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Reachability in Restricted Walk on Integers

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Abstract: We prove that two conditions are sufficient, and with three exceptions also necessary, for reachability of any position in restricted walk on integers in which the sizes of the moves to the left and to the right are constant but need not be equal. A method to compute the length of the shortest path between any two positions, as well as a shortest path algorithm when the reachability conditions are true are given. Also a complete characterization for Hamiltonian restricted walks between absorbing boundaries is given.

Key Words: Reachability, random walk, shortest path, Hamiltonian path, strong connectivity.
Category: G.2.2, C.3

1 Introduction

While implementing the specification of message traffic monitor that is based on up/down counter and is used to detect connection failures when the counter reaches a threshold [ITU–T 1996, Schmidt 1994], we have noticed that the recommended parameters (up step, down step and the thresholds that limit the counter movement) are such that most of the counter values are never reached, and we were lead to ask: "What kind of parameters would ensure reachability of *all* counter values?"

The needed constraints on the error traffic monitor parameters were not difficult to spot. For instance, it is clear that the greatest common divisor of the up and down steps should be one. However, we wanted a general answer that is not restricted to a single application. Therefore, we have modeled the movement of the counter between its threshold values as a random walk on integers in which the sizes of the moves to the left and to the right are constant but need not be equal. The walk is restricted in each direction by a boundary, which can be absorbing, or reflecting.[1]

[1] The walk cannot leave an absorbing boundary. The walk can leave a reflective boundary, but only in direction that is opposite to that of its arrival.

A proof of reachability conditions for graphs of such walks is the first new result of this paper.

In a walk between reflecting boundaries the reachability of any position from any other in a finite number of moves implies ergodicity: the probability of the particle being in position j after a large number of moves approaches a constant value that is greater than zero and that does not depend on the initial position of the particle. (See, e.g., [Shiryaev 1995] for a proof of this fact.) This property is important from the point of view of applications: We have used it in designing a traffic filter based on the random walk, were the source of randomness is the traffic itself. But the description of that application has to be left to another publication.

Secondly, while proving the reachability conditions, we have found a shortest–path algorithm for graphs that fulfill those conditions. This algorithm does not need ancillary data structures and the upper bound on its run–time grows linearly with the number of graph's vertices.

Thirdly, assuming absorbing boundaries, we have found a complete characterization of cases where the restricted walk is also a Hamiltonian path or a Hamiltonian circuit in the part of the graph that excludes the boundaries.

Those results apply in general to restricted walks on the members of integer–indexed sequence of states $\{E_i\}$: $\ldots$, E_{-1}, E_0, E_1, E_2, $\ldots$. The walk on integer values, in which $E_i = i$, is the simplest example of such sequence. As another example, the sequence $\{E_i\}$ could model the states of a thermostat that automatically responds to the temperature changes and activates switches controlling some equipment. A requirement for faster response to increase in the temperature than to its decrease, results in unequal size of moves between the states of the thermostat. This kind of asymmetrical response to inputs is typical in many control applications.

In implementations that can be modeled by restricted walk on the above sequence, a cost is associated with each E_i. That cost may be, for instance, in the memory needed to hold E_i, in the additional hardware needed by E_i, and in the labor needed to design and test E_i. By knowing the reachability conditions we can eliminate unreachable members from the implementation and thus reduce costs.

For instance, only 16 533 out of 739 545 counter values in the [ITU–T 1996] message traffic monitor for 2 Mb/s links are reachable. Therefore, the number of bits allotted for the counter value could be reduced from 20 to 15 in this particular case.

Also knowledge of graph algorithms, like shortest path and Hamiltonian path (if it exists) may help to reduce costs during maintenance, when the equipment needs to be moved from one state to another in a controlled way.

2 Restricted walk model

In this section we describe the restricted walk model on integer values. Starting at some position i a particle moves l steps to the left with probability p_i and r steps to the right with probability $1 - p_i$. The probability of moving to either direction is not zero: $0 < p_i < 1$. The position of the particle after t moves is c_t.

The movement of the particle is restricted to an interval the ends of which can be reflecting or absorbing.[2] The leftmost non–absorbing position in that interval is L and the rightmost non–absorbing position is R. Thus, the number P of non–absorbing positions of the particle is $R-L+1$. In order to shorten our discussions we exclude the most trivial case of $R = L$ from further considerations, i. e. we assume $P \geq 2$. If the left boundary is absorbing, then the particle moves to it with probability p_i from positions L, $L + 1$, $\ldots$, $L + l - 1$; if the right boundary is absorbing, then the particle moves to it with probability $1 - p_i$ from positions $R - r + 1$, $R - r + 2$, $\ldots$, R; if the left boundary is reflecting, then the particle moves to L with probability p_i from positions L, $L + 1$, $\ldots$, $L + l$; if the right boundary is reflecting, then the particle moves to R with probability $1 - p_i$ from positions $R - r$, $R - r + 1$, $\ldots$, R.

