, where $\Sigma = (S,F,F^c,P)$ is a **CHCL** signature, X_i is a set of constant symbols disjoint from the constants of F, and $\Sigma(X_i) = (S,F\cup X_i,F^c,P)$. Any substitution $\theta : X_1 \to T_{(S,F)}(X_2)$ can be extended to a function $\mathbb{S}en(\theta) : \mathbb{S}en(\Sigma(X_1)) \to \mathbb{S}en(\Sigma(X_2))$ that replaces all the symbols in X_1 by the corresponding $(S,F\cup X_2)$-terms, according to θ. This can be formally defined as follows:

(1) $\mathbb{S}en(\theta)(t = t') = (\theta^{tm}(t) = \theta^{tm}(t'))$, for all equational $\Sigma(X_1)$-atoms $t = t'$, where $\theta^{tm} : T_{(S,F)}(X_1) \to T_{(S,F)}(X_2)$ is the unique extension of θ to a (S,F)-morphism.

(2) $\mathbb{S}en(\theta)(\pi(t_1,\dots,t_n)) = \pi(\theta^{tm}(t_1),\dots,\theta^{tm}(t_n))$, for all $\Sigma(X_1)$-atoms $\pi(t_1,\dots,t_n)$.

(3) $\mathbb{S}en(\theta)(\bigwedge H \Rightarrow C) = \bigwedge \mathbb{S}en(\theta)(H) \Rightarrow \mathbb{S}en(\theta)(C)$, for all $\Sigma(X_1)$-sentences $\bigwedge H \Rightarrow C$.

(4) $\mathbb{S}en(\theta)((\forall Z_1)\rho) = (\forall Z_2)\mathbb{S}en(\theta_b)(\rho)$, for all $\Sigma(X_1)$-sentences $(\forall Z_1)\rho$, where

- Z_2 is obtained by renaming the constants in Z_1, i.e. there is a bijection $b : Z_1 \to Z_2$, such that Z_2 is disjoint from the constants of $\Sigma(X_2)$, and
- $\theta_b : \Sigma(X_1 \cup Z_1) \to \Sigma(X_2 \cup Z_2)$ works like θ on $\Sigma(X_1)$ and maps every $z_1 \in Z_1$ to $b(z_1) \in Z_2$.

On the semantics side, θ determines a functor $\mathbb{M}od(\theta) : \mathbb{M}od(\Sigma(X_2)) \to \mathbb{M}od(\Sigma(X_1))$ such that for all $\Sigma(X_2)$-models M we have

(1) $\mathbb{M}od(\theta)(M)_z = M_z$, for each sort $z \in S$, or operation symbol $z \in F$, or relation symbol $z \in P$, and

(2) $\mathbb{M}od(\theta)(M)_z = M_{\theta(z)}$ for each $z \in X_1$.

Proposition 3. *For every* **CHCL** *signature* Σ *and each substitution* $\theta : X_1 \to \Sigma(X_2)$

$$\mathbb{M}od(\theta)(M) \models \rho \text{ iff } M \models \mathbb{S}en(\theta)(\rho)$$

for all $\Sigma(X_2)$*-models* M *and all* $\Sigma(X_1)$*-sentences* ρ.

The proof is the same as the one for **FOL**, which can be found in [Diaconescu, 2004].

Assumption 2.1 *Throughout this paper, for all institutions above, we assume that signature morphisms allow mappings of constants to terms.*

The above assumption makes it possible to treat first-order substitutions in the comma category [1] of signature morphisms.

Definition 4. Consider two signature morphisms $\Sigma \xrightarrow{\chi_1} \Sigma_1$ and $\Sigma \xrightarrow{\chi_2} \Sigma_2$ of an institution. A signature morphisms $\theta : \Sigma_1 \to \Sigma_2$ such that $\chi_1;\theta = \chi_2$ is called a Σ-*substitution* between χ_1 and χ_2.

2.2 Entailment systems

A *sentence system* $(\mathbb{S}ig, \mathbb{S}en)$ consists of a category of signatures $\mathbb{S}ig$ and a sentence functor $\mathbb{S}en : \mathbb{S}ig \to \mathbb{S}et$. An *entailment system* [Meseguer, 1989] $\mathcal{E} = (\mathbb{S}ig, \mathbb{S}en, \vdash)$ consists of a sentence system $(\mathbb{S}ig, \mathbb{S}en)$ and a family of entailment relations $\vdash = (\vdash_\Sigma)_{\Sigma \in |\mathbb{S}ig|}$ between sets of sentences with the following properties:

(Monotonicity) $E_1 \vdash_\Sigma E_2$ if $E_2 \subseteq E_1$,
(Transitivity) $E_1 \vdash_\Sigma E_3$ if $E_1 \vdash_\Sigma E_2$ and $E_2 \vdash_\Sigma E_3$,
(Unions) $E_1 \vdash_\Sigma E_2 \cup E_3$ if $E_1 \vdash_\Sigma E_2$ and $E_1 \vdash_\Sigma E_3$, and
(Translation) $\varphi(E_1) \vdash_{\Sigma'} \varphi(E_2)$ if $E_1 \vdash_\Sigma E_2$, for all $\Sigma \xrightarrow{\varphi} \Sigma' \in \mathbb{S}ig$.

[1] Given a category $\mathcal{C}$ and an object $A \in |\mathcal{C}|$, the comma category $A/\mathcal{C}$ has arrows $A \xrightarrow{f} B \in \mathcal{C}$ as objects, and $h \in \mathcal{C}(B, B')$ with $f;h = f'$ as arrows.

The semantic entailment system of an institution $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ consists of $(\mathbb{S}ig, \mathbb{S}en, \models)$. When there is no danger of confusion we may omit the subscript Σ from $\vdash_\Sigma$. For every signature morphism $\varphi \in \mathbb{S}ig$, we sometimes let φ denote the sentence translation $\mathbb{S}en(\varphi)$. An entailment system $(\mathbb{S}ig, \mathbb{S}en, \vdash)$ is *sound* (resp. *complete*) for an institution $(\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ if $\Gamma \vdash_\Sigma \rho$ implies $\Gamma \models_\Sigma \rho$ (resp. $\Gamma \models_\Sigma \rho$ implies $\Gamma \vdash_\Sigma \rho$) for every signature Σ, each set of Σ-sentences Γ and any Σ-sentence ρ. We call the entailment system $\mathcal{E} = (\mathbb{S}ig, \mathbb{S}en, \vdash)$ *compact* whenever for every $\Gamma \subseteq \mathbb{S}en(\Sigma)$ and each finite $E_f \subseteq \mathbb{S}en(\Sigma)$ if $\Gamma \vdash_\Sigma E_f$ then there exists $\Gamma_f \subset \Gamma$ finite such that $\Gamma_f \vdash_\Sigma E_f$. For each entailment system $\mathcal{E} = (\mathbb{S}ig, \mathbb{S}en, \vdash)$ one can easily construct the *compact entailment subsystem* [Diaconescu, 2006] $\mathcal{E}^c = (\mathbb{S}ig, \mathbb{S}en, \vdash^c)$ by defining the entailment relation $\vdash^c$ as follows: $\Gamma \vdash^c_\Sigma E$ iff for each finite set $E_f \subseteq E$ there exists a finite set $\Gamma_f \subseteq \Gamma$ such that $\Gamma_f \vdash_\Sigma E_f$.

Lemma 5 (Compact entailment subsystems) [Diaconescu, 2006]. $(\mathbb{S}ig, \mathbb{S}en, \vdash^c)$ *is an entailment system.*

2.3 Basic sentences

A set of sentences $B \subseteq Sen(\Sigma)$ is called *basic* [Diaconescu, 2003] if there exists a Σ-model M_B such that, for all Σ-models M, $M \models B$ iff there exists a morphism $M_B \to M$. We say that M_B is a *basic model* of B. If in addition the morphisms $M_B \to M$ is unique then the set B is called *epi basic*. It is well-known that any set of atoms in **FOL** and **POA** is epi basic (see for example [Diaconescu, 2003] or [Diaconescu, 2008]).

Lemma 6. *Any set of atomic sentences in* **FOL**, **OSA**, **POA** *and* **PA** *is epi basic.*

Proof. In **FOL**, for a set E of atomic (S,F,P)-sentences there exists a basic model M_E. Actually it is the initial model for E. This is constructed as follows: on the quotient $(T_{(S,F)})_{=E}$ of the term model $T_{(S,F)}$ by the congruence generated by the equational atoms of E, we interpret each relation symbol $\pi \in P$ by $(M_E)_\pi = \{(t_1/_{=E}, \dots, t_n/_{=E}) \mid \pi(t_1, \dots, t_n) \in E\}$. A similar argument as the preceding holds for **OSA** and **POA** too.

In **PA** for a set of atomic sentences E we define S_E as the set of sub-terms appearing in E. Note that S_E is a partial algebra. The basic model M_E will be the quotient of S_E by the partial congruence induced by the equalities in E. $\square$

2.4 Internal logic

The following institutional notions dealing with logical connectives and quantifiers were defined in [Tarlecki, 1986].

Let Σ be a signature of an institution, a Σ-sentence

- $\neg e$ is a *(semantic) negation* of the Σ-sentence e when for every Σ-model M we have $M \models_\Sigma \neg e$ iff $M \not\models_\Sigma e$,

– $e_1 \wedge e_2$ is a *(semantic) conjunction* of the Σ-sentences e_1 and e_2 when for every Σ-model M we have $M \models_\Sigma e_1 \wedge e_2$ iff $M \models_\Sigma e_1$ and $M \models_\Sigma e_2$,

– $e_1 \Rightarrow e_2$ is a *(semantic) implication* of the Σ-sentences e_1 and e_2 when for every Σ-model M we have $M \models_\Sigma e_1 \Rightarrow e_2$ iff $M \models_\Sigma e_1$ implies $M \models_\Sigma e_2$, and

– $(\forall\chi)e'$, where $\Sigma \xrightarrow{\chi} \Sigma' \in \mathbb{S}ig$ and $e' \in \mathbb{S}en(\Sigma')$, is a *(semantic) universal χ-quantification* of e' when for every Σ-model M we have $M \models_\Sigma (\forall\chi)e'$ iff $M' \models_{\Sigma'} e'$ for all χ-expansions M' of M.

Very often quantification is considered only for a restricted class of signature morphisms. For example, quantification in **FOL** considers only the finitary signature extensions with constants. For a class $\mathcal{D} \subseteq \mathbb{S}ig$ of signature morphisms, we say that the institution has universal $\mathcal{D}$-quantifications when for each $\Sigma \xrightarrow{\chi} \Sigma' \in \mathcal{D}$, each Σ'-sentence has a universal χ-quantification.