Figure 1 illustrates the transition diagram of two walks with parameters $l = 2$, $r = 3$ and $P = 5$. In subsequent figures, the labels p_i and $1 - p_i$ indicating transition probability values are omitted, since they are not relevant to the topic of this paper. We will call this kind of transition diagram simply as a graph although it is actually a digraph, i.e. a directed graph.

In the following, we denote the sum of l and r by m, the greatest common divisor of l and r, $\gcd(l, r)$, by g and a distance between two positions i and j by d: $d(i, j) = j - i$. Please note that a distance can also be negative by our definition. This is intentional because we have asymmetry between distance to the right and distance to the left.

3 Reachability conditions

Assume that the initial position i of the particle is non–absorbing: $L \leq i \leq R$. We will prove in section 6 that for the particle to be able to reach any other position $j \neq i$ in the interval $[L, R]$ from any such initial position i it is sufficient and, with three exceptions listed below, also necessary that

i. r and l have no common factor: $g = 1$, and

ii. $P \geq m$.

[2] In the message traffic monitor application one boundary is reflecting and the other one is absorbing. Our model covers also the case where both boundaries are reflecting/absorbing.

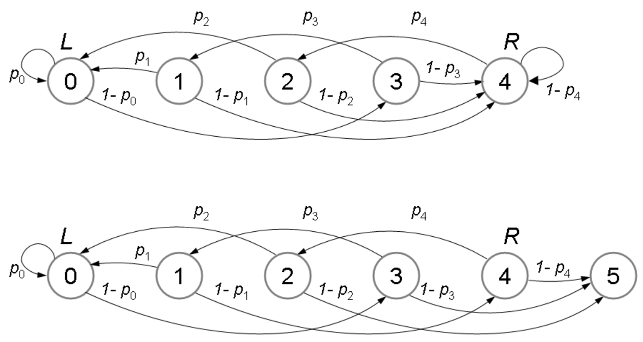

Figure 1: Possible moves and their probabilities in two walks with parameters $l = 2$, $r = 3$ and $P = 5$. Above, there are reflecting boundaries at positions 0 and 4. Below, there is reflecting boundary at position 0 and absorbing boundary at position 5.

The three exceptions to the necessity of conditions (i) and (ii) are illustrated in Figure 2. They are caused by reflections from interval ends:

a. If $P = 2$ and both boundaries are reflecting, then the particle may reach R from L and L from R even if one, or both of conditions (i) and (ii) are false.

b. If $r = 1$ and L is reflecting, or if $l = 1$ and R is reflecting, then the particle may reach every position (from any initial position) even if condition (ii) is false.

c. If $g = 2$, condition (ii) is true and in addition P is even, then every position is reachable (from any initial position) when both boundaries are reflecting, even though condition (i) is false.

The reachability conditions and the exceptional cases are summarized in Table 1.

Using common terminology in graph theory, what we call graph reachability is equivalent to the digraph (of non-absorbing nodes) being *strongly connected*.

4 Related work

A related problem in two dimensions has been investigated by Knuth [Knuth 1994]. He gives three conditions for connectivity of a graph created by moves of a "leaper", i. e. a generalized chess piece that can jump from position (x, y) on a rectangular, $m \times n$ board, to positions $(x \pm r, y \pm s)$ or $(x \pm s, y \pm r)$, where $2 \leq m \leq n$:

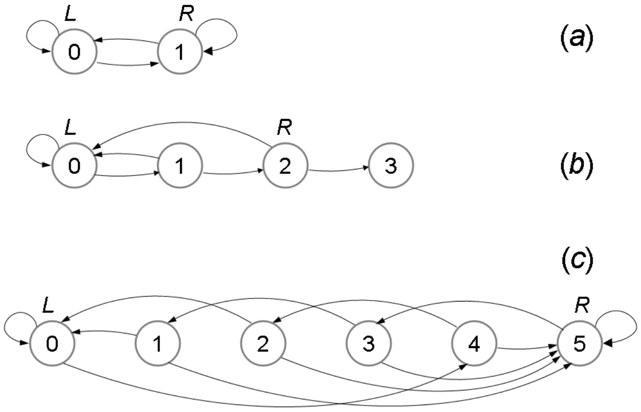

Figure 2: Graphs of three restricted walks that illustrate the exceptions to the necessity of conditions (i) and (ii): In (a) l and r may have any values, thus g and m may also have any values; in (b) $r = 1$ but l may have any value greater than one; in (c) $l = 2$ and $r = 4$, i. e. $g = 2$.

i. $\gcd(r + s, r - s) = 1$,

ii. $n \geq 2s$, and

iii. $m \geq r + s$.