2.5 Reachable models

In this subsection we give an abstract characterizations of reachable models.

Definition 7. Let $\mathcal{D}$ be a broad subcategory [2] of signature morphisms of an institution $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$. We say that a Σ-model M is $\mathcal{D}$-reachable if for each span of signature morphisms $\Sigma_1 \xleftarrow{\chi_1} \Sigma_0 \xrightarrow{\chi} \Sigma$ in $\mathcal{D}$, each χ_1-expansion M_1 of $M\upharpoonright_\chi$ determines a substitution $\theta : \chi_1 \to \chi$ such that $M\upharpoonright_\theta = M_1$.

In concrete examples of institutions, $\mathcal{D}$-reachable models correspond to models with elements reachable by ground terms.

Proposition 8. *In* **FOL**, **OSA**, **POA** *and* **PA**, *assume that $\mathcal{D}$ is the class of signature extensions with constants. A model M is $\mathcal{D}$-reachable iff its elements are the interpretations of terms.*

Proof. In **FOL**, let $\Sigma = (S, F, P)$ be a signature, X and Y two disjoint sets of constants with elements that are different from the constants in F, and (M, h) a $\Sigma(Y)$-model with elements that are interpretation of terms, i.e. the unique extension $h^\# : T_\Sigma(Y) \to M$ of h to a Σ-morphism is surjective. Then for every $\Sigma(X)$-model (M, g) there exists a function $\theta : X \to T_\Sigma(Y)$ such that $\theta; h^\# = g$. Note that for any $x \in X$ we have $((M, h)\upharpoonright_\theta)_x = h^\#(\theta(x)) = g(x) = (M, g)_x$. Therefore, $(M, h)\upharpoonright_\theta = (M, g)$.

For the converse implication, let $\Sigma = (S, F, P)$ be a signature and assume a Σ-model M that is $\mathcal{D}$-reachable. We prove that $T_\Sigma \to M$ is surjective, i.e. for every $m \in M$ there exists $t \in T_\Sigma$ such that $M_t = m$. Let $m \in M_s$ be an arbitrary element of M. Consider a variable x of sort s and let N be an expansion of M along $\Sigma \hookrightarrow \Sigma(x)$ (where $\Sigma(x) =$

[2] $\mathcal{C}'$ is a broad subcategory of $\mathcal{C}$ if $|\mathcal{C}'| = |\mathcal{C}|$.

$(S, F \cup \{x\}, P))$ which interprets the constant symbol x as m. Since M is $\mathcal{D}$-reachable there exists a substitution $\theta : \{x\} \to T_\Sigma$ such that $M\!\upharpoonright_\theta = N$. Take $t = \theta(x)$ and we have $M_t = M_{\theta(x)} = (M\!\upharpoonright_\theta)_x = N_x = m$.

One can replicate the arguments above for **OSA** and **POA** too.

In **PA**, let $\Sigma = (S, F)$ be a signature, X and Y two sets of constants with elements that are different from the constants in F, and M a $\Sigma(Y)$-model, with elements that are interpretations of terms, i.e. the unique $\Sigma(Y)$-morphism $h : T_M \to M$ is surjective, where $T_M \subseteq T_{F \cup Y}$ is the maximal algebra of terms such that $M \models def(t)$ for all $t \in T_M$. Let M' be a $\Sigma(X)$-expansion of $M\!\upharpoonright_\Sigma$, where $M' = (M\!\upharpoonright_\Sigma, g)$, $g = (g_s : X'_s \to M_s)_{s \in S}$ and $X' \subseteq X$ are all constants of X defined in M'. Since h is surjective, there exists a function $\theta' : X' \to T_M$ such that $\theta'; h = g$. We straightforwardly extend θ' to a substitution $\theta : X \to T_\Sigma(Y)$ such that θ is equal to θ' on X', and $\theta(x) = \perp$ for all $x \in (X \backslash X')$. Note that for every

- $x \in X'$ we have: $(M\!\upharpoonright_\theta)_x = M_{\theta(x)} = M_{\theta'(x)} = h(\theta'(x)) = g(x) = M'_x$;
- $x \in (X \backslash X')$ we have: $(M\!\upharpoonright_\theta)_x = M_{\theta(x)} = M_\perp = M'_\perp = M'_x$.

Hence $M\!\upharpoonright_\theta = M'$.

For the converse implication, consider a partial (S, F)-algebra M that is $\mathcal{D}$-reachable. We prove that for all $m \in M$ there exists a term $t \in T_\Sigma$ such that $M_t = m$. Let $m \in M_s$ be an arbitrary element of M. Consider a variable x of sort s and let N be an expansion of M along $\Sigma \hookrightarrow \Sigma(x)$ which interprets the constant symbol x as m. Since M is $\mathcal{D}$-reachable, there exists a substitution $\theta : \{x\} \to T_\Sigma$ such that $M\!\upharpoonright_\theta = N$. Take $t = \theta(x)$, and we have $M_t = M_{\theta(x)} = (M\!\upharpoonright_\theta)_x = N_x = m$. □

Remark. Note that for each set E of atomic sentences in **FOL**, **OSA**, **POA** and **PA** the basic model M_E is $\mathcal{D}$-reachable, where $\mathcal{D}$ is the class of signature extensions with constants.

Definition 9. We say that a signature morphism $\varphi : \Sigma \to \Sigma'$ is finitary if it is finitely presented [3] in the comma category $\Sigma/\mathbb{S}ig$.

In typical institutions the extensions of signatures with a finite number of symbols are finitary.

Definition 10. Assume two broad subcategories of signature morphisms $\mathcal{D}$ and $\mathcal{D}^l$. A Σ-model M is $(\mathcal{D}, \mathcal{D}^l)$-reachable if there exists a signature morphism $\Sigma \xrightarrow{\varphi} \Sigma' \in \mathcal{D}$ and a $\mathcal{D}$-reachable Σ'-model M' such that

[3] An object A in a category $\mathcal{C}$ is *finitely presented* ([Adámek and Rosický, 1994]) if

- for each directed diagram $D : (J, \leq) \to \mathcal{C}$ with co-limit $\{Di \xrightarrow{\mu_i} B\}_{i \in J}$, and for each morphism $A \xrightarrow{g} B$, there exists $i \in |J|$ and $A \xrightarrow{g_i} Di$ such that $g_i; \mu_i = g$,
- for any two arrows g_i and g_j as above, there exists $k \in |J|$ such that $i \leq k$, $j \leq k$ and $g_i; D(i \leq k) = g_j; D(j \leq k)$.

1. $M'\upharpoonright_{\varphi} = M$, and

2. φ is the vertex of a co-limit $(\varphi_i \xrightarrow{u_i} \varphi)_{i\in J}$ of a directed diagram $(\varphi_i \xrightarrow{u_{i,j}} \varphi_j)_{(i\leq j)\in(J,\leq)}$ in $\Sigma/\mathbb{S}ig$, with $\varphi_i \in \mathcal{D}^l$ for all $i \in J$.

Throughout this paper we implicitly assume that $\mathcal{D}$ represents the subcategory of signature morphisms that consists of signature extensions with constants, and $\mathcal{D}^l$ represents the subcategory of signature morphisms that consists of signature extensions with a finite number of constants of loose sorts. In institutions such as **HCL**, **HOSA**, **HPOA** and **HPA**, for each signature we consider the set of constrained sorts empty; therefore all sorts are regarded as loose, and $\mathcal{D}$ consists of signature extensions with constants of any sort, and $\mathcal{D}^l$ consists of signature extensions with a finite number of constants of any sort.

Proposition 11. *In* **CHCL***, all models are* $(\mathcal{D},\mathcal{D}^l)$*-reachable.*

Proof. Let M be a Σ-model, where $\Sigma = (S,F,F^c,P)$. There exists an assignment $f : Y \to M$, where Y is a set of loose variables that are different from the constants in F, such that the unique extension $f^\# : T_{(S,F^c)}(Y) \to M\upharpoonright_{(S,F^c)}$ of f to a (S,F^c)-morphism is surjective. The inclusion $\Sigma \overset{\varphi}{\hookrightarrow} \Sigma(Y)$, where $\Sigma = (S,F\cup Y,F^c,P)$, is the vertex of the directed co-limit $((\Sigma \overset{\varphi_i}{\hookrightarrow} \Sigma(Y_i)) \overset{u_i}{\hookrightarrow} (\Sigma \overset{\varphi}{\hookrightarrow} \Sigma(Y)))_{Y_i\subseteq Y\, finite}$ of the directed diagram $((\Sigma \overset{\varphi_i}{\hookrightarrow} \Sigma(Y_i)) \overset{u_{i,j}}{\hookrightarrow} (\Sigma \overset{\varphi_j}{\hookrightarrow} \Sigma(Y_j)))_{Y_i\subseteq Y_j\subseteq Y\, finite}$. Since for all $s \in S$ the function $f^\#_s : (T_{(S,F^c)}(Y))_s \to M_s$ is surjective, the elements of (M,f) consist only of interpretations of terms. By Proposition 8, (M,f) is $\mathcal{D}$-reachable, and by Definition 10, M is $(\mathcal{D},\mathcal{D}^l)$-reachable. □

In the following we prove an important property of $(\mathcal{D},\mathcal{D}^l)$-reachable models.

Proposition 12. *Assume a* $(\mathcal{D},\mathcal{D}^l)$*-reachable* Σ*-model* M *as in Definition 10, and a finitary signature morphism* $\Sigma \xrightarrow{\chi} \Sigma_1 \in \mathcal{D}$*. Then every* χ*-expansion* M_1 *of* M *generates a substitution* $\theta : \chi \to \psi$*, and a* ψ*-expansion* M_2 *of* M *such that* $\Sigma \xrightarrow{\psi} \Sigma_2 \in \mathcal{D}^l$ *and* $M_2\upharpoonright_{\theta} = M_1$*.*

Proof. Let M be a $(\mathcal{D},\mathcal{D}^l)$-reachable model as in Definition 10 and let M_1 be a χ-expansion of M, where $\Sigma \xrightarrow{\chi} \Sigma_1 \in \mathcal{D}$ is finitary. Since M' is $\mathcal{D}$-reachable there exists a substitution $v : \chi \to \varphi$ such that $M'\upharpoonright_v = M_1$. Because χ is finitary there exists a substitution $v_i : \chi \to \varphi_i$ such that $v_i; u_i = v$. Then take $\psi = \varphi_i$, $\theta = v_i$ and $M_2 = M'\upharpoonright_{u_i}$. □

Since the subcategories $\mathcal{D}$ and $\mathcal{D}^l$ of signature morphisms are fixed in concrete institutions, we will refer to $\mathcal{D}$-reachable model(s) as ground reachable model(s), and to $(\mathcal{D},\mathcal{D}^l)$-reachable model(s) as reachable model(s).

3 Universal Institutions

The *reachable universal entailment system* (*RUES*) developed in this section consists of four layers:

1. the "atomic" layer of the *atomic entailment system* (*AES*), which in abstract settings is assumed but is developed in concrete examples,
2. the layer of the *entailment system with implications* (*IES*) obtained by adding to the AES the rules of implication,
3. the layer of the *generic universal entailment system* (*GUES*) obtained by adding to the IES the rules for the quantification over variables of loose sorts, and
4. the upmost layer of the RUES of I is obtained by adding to the GUES the rules for the quantification over variables of constrained sorts.

Soundness and completeness at each layer are obtained relatively to the soundness and completeness of the layer immediately below.

Definition 13. Consider an institution $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$, and let $\mathcal{D}^\bullet, \mathbb{S}ig_n \subseteq \mathbb{S}ig$ be two broad subcategories of signature morphisms, and $\mathbb{S}en_n : \mathbb{S}ig_n \to \mathbb{S}et$ a sub-functor of $\mathbb{S}en$ [4]. We denote by I_n the institution $(\mathbb{S}ig_n, \mathbb{S}en_n, \mathbb{M}od_n, \models_n)$, where

- $\mathbb{M}od_n : \mathbb{S}ig_n \to \mathbb{C}at^{op}$ is the restriction of $\mathbb{M}od : \mathbb{S}ig \to \mathbb{C}at^{op}$ to $\mathbb{S}ig_n$, and
- for all $\Sigma \in |\mathbb{S}ig|$, $(\models_n)_\Sigma$ is the restriction of $\models_\Sigma$ to $|\mathbb{M}od(\Sigma)| \times \mathbb{S}en_n(\Sigma)$.

We say that I is a $\mathcal{D}^\bullet$*-universal institution over* I_n when

1. $(\forall\chi)\rho \in \mathbb{S}en(\Sigma)$ for all $\Sigma \xrightarrow{\chi} \Sigma' \in \mathcal{D}^\bullet$ and $\rho \in \mathbb{S}en_n(\Sigma')$, and
2. any sentence of I is of the form $(\forall\chi)\rho$ as above.

Fact 3.1 *If* $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ *is a* $\mathcal{D}^\bullet$*-universal institution over the institution* $I_n = (\mathbb{S}ig_n, \mathbb{S}en_n, \mathbb{M}od_n, \models_n)$ *then for all* $\Sigma \in |\mathbb{S}ig| = |\mathbb{S}ig_n|$*,* $\Gamma \subseteq \mathbb{S}en_n(\Sigma)$ *and* $\rho \in \mathbb{S}en_n(\Sigma)$ *we have* $\Gamma \models_n \rho$ *iff* $\Gamma \models \rho$.

For example **CHCL** is a $\mathcal{D}^c$-universal institution over its restriction $\mathbf{CHCL}_2$ to

(a) signature morphisms that do not add constructors on "old" sorts, and

(b) sentences quantified over finite sets of variables of loose sorts,

[4] For each signature $\Sigma \in |\mathbb{S}ig_n|$ we have $\mathbb{S}en_n(\Sigma) \subseteq \mathbb{S}en(\Sigma)$ and for any signature morphism $\Sigma \xrightarrow{\varphi} \Sigma' \in \mathbb{S}ig_n$ we have $\varphi(\mathbb{S}en_n(\Sigma)) \subseteq \mathbb{S}en_n(\Sigma')$).

where $\mathcal{D}^c$ consists of signature extensions with a finite number of constants of constrained sorts. If $\mathbb{S}ig^{\mathbf{CHCL}_2} = \mathbb{S}ig^{\mathbf{CHCL}}$ then the translation of a sentence quantified over a variable of loose sort may be a sentence quantified over a variable of constrained sort, which implies that $\mathbf{CHCL}_2$ is not an institution.

Example 10. Consider the following example of **CHCL** signature morphism:

$$\boxed{\text{sort Triv}} \xrightarrow[\mathit{Triv} \mapsto \mathit{Nat}]{\varphi} \boxed{\begin{array}{ll} \text{sort Nat} & \\ \text{op } 0 : \to \text{Nat} & \{\text{constr}\} \\ \text{op s\_} : \text{Nat} \to \text{Nat} & \{\text{constr}\} \end{array}}$$

Note that $(\forall x : \mathit{Triv})x = x$ is a sentence quantified over the variable $(x : \mathit{Triv})$, which is loose, but $\varphi((\forall x : \mathit{Triv})x = x)$ is equal to $(\forall x : \mathit{Nat})x = x$, a sentence quantified over the variable $(x : \mathit{Nat})$ which is constrained.

3.1 Reachable universal entailment systems

Our approach is top-down, and we start by defining the proof rules for the sentences quantified over constrained variables.

Assumption 3.1 *Throughout this subsection, we assume a $\mathcal{D}^c$-universal institution $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ over $I_2 = (\mathbb{S}ig_2, \mathbb{S}en_2, \mathbb{M}od_2, \models_2)$, and two broad subcategories of signature morphisms $\mathcal{D}^l \subseteq \mathcal{D} \subseteq \mathbb{S}ig$ such that $\mathcal{D}^c \subseteq \mathcal{D}$, and I_2 has $\mathcal{D}^l$-quantifications.*

CHCL is a $\mathcal{D}^c$-universal institution over $\mathbf{CHCL}_2$, where $\mathcal{D}^c$ consists of signature extensions with a finite number of constants of constrained sorts. Assume that $\mathcal{D}$ and $\mathcal{D}^l$ are as in subsection 2.5. It follows that **CHCL** is an example of $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$.

We define the following proof rules, for the RUES of I:

(C-Abstraction) $\dfrac{\{\Gamma \vdash_\Sigma (\forall \varphi)\theta(\rho) \mid \varphi \in \mathcal{D}^l,\ \theta : \chi \to \varphi \text{ is a substitution}\}}{\Gamma \vdash_\Sigma (\forall \chi)\rho}$

where $\Gamma \subseteq \mathbb{S}en(\Sigma)$ and $(\forall \chi)\rho \in \mathbb{S}en(\Sigma)$ with $\chi \in \mathcal{D}^c$.

(Substitutivity) $\dfrac{}{(\forall \chi)\rho \vdash_\Sigma (\forall \varphi)\theta(\rho)}$

where $(\forall \chi)\rho \in \mathbb{S}en(\Sigma)$ and $\theta : \chi \to \varphi$ is a substitution.

In **CHCL**, consider a set Γ of Σ-sentences and a Σ-sentence $(\forall x)\rho$ such that x is a constrained variable. In this case *C-Abstraction* says that if for any term t of the same sort as x formed with loose variables and operation symbols from Σ, we have $\Gamma \vdash (\forall Y)\rho(x \leftarrow t)$ [5], where Y are all (loose) variables that occur in t, then we have proved $\Gamma \vdash (\forall x)\rho$. In most of the cases the set of terms t is infinite, which implies that the premises of *C-Abstraction* are infinite, and thus, the corresponding entailment system is not compact.

Given a compact entailment system $\mathcal{E}_2 = (\mathbb{S}ig_2, \mathbb{S}en_2, \vdash_2)$ for I_2, the RUES of I consists of the least entailment system over $\mathcal{E}_2$ closed under *C-Abstraction* and *Substitutivity*. Note that the resulting entailment system is not compact (even if $\mathcal{E}_2$ is compact)

[5] $\rho(x \leftarrow t)$ is the formula obtained from ρ by substituting t for x.

since the premises of *C-Abstraction* may be infinite. The abstract RUES above can be instantiated to **CHCL**, **CHOSA**, **CHPOA** and **CHPA**.

Proposition 14 (R-soundness). *The RUES of I is sound if the entailment system of I_2 is sound, the signature morphisms in $\mathcal{D}^c$ are finitary, and all models of I are $(\mathcal{D},\mathcal{D}^l)$-reachable.*

Proof. Firstly, we show that

1. *C-Abstraction* is sound, i.e. for a set $\Gamma \subseteq \mathbb{S}en(\Sigma)$ of sentences and any sentence $(\forall\chi)\rho \in \mathbb{S}en(\Sigma)$ with $\chi \in \mathcal{D}^c$ we have $\Gamma \models_\Sigma (\forall\chi)\rho$ whenever $\Gamma \models_\Sigma (\forall\varphi)\theta(\rho)$ for all substitutions $\theta : \chi \to \varphi$ with $\varphi \in \mathcal{D}^l$. Fix Γ and $(\forall\chi)\rho$ as above, and assume $M \models \Gamma$. Let M' be an χ-expansion of M. By Proposition 12, there exists a signature morphism $\Sigma \xrightarrow{\varphi} \Sigma'' \in \mathcal{D}^l$, a substitution $\theta : \chi \to \varphi$, and a φ-expansion M'' of M such that $M''\restriction_\theta = M'$. We have $M \models (\forall\varphi)\theta(\rho)$ which implies $M'' \models \theta(\rho)$, and by the satisfaction condition, $M' \models \rho$. Since M' is an arbitrary expansion of M, we have thus proved $M \models (\forall\chi)\rho$.
2. *Substitutivity* is sound, i.e. $(\forall\chi)\rho \models_\Sigma (\forall\varphi)\theta(\rho)$ for all sentences $(\forall\chi)\rho \in \mathbb{S}en(\Sigma)$ and substitutions $\theta : \chi \to \varphi$. Let M be a Σ-model such that $M \models (\forall\chi)\rho$. Assume a substitution $\theta : \chi \to \varphi$, and let M_2 be any φ-expansion of M. We have $M_2\restriction_\theta \models \rho$ and by the satisfaction condition, $M_2 \models \theta(\rho)$. Since M_2 is an arbitrary expansion of M, we have thus proved $M \models (\forall\varphi)\theta(\rho)$.

Secondly, since $\mathcal{E}_2$, *C-Abstraction* and *Substitutivity* are sound then the least entailment system closed to *C-Abstraction* and *Substitutivity* is sound too. □

CHCL with the RUES defined above is not complete. If we restrict the signatures in such a way that all operations of constrained sorts are constructors then we can prove completeness. Concretely, we will prove that the RUES of **RHCL** is complete.

Assumption 3.2 *In addition to Assumption 3.1, we also consider a full subcategory* [6] *of signature morphisms $\mathbb{S}ig^r \subseteq \mathbb{S}ig$.*

Let $\mathbb{S}ig_2^r \subseteq \mathbb{S}ig_2$ be the full subcategory such that $|\mathbb{S}ig_2^r| = |\mathbb{S}ig^r|$. Note that the definition of $\mathbb{S}ig_2^r$ is correct because $\mathbb{S}ig^r \subseteq \mathbb{S}ig$ and $|\mathbb{S}ig| = |\mathbb{S}ig_2|$. We define

1. $\mathbb{S}en^r : \mathbb{S}ig^r \to \mathbb{S}et$ and $\mathbb{M}od^r : \mathbb{S}ig^r \to \mathbb{C}at^{op}$ as the restrictions of $\mathbb{S}en : \mathbb{S}ig \to \mathbb{S}et$ and $\mathbb{M}od : \mathbb{S}ig \to \mathbb{C}at^{op}$, respectively, to $\mathbb{S}ig^r$.
2. $\mathbb{S}en_2^r : \mathbb{S}ig_2^r \to \mathbb{S}et$ and $\mathbb{M}od_2^r : \mathbb{S}ig_2^r \to \mathbb{C}at^{op}$ as the restrictions of $\mathbb{S}en_2 : \mathbb{S}ig_2 \to \mathbb{S}et$ and $\mathbb{M}od : \mathbb{S}ig_2 \to \mathbb{C}at^{op}$, respectively, to $\mathbb{S}ig_2^r$.
3. $\models^r \overset{def}{=} (\models_\Sigma)_{\Sigma\in|\mathbb{S}ig^r|}$ and $\models_2^r \overset{def}{=} ((\models_2)_\Sigma)_{\Sigma\in|\mathbb{S}ig^r|}$.

[6] $\mathcal{C}'$ is a full subcategory of $\mathcal{C}$ if for all objects $A,B \in |\mathcal{C}'|$ we have $\mathcal{C}(A,B) = \mathcal{C}'(A,B)$.

Fact 3.2 *$I^r = (\mathbb{S}ig^r, \mathbb{S}en^r, \mathbb{M}od^r, \models^r)$ and $I_2^r = (\mathbb{S}ig_2^r, \mathbb{S}en_2^r, \mathbb{M}od_2^r, \models_2^r)$ are institutions.*

In concrete examples,

1. I is **CHCL**, I^r is **RHCL**,
2. I_2 is **CHCL**$_2$ and I_2^r is **RHCL**$_2$, the restriction of **RHCL** to
   (a) signature morphisms that do not add constructors on "old" sorts, and
   (b) sentences quantified over variables of loose sorts.

Theorem 15 (R-completeness). *Assume that every signature morphism in $\mathcal{D}^c$ is finitary, and each model of I is $(\mathcal{D}, \mathcal{D}^l)$-reachable. Then the RUES $\mathcal{E}^r = (\mathbb{S}ig^r, \mathbb{S}en^r, \vdash^r)$ of $I^r = (\mathbb{S}ig^r, \mathbb{S}en^r, \mathbb{M}od^r, \models^r)$ is complete if the entailment system $\mathcal{E}_2^r = (\mathbb{S}ig_2^r, \mathbb{S}en_2^r, \vdash_2^r)$ of $I_2^r = (\mathbb{S}ig_2^r, \mathbb{S}en_2^r, \mathbb{M}od_2^r, \models_2^r)$ is complete and compact.*

Proof. Let $\Sigma \in |\mathbb{S}ig^r|$ be a signature of I^r, $\Gamma \subseteq \mathbb{S}en^r(\Sigma)$ a set of sentences and $(\forall\chi)e' \in \mathbb{S}en^r(\Sigma)$ any sentence, where $\Sigma \xrightarrow{\chi} \Sigma' \in \mathcal{D}^c$, such that $\Gamma \models^r (\forall\chi)e'$. Suppose towards a contradiction that $\Gamma \nvdash^r (\forall\chi)e'$. Then there exists a signature morphism $\Sigma \xrightarrow{\varphi} \Sigma'' \in \mathcal{D}^l$ and a substitution $\theta : \chi \to \varphi$ in I such that $\Gamma \nvdash^r (\forall\varphi)\theta(e')$.

We define the set of Σ-sentences $\Gamma_2 = \{\rho \in \mathbb{S}en_2^r(\Sigma) \mid \Gamma \vdash^r \rho\}$. We show that $\Gamma_2 \nvdash_2^r (\forall\varphi)\theta(e')$. Assume that $\Gamma_2 \vdash_2^r (\forall\varphi)\theta(e')$. Since the entailment system of I_2^r is compact, there exists $\Gamma_f \subseteq \Gamma_2$ finite such that $\Gamma_f \vdash_2^r (\forall\varphi)\theta(e')$. It follows that $\Gamma_f \vdash^r (\forall\varphi)\theta(e')$ and, since $\Gamma \vdash^r \rho$ for all $\rho \in \Gamma_f$, by *Unions* we obtain $\Gamma \vdash^r \Gamma_f$. Therefore, $\Gamma \vdash^r (\forall\varphi)\theta(e')$ which is a contradiction with our assumption.

We have $\Gamma_2 \nvdash_2^r (\forall\varphi)\theta(e')$, and because $\mathcal{E}_2^r$ is complete, there exists a Σ-model M such that $M \models_2^r \Gamma_2$ and $M \not\models_2^r (\forall\varphi)\theta(e')$. It follows that $M \models^r \Gamma_2$ and $M \not\models^r (\forall\varphi)\theta(e')$. Note that $M \not\models^r (\forall\varphi)\theta(e')$ implies $M \not\models^r (\forall\chi)e'$. If we prove that $M \models^r \Gamma$ then we reach a contradiction with $\Gamma \models^r (\forall\chi)e'$ and therefore we can conclude $\Gamma \vdash^r (\forall\chi)e'$.

Let $(\forall\chi_1)e_1 \in \Gamma$, where $\Sigma \xrightarrow{\chi_1} \Sigma_1 \in \mathcal{D}^c$, and N be any χ_1-expansion of M. Since M is $(\mathcal{D}, \mathcal{D}^l)$-reachable, by Proposition 12, there exists a signature morphism $\Sigma \xrightarrow{\varphi_1} \Sigma_1' \in \mathcal{D}^l$, a substitution $\psi : \chi_1 \to \varphi_1$ in I, and a φ_1-expansion N' of M such that $N'\upharpoonright_\theta = N$. By *Substitutivity*, we have $(\forall\varphi_1)\psi(e_1) \in \Gamma_2$, and since $M \models^r \Gamma_2$, we obtain $M \models^r (\forall\varphi_1)\psi(e_1)$. Since N' is a φ_1-expansion of M, $N' \models^r \psi(e_1)$ and by satisfaction condition, $N \models^r e_1$. $\square$

One may wonder what is the role played by I in the abstract setting. The answer is simple: I provides the subcategory $\mathcal{D}^c$ of signature morphisms and the satisfaction relation for the sentences quantified over the signature morphisms in $\mathcal{D}^c$. If I is **CHCL** and I^r is **RHCL** then it is easy to notice that a signature extension with constants of constrained sorts is not a signature morphism in **RHCL**. Therefore, in concrete examples, we have $\mathcal{D}^c \nsubseteq \mathbb{S}ig^r$.

3.2 Generic universal entailment systems

We define the proof rules for the sentences quantified over loose variables.

Assumption 3.3 *Throughout this subsection, we assume a $\mathcal{D}^l$-universal institution $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ over $I_1 = (\mathbb{S}ig_1, \mathbb{S}en_1, \mathbb{M}od_1, \models_1)$.*

HCL is a $\mathcal{D}^l$-universal institution over its restriction $\mathbf{HCL}_1$ to quantifier-free sentences, where $\mathcal{D}^l$ consists of signature extensions with a finite number of constants. Also, $\mathbf{RHCL}_2$ is a $\mathcal{D}^l$-universal institution over its restriction $\mathbf{RHCL}_1$ to quantifier-free sentences, where $\mathcal{D}^l$ consists of signature extensions with a finite number of constants of loose sorts.

We define the proof rules for the GUES of I.

(Generalization) $\dfrac{\Gamma \vdash_\Sigma (\forall\varphi)e'}{\varphi(\Gamma) \vdash_{\Sigma'} e'}$ and $\dfrac{\varphi(\Gamma) \vdash_{\Sigma'} e'}{\Gamma \vdash_\Sigma (\forall\varphi)e'}$

where $\Gamma \subseteq \mathbb{S}en(\Sigma)$, $(\forall\varphi)e' \in \mathbb{S}en(\Sigma)$ and $\Sigma \xrightarrow{\varphi} \Sigma' \in \mathcal{D}^l$.

Given a compact entailment system $\mathcal{E}_1 = (\mathbb{S}ig_1, \mathbb{S}en_1, \vdash_1)$ for I_1, the GUES of I consists of the least entailment system over $\mathcal{E}_1$ closed under *Substitutivity* and *Generalization*.

Proposition 16 (G-soundness). *The GUES of I is sound whenever the entailment system of I_1 is sound.*

Proof. Note that *Generalization* is sound, i.e. for every set Γ of Σ-sentences and each Σ-sentence $(\forall\varphi)e'$ we have $\Gamma \models (\forall\varphi)e'$ iff $\varphi(\Gamma) \models e'$. Since *Substitutivity* and $\mathcal{E}_1$ are also sound, the least entailment system over $\mathcal{E}_1$ closed under *Substitutivity* and *Generalization* is sound too. $\square$

Theorem 17 (G-completeness). *Assume that*

1. *the entailment system of I_1 is complete and compact, and*
2. *for every set of sentences $E \subseteq \mathbb{S}en_1(\Sigma)$ and each sentence $e \in \mathbb{S}en_1(\Sigma)$,*

$$E \models_1 e \text{ iff for all } \mathcal{D}^l\text{-reachable models } M, \text{ we have } M \models_1 E \text{ implies } M \models_1 e$$

Then the GUES $\mathcal{E} = (\mathbb{S}ig, \mathbb{S}en, \vdash)$ of I is complete and compact.

Proof. Let $\Sigma \in |\mathbb{S}ig|$, $\Gamma \subseteq \mathbb{S}en(\Sigma)$ and $(\forall\varphi)e' \in \mathbb{S}en(\Sigma)$, where $\Sigma \xrightarrow{\varphi} \Sigma' \in \mathcal{D}^l$ and $e' \in \mathbb{S}en_1(\Sigma')$, such that $\Gamma \models_\Sigma (\forall\varphi)e'$. We show that $\Gamma \vdash_\Sigma (\forall\varphi)e'$. Suppose towards a contradiction that $\Gamma \not\vdash_\Sigma (\forall\varphi)e'$.

We define the set of Σ'-sentences $\Gamma_1^\varphi = \{\rho' \in \mathbb{S}en_1(\Sigma') | \varphi(\Gamma) \vdash \rho'\}$. Suppose that $\Gamma_1^\varphi \vdash_1 e'$. Since the entailment system of I_1 is compact, there exists a finite $\Gamma_f \subseteq \Gamma_1^\varphi$ such that $\Gamma_f \vdash_1 e'$, which implies $\Gamma_f \vdash e'$. We have $\varphi(\Gamma) \vdash \rho_f$ for all $\rho_f \in \Gamma_f$, which implies $\varphi(\Gamma) \vdash \Gamma_f$. By *Transitivity* property, $\varphi(\Gamma) \vdash e'$, and by *Generalization*, $\Gamma \vdash (\forall\varphi)e'$, which contradicts our assumption. Hence, $\Gamma_1^\varphi \not\vdash_1 e'$.

By completeness of I_1, $\Gamma_1^{\varphi} \not\models_1 e'$. There exists a $\mathcal{D}^l$-reachable model M such that $M \models_1 \Gamma_1^{\varphi}$ but $M \not\models_1 e'$. It follows that $M \models \Gamma_1^{\varphi}$ and $M \not\models e'$. This implies $M\!\upharpoonright_{\varphi} \not\models (\forall\varphi)e'$. If we prove $M\!\upharpoonright_{\varphi} \models \Gamma$ then we reach a contradiction with $\Gamma \models (\forall\varphi)e'$. We will focus on proving $M\!\upharpoonright_{\varphi} \models \Gamma$.

Consider $(\forall\varphi_1)e_1 \in \Gamma$, where $\Sigma \xrightarrow{\varphi_1} \Sigma_1 \in \mathcal{D}^l$, and let N be any φ_1-expansion of $M\!\upharpoonright_{\varphi}$. We prove that $N \models e_1$. Since M is $\mathcal{D}^l$-reachable there exists a substitution $\theta : \varphi_1 \to \varphi$ such that $M\!\upharpoonright_{\theta} = N$. By *Substitutivity*, $\Gamma \vdash (\forall\varphi)\theta(e_1)$, and by *Generalization*, $\varphi(\Gamma) \vdash \theta(e_1)$, which implies $\theta(e_1) \in \Gamma_1^{\varphi}$. Since $M \models \Gamma_1^{\varphi}$, we have $M \models \theta(\rho)$ and by the satisfaction condition $M\!\upharpoonright_{\theta} = N \models e_1$.

For the compactness of the GUES of I, we consider the compact entailment subsystem $\mathcal{E}^c = (\mathbb{S}ig, \mathbb{S}en, \vdash^c)$ of $\mathcal{E} = (\mathbb{S}ig, \mathbb{S}en, \vdash)$. It contains $\mathcal{E}_1$ because $\mathcal{E}_1$ is compact. It is straightforward to check that $\mathcal{E}^c$ satisfies *Substitutivity*. If we prove that $\mathcal{E}^c$ satisfies *Generalization* then because $\mathcal{E}$ is the least entailment system over $\mathcal{E}_1$ satisfying the rules of *Substitutivity* and *Generalization* we obtain $\mathcal{E}^c = \mathcal{E}$.

If $\Gamma \vdash^c (\forall\varphi)e'$ then there exists a finite subset $\Gamma_f \subseteq \Gamma$ such that $\Gamma_f \vdash (\forall\varphi)e'$. By *Generalization*, $\varphi(\Gamma_f) \vdash e'$ which means $\varphi(\Gamma) \vdash^c e'$. Now if $\varphi(\Gamma) \vdash^c e'$ then there is a finite subset $\Gamma_f \subseteq \Gamma$ such that $\varphi(\Gamma_f) \vdash e'$. By *Generalization*, $\Gamma_f \vdash (\forall\varphi)e'$ which implies $\Gamma \vdash^c (\forall\varphi)e'$. □

3.3 Entailment systems with implications

All the results in this subsection can be found in [Codescu and Găină, 2008].

Assumption 3.4 *Throughout this subsection, we assume an institution* $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$*, and a sub-functor* $\mathbb{S}en_0 : \mathbb{S}ig \to \mathbb{S}et$ *of* $\mathbb{S}en$ *such that*

- $(\bigwedge H \Rightarrow C) \in \mathbb{S}en(\Sigma)$ *for all finite sets* $H \subseteq \mathbb{S}en_0(\Sigma)$ *and any* $C \in \mathbb{S}en_0(\Sigma)$*,*
- *any sentence of* I *is of the form* $\bigwedge H \Rightarrow C$ *as above.*

An example of institution $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ is the restriction of **HCL** to quantifier-free sentences, where $I_0 = (\mathbb{S}ig, \mathbb{S}en_0, \mathbb{M}od, \models)$ is the restriction of **HCL** to atoms. Another example is the restriction of **RHCL**$_1$ to quantifier-free sentences, where I_0 is its restriction to atoms.

We define the following proof rules for the IES of I:

(*Implications*) $\dfrac{\Gamma \vdash_\Sigma (\bigwedge H \Rightarrow C)}{\Gamma \cup H \vdash_\Sigma C}$ and $\dfrac{\Gamma \cup H \vdash_\Sigma C}{\Gamma \vdash_\Sigma (\bigwedge H \Rightarrow C)}$,

where $\Gamma \subseteq \mathbb{S}en(\Sigma)$ and $\bigwedge H \Rightarrow C \in \mathbb{S}en(\Sigma)$.

Given a compact entailment system $\mathcal{E}_0 = (\mathbb{S}ig, \mathbb{S}en_0, \vdash_0)$ for I_0, the IES of I consists of the least entailment system over $\mathcal{E}_0$ closed to *Implications*.

Proposition 18 (I-soundness). *The IES of* I *is sound when the entailment system of* I *is sound.*

Theorem 19 (I-completeness). *Let $\mathcal{D} \subseteq \mathbb{S}ig$ be a broad subcategory of signature morphisms such that the entailment system of I_0 is complete and compact, every set of sentences of I_0 is basic, and for all sets $B \subseteq \mathbb{S}en_0(\Sigma)$ there exists a $\mathcal{D}$-reachable model M_B defining B as basic set of sentences. Then*

1. *the IES of I is complete and compact,*
2. *for every set $\Gamma \subseteq \mathbb{S}en(\Sigma)$ and each sentence $\rho \in \mathbb{S}en(\Sigma)$ we have*

$$\Gamma \models_\Sigma \rho \text{ iff for all } \mathcal{D}\text{-reachable } \Sigma\text{-models } M, \text{ we have } M \models_\Sigma \Gamma \text{ implies } M \models \rho$$

3.4 Atomic entailment systems

In order to develop concrete universal entailment systems we need to define the entailment systems for the "atomic" layers of institutions.

Proposition 20 [Petria, 2007]. *Let* $\mathbf{FOL}_0$ *be the restriction of* $\mathbf{FOL}$ *to the atomic sentences. The entailment system of* $\mathbf{FOL}_0$ *generated by the rules below is sound, complete and compact.*
(Reflexivity) $\emptyset \vdash t = t$*, where t is a term.*
(Symmetry) $t = t' \vdash t' = t$ *, where t, t' are terms.*
Transitivity $\{t = t', t' = t''\} \vdash t = t''$*, where t, t', t'' are terms.*
(Congruence) $\{t_i = t_i' | 1 \leq i \leq n\} \vdash \sigma(t_1, ..., t_n) = \sigma(t_1', ..., t_n')$*, where t_i, t_i' are terms and σ is an operation symbol.*
(P-Congruence) $\{t_i = t_i' | 1 \leq i \leq n\} \cup \{\pi(t_1, ..., t_n)\} \vdash \pi(t_1', ..., t_n')$*, where t_i, t_i' are terms and π is a predicate symbol.*

Proposition 21 [Codescu and Găină, 2008]. *Let* $\mathbf{OSA}_0$ *be the restriction of* $\mathbf{OSA}$ *to the atomic sentences. The entailment system of* $\mathbf{OSA}_0$ *generated by the rules below is sound, complete and compact.*
(Reflexivity) $\emptyset \vdash t = t$*, where t is a term.*
(Symmetry) $t = t' \vdash t' = t$ *, where t, t' are terms.*
Transitivity $\{t = t', t' = t''\} \vdash t = t''$*, where t, t', t'' are terms.*
(Congruence) $\{t_i = t_i' | 1 \leq i \leq n\} \vdash \sigma(t_1, ..., t_n) = \sigma(t_1', ..., t_n')$*, where t_i, t_i' are terms and σ is an operation symbol.*

Proposition 22 [Codescu and Găină, 2008]. *Let* $\mathbf{POA}_0$ *be the restriction of* $\mathbf{POA}$ *to the atomic sentences. The entailment system of* $\mathbf{POA}_0$ *generated by the rules below is sound, complete and compact.*
(Reflexivity) $\emptyset \vdash t = t$ *for each term t*
(Symmetry) $t = t' \vdash t' = t$ *for any terms t, t'*
Transitivity $\{t = t', t' = t''\} \vdash t = t''$ *for any terms t, t', t''*

(Congruence) $\{t_i = t_i' | 1 \leq i \leq n\} \vdash \sigma(t_1,...,t_n) = \sigma(t_1',...,t_n')$, *where* t_i, t_i' *are terms and* σ *is an operation symbol.*
(Reflexivity') $\emptyset \vdash t \leq t$ *for each term* t
(Transitivity') $\{t \leq t', t' \leq t''\} \vdash t \leq t''$ *for any terms* t, t', t''
(Congruence') $\{t_i \leq t_i' | 1 \leq i \leq n\} \vdash \sigma(t_1,...,t_n) \leq \sigma(t_1',...,t_n')$, *where* t_i, t_i' *are terms and* σ *is an operation symbol.*
(ET) $\{t_1 = t_2,\ t_2 \leq t_3,\ t_3 = t_4\} \vdash t_1 \leq t_4$ *for any terms* t_1, t_2, t_3, t_4

Proposition 23 [Petria, 2007]. *Let* $\mathbf{PA}_0$ *be the restriction of* **PA** *to the atomic sentences. The entailment system of* $\mathbf{PA}_0$ *generated by the rules below is sound, complete and compact.*
(Symmetry) $t \overset{e}{=} t' \vdash t' \overset{e}{=} t$ *for any terms* t, t'
Transitivity $\{t \overset{e}{=} t',\ t' \overset{e}{=} t''\} \vdash t \overset{e}{=} t''$ *for any terms* t, t', t''
(Congruence) $\{t_i \overset{e}{=} t_i',\ def(\sigma(t_1,\ldots,t_n)),\ def(\sigma(t_1',\ldots,t_n'))\} \vdash$
$\sigma(t_1,\ldots,t_n) \overset{e}{=} \sigma(t_1',\ldots,t_n')$, *where* t_i, t_i' *are terms and* σ *is an operation symbol.*
(Subterm) $def(\sigma(t_1,\ldots,t_n)) \vdash \{def(t_i) \mid i \in \overline{1,n}\}$, *where* t_i *are terms and* σ *is an operation symbol.*

The proof rules defined above are sound and complete for the "atomic" layers of "reachable" institutions, and sound but not complete for the "atomic" layers of the constructor-based variants. For example, the proof rules of Proposition 20 are

1. sound and complete for $\mathbf{RHCL}_0$, the restriction of **RHCL** to
   (a) signature morphisms that do not add constructors on "old" sorts and
   (b) atomic sentences, and
2. sound but not complete for $\mathbf{CHCL}_0$, the restriction of **CHCL** to
   (a) signature morphisms that do not add constructors on "old" sorts and
   (b) atomic sentences.

The following is a corollary of Theorems 17, 19 and Proposition 20.

Corollary 24. *[Completeness of* **HCL***] The GUES of* **HCL** *generated by the rules of* Substitutivity, Generalization, Implications, Reflexivity, Symmetry, Transitivity, Congruence *and* P-Congruence *is complete and compact.*

Similar corollaries as above can be formulated for **HOSA**, **HPOA**, and **HPA**. The following is a corollary of Theorems 15, 17, 19 and Proposition 20.

Corollary 25. *[Completeness of* **RHCL***] The RUES of* **RHCL** *generated by the rules of* C-Abstraction, Substitutivity, Generalization, Implications, Reflexivity, Symmetry, Transitivity, Congruence *and* P-Congruence *is complete.*

Similar corollaries as above can be formulated for **RHOSA**, **RHPOA**, and **RHPA**.

4 Sufficient Completeness

If we instantiate *C-Abstraction* to **CHCL** then we obtain the following proof rule:

$$\textit{(C-Abstraction)}\ \frac{\{\Gamma \vdash (\forall Y)\theta(\rho) \mid \theta : X \to T_{(S,F)}(Y),\ Y\text{-finite set of loose vars}\}}{\Gamma \vdash (\forall X)\rho}$$

where Γ is a set of (S,F,F^c,P)-sentences, and $(\forall X)\rho$ is a (S,F,F^c,P)-sentence such that X is a finite set variables of constrained sorts. Below is a refined version of the above rule:

$$\textit{(C-Abstraction')}\ \frac{\{\Gamma \vdash (\forall Y)\theta(\rho) \mid \theta : X \to T_{(S,F^c)}(Y),\ Y\text{-finite set of loose vars}\}}{\Gamma \vdash (\forall X)\rho}$$

where Γ is a set of (S,F,F^c,P)-sentences, and $(\forall X)\rho$ is a (S,F,F^c,P)-sentence such that X is a finite set variables of constrained sorts.

Let $(\mathbb{S}ig^{\mathbf{CHCL}},\mathbb{S}en^{\mathbf{CHCL}},\vdash^{\mathbf{CHCL}})$ be the entailment system of **CHCL** generated with *C-Abstraction*, and $(\mathbb{S}ig^{\mathbf{CHCL}},\mathbb{S}en^{\mathbf{CHCL}},\vdash')$ be the entailment system of **CHCL** generated with *C-Abstraction'*. When there is no danger of confusion we may drop the superscript **CHCL** from notations.

Proposition 26. *We have* $\vdash^{\mathbf{CHCL}}\subseteq\vdash'\subseteq\models^{\mathbf{CHCL}}$.

Proof. We prove that $\vdash^{\mathbf{CHCL}}\subseteq\vdash'$ and $\vdash'\subseteq\models^{\mathbf{CHCL}}$.

$\vdash\subseteq\vdash'$. It suffices to show that $(\mathbb{S}ig^{\mathbf{CHCL}},\mathbb{S}en^{\mathbf{CHCL}},\vdash')$ satisfies *C-Abstraction*. Consider a **CHCL** signature (S,F,F^c,P), and let Γ be a set of (S,F,F^c,P)-sentences and $(\forall X)\rho$ a (S,F,F^c,P)-sentence such that the sorts of the variables in X are constrained. Assume $\Gamma\vdash'_{(S,F,F^c,P)}(\forall Y)\theta(\rho)$ for all $\theta : X\to T_{(S,F)}(Y)$ such that Y consists only of loose variables. In particular, $\Gamma\vdash'_{(S,F,F^c,P)}(\forall Y)\theta(\rho)$ for all $\theta : X\to T_{(S,F^c)}(Y)$ such that Y consists only of loose variables. By *C-Abstraction'*, we obtain $\Gamma\vdash'_{(S,F,F^c,P)}(\forall X)\rho$.

$\vdash'\subseteq\models$. By soundness of *C-Abstraction'*. □

The entailment system generated with *C-Abstraction'* is more expressive. Hereafter, we replace the definition of *C-Abstraction* by the definition of *C-Abstraction'*. Let $(\mathbb{S}ig^{\mathbf{RHCL}},\mathbb{S}en^{\mathbf{RHCL}},\vdash^{\mathbf{RHCL}})$ be the entailment system of **RHCL**. Recall that for any **CHCL** signature (S,F,F^c,P), $F^{S^c}_{w\to s}=\begin{cases}F_{w\to s} & : & s\in S^c\\ \emptyset & : & s\notin S^c\end{cases}$.

Proposition 27. *For any* **CHCL** *signature* (S,F,F^c,P) *we have* $\vdash^{\mathbf{RHCL}}_{(S,S^c,F,P)}\subseteq\vdash^{\mathbf{CHCL}}_{(S,F,F^c,P)}$.

Proof. By induction in the definition of $\vdash^{\mathbf{RHCL}}$. All cases are trivial except *Translation* and *C-Abstraction*.

Translation. Let (S,F,F^c,P) be a **CHCL** signature, $\varphi : (S_0,S_0^c,F_0,P_0)\to(S,S^c,F,P)$ a **RHCL** signature morphism, and $E_1,E_2\subseteq\mathbb{S}en(S_0,S_0^c,F_0,P_0)$ two sets of sentences such that $E_1\vdash^{\mathbf{RHCL}}_{(S_0,S_0^c,F_0,P_0)}E_2$. We define $F_0^c=((F_0^c)_{w_0\to s_0})_{(w_0,s_0)\in S_0^*\times S_0}$: for all $(w_0,s_0)\in S_0^*\times S_0$, $(F_0^c)_{w_0\to s_0}=\varphi^{-1}(F^c_{\varphi(w_0)\to\varphi(s_0)})$. Since $\varphi : (S_0,S_0^c,F_0,P_0)\to(S,S^c,F,P)$ is a **RHCL** signature morphism, $\varphi : (S_0,F_0,F_0^c,P_0)\to(S,F,F^c,P)$ is a **CHCL** signature morphism. By the induction hypothesis, $E_1 \vdash^{\mathbf{RHCL}}_{(S_0,S_0^c,F_0,P_0)} E_2$ implies $E_1 \vdash^{\mathbf{CHCL}}_{(S_0,F_0,F_0^c,P_0)} E_2$, and by *Translation* we obtain $\varphi(E_1) \vdash^{\mathbf{CHCL}}_{(S,F,F^c,P)} \varphi(E_2)$.

C-Abstraction. Let (S,F,F^c,P) be a **CHCL** signature, $\Gamma \subseteq \mathbb{S}en(S,F,F^c,P)$ a set of sentences, and $(\forall X)\rho \in \mathbb{S}en(S,F,F^c,P)$ a sentence such that X consists only of constrained variables. Assume that $\Gamma \vdash^{\mathbf{RHCL}}_{(S,S^c,F,P)} (\forall Y)\theta(\rho)$ for all substitutions $\theta : X \to T_{(S,F^{S^c})}(Y)$ such that Y consists of loose variables. By induction hypothesis, $\Gamma \vdash^{\mathbf{CHCL}}_{(S,F,F^c,P)} (\forall Y)\theta(\rho)$ for all substitutions $\theta : X \to T_{(S,F^{S^c})}(Y)$ such that Y consists of loose variables. In particular, $\Gamma \vdash^{\mathbf{CHCL}}_{(S,F,F^c,P)} (\forall Y)\theta(\rho)$ for all substitutions $\theta : X \to T_{(S,F^c)}(Y)$ such that Y consists of loose variables. By *C-Abstraction*, $\Gamma \vdash^{\mathbf{CHCL}}_{(S,F,F^c,P)} (\forall X)\rho$. □

Definition 28. A **CHCL** presentation $((S,F,F^c,P),E)$ is *sufficient complete*, where (S,F,F^c,P) is a **CHCL** signature and E is a set of (S,F,F^c,P)-sentences, if for all models $M \in \mathbb{M}od(S,F,F^{S^c},P)$ that satisfies E we have $M \in \mathbb{M}od(S,F,F^c,P)$.

Theorem 29. *The entailment system of* **CHCL** *generated by* C-Abstraction*,* Substitutivity*,* Generalization*,* Implications*,* Reflexivity*,* Symmetry*,*Transitivity*,* Congruence *and* P-Congruence *is complete with respect to the sufficient complete presentations, i.e. for all sufficient complete presentations* $((S,F,F^c,P),\Gamma)$ *and* (S,F,F^c,P)*-sentences* ρ*,* $\Gamma \models^{\mathbf{CHCL}}_{(S,F,F^c,P)} \rho$ *implies* $\Gamma \vdash^{\mathbf{CHCL}}_{(S,F,F^c,P)} \rho$.

Proof. Let $((S,F,F^c,P),\Gamma)$ be a sufficient complete presentation and ρ a (S,F,F^c,P)-sentence . If $\Gamma \models^{\mathbf{CHCL}}_{(S,F,F^c,P)} \rho$ then since $((S,F,F^c,P),\Gamma)$ is sufficient complete, we have $\Gamma \models^{\mathbf{RHCL}}_{(S,S^c,F,P)} \rho$. By Corollary 25, $\Gamma \vdash^{\mathbf{RHCL}}_{(S,S^c,F,P)} \rho$, and by Proposition 27, $\Gamma \vdash^{\mathbf{CHCL}}_{(S,F,F^c,P)} \rho$. □

Similar results as Theorem 29 can be formulated for **CHOSA**, **CHPOA** and **CHPA**. Below there is an example of sufficient complete specification.

Example 11. Consider the following signature

$$\Sigma = \boxed{\begin{array}{ll} \text{sort Nat} & \\ \text{op } 0 : \to \text{Nat} & \{\text{constr}\} \\ \text{op s\_ : Nat} \to \text{Nat} & \{\text{constr}\} \\ \text{op \_+\_ : Nat Nat} \to \text{Nat} & \end{array}}$$

If $\rho_1 = (\forall x)x+0 = x$ and $\rho_2 = (\forall \{x,x'\})x+(s\ x') = s(x+x')$ then $(\Sigma,\{\rho_1,\rho_2\})$ is sufficient complete. Intuitively, if (Σ,Γ) specifies that non-constructor operators are inductively defined with respect to the constructors then (Σ,Γ) is sufficient complete.

In general, the proof rules for the constructor-based institutions are not complete, as one can see in the following example.

Example 12. Consider the following signatures

$$\Sigma = \boxed{\begin{array}{l} \text{sort s} \\ \text{op a} : \to \text{s} \\ \text{op b} : \to \text{s } \{\text{constr}\} \end{array}} \text{ and } \Sigma^r = \boxed{\begin{array}{l} \text{sort s} \\ \text{op a} : \to \text{s } \{\text{constr}\} \\ \text{op b} : \to \text{s } \{\text{constr}\} \end{array}}$$

It is easy to notice that $\emptyset \models_\Sigma a = b$ but there is no way to prove $\emptyset \vdash_\Sigma a = b$ because $(\Sigma, \emptyset)$ is not sufficient complete. Indeed, if $\emptyset \vdash_\Sigma a = b$ then since there are no sentences quantified over constrained variables, we have $\emptyset \vdash_{\Sigma^r} a = b$, and we get $\emptyset \models_{\Sigma^r} a = b$, which is not true.

4.1 An induction scheme

One big problem raised by *C-Abstraction* is its premises which are infinite, in general. One needs to perform infinitely many proofs for checking the premises of *C-Abstraction*, one for each substitution θ. In order to make proofs it is mandatory to have a finitary procedure to deal with the infinite conditions of *C-Abstraction*. The standard one is the so-called method of *Structural Induction*.

For a better understanding of the following proposition, we will give some explanations in advance. Assume that we need to prove a property $(\forall X)\rho$ for a **CHCL** presentation (Σ, Γ), where $\Sigma = (S, F, F^c, P)$ and X consists of constrained variables. Let $\mathtt{CON}$ be a sort-preserving mapping $X \to F^c$ (i.e. for all $x \in X$, $\mathtt{CON}(x)$ has the same sort as x). For each $x \in X$, let $Z_{x,\mathtt{CON}} = z^1_{x,\mathtt{CON}} \dots z^n_{x,\mathtt{CON}}$ be a string of arguments for the constructor $\mathtt{CON}(x)$. We define the set $Z_{\mathtt{CON}} = \cup_{x \in X} Z_{x,\mathtt{CON}}$ [7] and the substitution $\mathtt{VAR}^\#_{\mathtt{CON}} : X \to T_{(S,F^c)}(Z_{\mathtt{CON}})$ by $\mathtt{VAR}^\#_{\mathtt{CON}}(x) = \mathtt{CON}(x)(Z_{x,\mathtt{CON}})$. We let $\mathtt{VAR}_{\mathtt{CON}}$ range over all sort-preserving substitutions $X \to T_{(S,F^c)}(Z_{\mathtt{CON}})$ with the following properties:

- for all $x \in X$ we have $\mathtt{VAR}_{\mathtt{CON}}(x) \in Z_{x,\mathtt{CON}}$ or $\mathtt{VAR}_{\mathtt{CON}}(x) = \mathtt{CON}(x)(Z_{x,\mathtt{CON}})$, and
- there exists $x \in X$ such that $\mathtt{VAR}_{\mathtt{CON}}(x) \in Z_{x,\mathtt{CON}}$.

The function $\mathtt{CON}$ will give all the induction cases, while the function $\mathtt{VAR}_{\mathtt{CON}}$ is used to define the induction hypothesis for each case.

Proposition 30 (Structural Induction). *The entailment system of* **CHCL** *is closed to* Structural Induction*:*

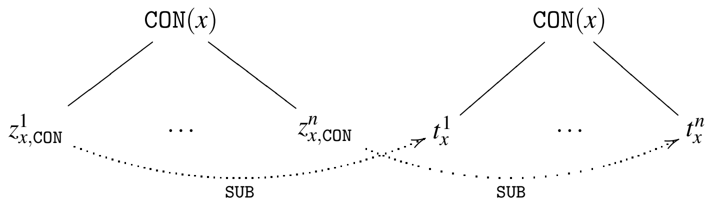
- (Induction Step) $\Gamma \cup \{\mathtt{VAR}_{\mathtt{CON}}(\rho) \mid \mathtt{VAR}_{\mathtt{CON}} : X \to T_{(S,F^c)}(Z_{\mathtt{CON}})\} \vdash_{\Sigma(Z_{\mathtt{CON}})} \mathtt{VAR}^\#_{\mathtt{CON}}(\rho)$ *for all* $\mathtt{CON} : X \to F^c$*, implies*
- (Induction Conclusion) $\Gamma \vdash_\Sigma (\forall X)\rho$.

Proof. If we prove that $\Gamma \vdash_\Sigma \theta(\rho)$ for all substitutions $\theta : X \to T_{(S,F^c)}(Y)$ such that Y consists of loose variables then by *C-Abstraction* we obtain $\Gamma \vdash_\Sigma (\forall X)\rho$.

We proceed by induction on the sum of depth of the terms in $\{\theta(x) \mid x \in X\}$ which exists as a consequence of X being finite. Let $\mathtt{CON} : X \to F^c$ be the sort-preserving mapping such that for all $x \in X$, $\mathtt{CON}(x)$ is the topmost constructor of $\theta(x)$. Let $T_x =$

[7] By a slightly abuse of notation, we let $Z_{x,\mathtt{CON}}$ to represent both the string $z^1_{x,\mathtt{CON}} \dots z^n_{x,\mathtt{CON}}$ and its corresponding set $\{z^1_{x,\mathtt{CON}}, \dots, z^n_{x,\mathtt{CON}}\}$.

$t_x^1 \dots t_x^n$ be the string of the immediate sub-terms of $\theta(x)$, and $Y = \bigcup_{x \in X} Y_x$, where Y_x are all variables of the terms in T_x. We define the substitution $\mathtt{SUB} : Z_{\mathtt{CON}} \to T_{(S,F^c)}(Y)$ by $\mathtt{SUB}(z^i_{x,\mathtt{CON}}) = t^i_x$.



Note that $\mathtt{SUB}(\Gamma \cup \{\mathtt{VAR}_{\mathtt{CON}}(\rho) \mid \mathtt{VAR}_{\mathtt{CON}} : X \to T_{(S,F^c)}(Z_{\mathtt{CON}})\}) = \Gamma \cup \{(\mathtt{VAR}_{\mathtt{CON}};\mathtt{SUB})(\rho) \mid \mathtt{VAR}_{\mathtt{CON}} : X \to T_{(S,F^c)}(Z_{\mathtt{CON}})\}$ and $\mathtt{SUB}(\mathtt{VAR}^{\#}_{\mathtt{CON}}(\rho)) = \theta(\rho)$. By our assumptions,

$$\Gamma \cup \{\mathtt{VAR}_{\mathtt{CON}}(\rho) \mid \mathtt{VAR}_{\mathtt{CON}} : X \to T_{(S,F^c)}(Z_{\mathtt{CON}})\} \vdash_{\Sigma(Z_{\mathtt{CON}})} \mathtt{VAR}^{\#}_{\mathtt{CON}}(\rho) \tag{1}$$

By *Translation* applied to (1),

$$\Gamma \cup \{(\mathtt{VAR}_{\mathtt{CON}};\mathtt{SUB})(\rho) \mid \mathtt{VAR}_{\mathtt{CON}} : X \to T_{(S,F^c)}(Z_{\mathtt{CON}})\} \vdash_{\Sigma(Y)} \theta(\rho) \tag{2}$$

The sum of depth of the terms in $\{(\mathtt{VAR}_{\mathtt{CON}};\mathtt{SUB})(x) \mid x \in X\}$ is strictly less than the sum of depth of the terms in $\{\theta(x) \mid x \in X\}$. By the induction hypothesis, we have that $\Gamma \vdash_\Sigma (\forall Y)(\mathtt{VAR}_{\mathtt{CON}};\mathtt{SUB})(\rho)$ for all $\mathtt{VAR}_{\mathtt{CON}} : X \to T_{(S,F^c)}(Z_{\mathtt{CON}})$. By *Generalization*, we obtain $\Gamma \vdash_{\Sigma(Y)} (\mathtt{VAR}_{\mathtt{CON}};\mathtt{SUB})(\rho)$ for all $\mathtt{VAR}_{\mathtt{CON}} : X \to T_{(S,F^c)}(Z_{\mathtt{CON}})$. By *Union*, we get $\Gamma \vdash_{\Sigma(Y)} \{(\mathtt{VAR}_{\mathtt{CON}};\mathtt{SUB})(\rho) \mid \mathtt{VAR}_{\mathtt{CON}} : X \to T_{(S,F^c)}(Z_{\mathtt{CON}})\}$. It follows that

$$\Gamma \vdash_{\Sigma(Y)} \Gamma \cup \{(\mathtt{VAR}_{\mathtt{CON}};\mathtt{SUB})(\rho) \mid \mathtt{VAR}_{\mathtt{CON}} : X \to T_{(S,F^c)}(Z_{\mathtt{CON}})\} \tag{3}$$

By *Transitivity* applied (3) and (2), $\Gamma \vdash_{\Sigma(Y)} \theta(\rho)$. By *Generalization*, $\Gamma \vdash_\Sigma (\forall Y)\theta(\rho)$. □

The above induction scheme was inspired from [Diaconescu, 2011].

4.2 An example of inductive proof

Consider the signature Σ defined in Example 11. We prove by structural induction that $\{\rho_1, \rho_2\} \vdash_\Sigma (\forall n) 0 + n = n$. The function CON range over all mappings $\{n\} \to \{0, s\_\}$. We have only two possibilities for the mapping $\mathtt{CON} : \{n\} \to \{0, s\_\}$:

1. $\mathtt{CON}(n) = 0$. By Proposition 30, we need $\Gamma \vdash_\Sigma 0 + 0 = 0$.
   (a) $\{\rho_1, \rho_2\} \vdash_\Sigma \rho_1$ by *Monotonicity* of the entailment relation.
   (b) $\rho_1 \vdash_\Sigma 0 + 0 = 0$ by *Substitutivity* for x substituted by 0.
   (c) $\{\rho_1, \rho_2\} \vdash_\Sigma 0 + 0 = 0$ from (1a) and (1b) by *Transitivity* of $\vdash_\Sigma$.

2. $\mathtt{CON}(n) = s\_$. By Proposition 30, we need $\{\rho_1, \rho_2, 0+z=z\} \vdash_{\Sigma(z)} 0+s\,z=z$.
   (a) $\{\rho_1, \rho_2, 0+z=z\} \vdash_{\Sigma(z)} \{\rho_1, \rho_2\}$ by *Monotonicity* of the entailment relation.

   (b) $\{\rho_1, \rho_2\} \vdash_{\Sigma(z)} 0+s\,z = s(0+z)$ by *Substitutivity* for x and x' substituted by 0 and z, respectively.

   (c) $\{\rho_1, \rho_2, 0+z=z\} \vdash_{\Sigma(z)} 0+s\,z = s(0+z)$ from (2a) and (2b) by *Transitivity* of the entailment relation.

   (d) $\{\rho_1, \rho_2, 0+z=z\} \vdash_{\Sigma(z)} 0+z=z$ by *Monotonicity* of the entailment relation.

   (e) $0+z=z \vdash_{\Sigma(z)} s(0+z) = s\,z$ by *Congruence*.

   (f) $\{\rho_1, \rho_2, 0+z=z\} \vdash_{\Sigma(z)} 0+s\,z = s\,z$ from (2c) and (2e) by *Transitivity* of the entailment relation.

By soundness, $\{\rho_1, \rho_2\} \models_\Sigma (\forall n) 0+n=n$. Notice that

– $\mathbb{N}$, the standard model of the natural numbers with the carrier set for the sort *Nat* consisting of elements $\{0,1,2,\ldots\}$ and interpreting the function symbol 0 as the element 0, $s\_$ as the successor function and $\_+\_$ as addition, and

– $\mathbb{Z}_n$, the model of integers modulo n with the carrier set for the sort *Nat* consisting of $\{\widehat{0}, \widehat{1}, \ldots, \widehat{n-1}\}$ and interpreting the function symbol 0 as the element $\widehat{0}$, $s\_$ as the successor function and $\_+\_$ as addition,

satisfy $\{\rho_1, \rho_2\}$. Hence, we have proved formally that $\mathbb{N}$ and $\mathbb{Z}_n$ satisfy $(\forall n) 0+n=n$. Similarly, one can prove that $\mathbb{N}$ and $\mathbb{Z}_n$ satisfy $(\forall\{m,n\}) s\,m+n = s(m+n)$, and the commutativity of the addition, i.e. $(\forall\{m,n\}) m+n = n+m$.

4.3 Case Analysis

One of the advantages of this approach is that we can reason about "inductive" properties of a given specification even if it is not sufficient complete. As we have seen in Example 12, the proof rules for **CHCL** are not complete, in general. Consider a set Γ of Σ-sentences, where $\Sigma = (S, F^c, F, P)$, a non-constructor operation symbol $\sigma \in F_{s_1 \ldots s_n \to s}$ such that s is a constrained sort, and a string of arguments for σ, $T = t_1 \ldots t_n$, such that the terms t_i are formed with constructors and variables of loose sorts from Y. We define the following proof rule:

$$\textit{(Case Analysis)}\ \frac{\{\Gamma \cup \{\sigma(T) = t\} \vdash_{\Sigma(Z)} \rho \mid t \in T_{(S,F^c)}(Z),\ Z \supseteq Y \text{ finite set of loose vars.}\}}{\Gamma \vdash_\Sigma \rho}$$

The rule above says that if $\sigma(T)$ cannot be "reduced" to a term formed with constructors and variables of loose sorts by the equations in Γ then we need to make a case analysis on the possible value of $\sigma(T)$. The set of terms t above may be infinite and therefore the premises of *Case Analysis* may be infinite too. In many examples, the case analysis is conducted on the possible value of a term of sort *Bool* (the sort of $\sigma(T)$ above can be *Bool*) that has two constructors: $true$ and $false$. In this case, the premises of *Case Analysis* are finite.

Proposition 31. *The entailment system of* **CHCL** *generated by the rules of* Case Analysis, C-Abstraction, Substitutivity, Generalization, Implications, Reflexivity, Symmetry, Transitivity, Congruence *and* P-Congruence *is sound.*

Proof. Firstly we prove that *Case Analysis* is sound. Let Γ be a set of Σ-sentences, ρ a Σ-sentence, where $\Sigma = (S,F,F^c,P)$, $\sigma \in F_{s_1 \ldots s_n \to s}$ a non-constructor operation symbol such that s is constrained, and $T = t_1 \ldots t_n$ a string of terms formed with constructors and loose variables. We prove that $\Gamma \models_\Sigma \rho$ whenever $\Gamma \cup \{\sigma(T) = t\} \models_{\Sigma(Y)} \rho$ for all terms t formed with constructors and loose variables, where Y is a finite set of loose variables such that $t_i \in T_{(S,F^c)}(Y)$ and $t \in T_{(S,F^c)}(Y)$. Assume a Σ-model M such that $M \models_\Sigma \Gamma$. Let Y' be all variables in T, $f : Y' \to M$ be a valuation of Y' into M and $m = \overline{f}(\sigma(T))$, where $\overline{f} : T_{(S,F^{S^c})}(Y') \to M$ is the unique extension of f to a (S,F^{S^c})-morphism. Since $M \in \mathbb{M}od(S,F,F^c,P)$ there exists a finite set of loose variables Y'', a valuation $g : Y'' \to M$ and a term $t \in T_{(S,F^c)}(Y'')$ such that $g^\#(t) = m$, where $g^\# : T_{(S,F^c)}(Y'') \to M$ is the unique extension of g to a (S,F^c)-morphism. Let $Y = Y' \cup Y''$ and $h : Y \to M$ such that $h|_{Y'} = f$ and $h|_{Y''} = g$. Note that $(M,h) \models_{\Sigma(Y)} \Gamma \cup \{\sigma(T) = t\}$, and since $\Gamma \cup \{\sigma(T) = t\} \models_{\Sigma(Y)} \rho$, we obtain $(M,h) \models_{\Sigma(Y)} \rho$. By the satisfaction condition, $M \models_\Sigma \rho$.

Since all the proof rules enumerated in the hypothesis are sound, the least entailment system closed to these rules is sound too. □

Consider the signature Σ defined in Example 12. The sort s has one constructor, and since $a = b \vdash_\Sigma a = b$, by *Case Analysis*, we obtain $\emptyset \vdash_\Sigma a = b$.

Case Analysis is often used in applications (see for example [Futatsugi et al., 2005] and [Futatsugi, 2006]) and it splits the initial goal $\Gamma \vdash_\Sigma \rho$ into subgoals $\Gamma \cup \{\sigma(T) = t\} \vdash_{\Sigma(Z)} \rho$, where the presentations $(\Sigma, \Gamma \cup \{\sigma(T) = t\})$ are expected to be sufficient complete. So, we can state that we provided all rules for proving the constructor-based properties. In some cases we need to iterate the process of splitting the goals with *Case Analysis* several times. Therefore, it is difficult to formulate a completeness result that does not depend on sufficient completeness.

5 Conclusions

Consider a **CHCL** signature (S,F,F^c) with no predicate symbols such that all operators are constructors (i.e. $F = F^c$) and the set of loose sorts is empty (i.e. $S^l = \emptyset$). The carrier sets of every (S,F,F^c)-algebra consist of interpretations of terms formed with constructors. Let Γ be a set of conditional (S,F)-equations. Since $F = F^c$, it follows that $((S,F,F^c),\Gamma)$ is sufficient complete, and Γ has an initial model O_Γ. Since all algebras consist of interpretations of terms, every morphism $O_\Gamma \to M$ is surjective. Further, since surjective morphisms preserve satisfaction of equations, for every (S,F,F^c)-morphism $O_\Gamma \to M$ and each (S,F)-equation $(\forall X)t = t'$ we have $O_\Gamma \models (\forall X)t = t'$ implies $M \models (\forall X)t = t'$. Therefore, $\Gamma \models (\forall X)t = t'$ iff $O_\Gamma \models (\forall X)t = t'$ for all $(\forall X)t = t' \in \mathbb{S}en(S,F)$. Because the entailment system of **CHCL** is complete, the proof rules generate a complete entailment relation to reason about the properties of the initial model O_Γ. We have defined *Structural induction* to deal with the infinitary premises of *C-Abstraction*. However, the infinitary rules cannot be replaced with the finitary ones in order to obtain a complete and compact entailment system; we would obtain complete and compact entailment relations to reason about the properties of the initial models of the specifications. Gödel incompleteness theorem shows that this is not possible even for the initial model of the specification of natural numbers.

The area of applications provided by the general framework of the present work is much wider. For example, we may consider variations of the institutions presented here, such as order-sorted algebra with transitions. The abstract results of this research can be applied also to constructor-based Horn variants of higher-order logic with intensional Henkin semantics, and membership algebra [Meseguer, 1997]. Higher order logic with Henkin semantics has been introduced and studied in [Church, 1940, Henkin, 1950] and intensionality is discussed in [Moggi, 1985] and [Astesiano and Cerioli, 1995]. The generic universal entailment system developed in this paper can be seen as a refined version of the one in [Codescu and Găină, 2008]. However, **HPA** cannot be captured by the abstract framework of [Codescu and Găină, 2008] due to the quantification over partial constant symbols (the signature extensions with a finite number of partial constant symbols are not *representable*). On the other hand, the Horn part of the partial algebra with both partial and total operation symbols is an instance of the framework of [Codescu and Găină, 2008] but it cannot be captured by the present framework.

Due to the abstract definition of reachable model, one can easily define a constructor-based institution on top of some base institution by enhancing the syntax with a subsignature of constructors and restricting the semantics to reachable models. This construction may be useful when lifting the interpolation and amalgamation properties from the base institution. The institution-independent rule of *C-Abstraction* allows somehow a uniform treatment of induction schemes in different institutions. Future work includes the development of the OTS/CafeOBJ method based on the theoretical framework defined in this paper.

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