Knuth also studies extensively the question of whether or not a leaper graph has a Hamiltonian circuit.

The reachability condition (i) of Knuth puts a restriction on the relation of different "leaps", and therefore corresponds to our condition (i). Respectively, conditions (ii) and (iii) put restrictions on the relation of "leaps" and total area available for movements, thus these correspond to the condition (ii) in this paper.

To understand better why conditions (i) and (ii) in this paper are similar to the set of reachability conditions of a leaper graph consider, for instance, the following transformation of leaper graphs from $m \times n$ to m nodes—illustrated in Figure 3 (a) and (b):

a) Project the edges leading to possible positions of the leaper after a single move to the m–sized axis of the board: for all $i, j \in [1, m]$ add a directed edge from node i to node j, if $j = i \pm r$, or $j = i \pm s$.

b) Remove edges from that graph, so that the projection of the leaper can move s steps only in one direction and r steps only in the other.

	$g > 2$	$g = 2$	$g = 1$
$P \geq m$	False	False, except if (c): P is even and both boundaries are reflecting	True
$2 < P < m$	False	False	False, except if (b): $r = 1$ and L is reflecting, or $l = 1$ and R is reflecting
$2 = P < m$	False, except if (a): both boundaries are reflecting	False, except if (a)	False, except if (a) or (b)

Table 1: The truth, or falsity of graph reachability (digraph strong connectivity) as it depends on g and P.

(a)

(b)

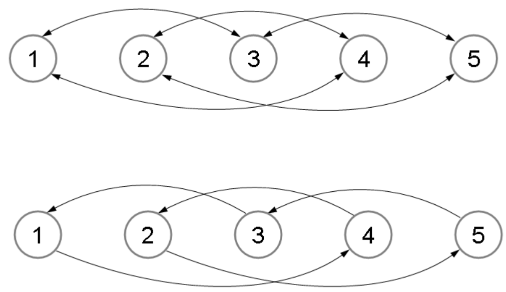

Figure 3: (a) one–dimensional projection of leaper moves with parameters $s = 3$, $r = 2$ and $m = 5$; (b) the projection graph after pruning edges so that only moves of r steps to the left and s steps to the right are possible.

Compare Figure 3 (b) to Figure 4, that illustrates a graph of restricted walk between absorbing boundaries studied in this paper: the one–dimensional graph obtained by the above transformation, differs from graphs of walks between absorbing boundaries studied in this paper only in that the absorbing positions and the edges leading to them are removed. Therefore, conditions (i) and (ii) of this paper are necessary[3] and sufficient for connectivity in the graph of Figure 3 (b).

[3] Since the exceptions to the necessity of conditions (i) and (ii) are caused by reflective boundaries, they do not apply to that graph.

An inverse transformation can be done in the other direction: by adding edges to graph in Figure 3 (b), we obtain the projection graph in Figure 3 (a).

Since adding edges can only improve connectivity, our conditions (i) and (ii) are sufficient for connectivity of the projection graph. The latter, in turn, implies reachability between the columns of the $m \times n$ board. However, this property does not guarantee that the leaper graph itself is connected.

5 Outline of results

Section 6 contains the sufficiency and the necessity proofs of reachability conditions (i) and (ii). The rest of the paper deals with walks between restricting boundaries for which those conditions are true. In section 7 we characterize the cases in which a graph of a walk between absorbing boundaries contains a Hamiltonian circuit or a Hamiltonian path. In section 8 we provide methods to estimate and to compute the length of the shortest walk between two positions. In section 9 two shortest path algorithms are given.

Below we informally summarize the main results and the methods used to obtain them.

The sufficiency of conditions (i) and (ii) is proved in section 6.1 by constructing a program, labeled algorithm A, that will move the particle from position i to position j, assuming that (i) and (ii) are true. The inputs of that program are the required numbers of moves to the left and to the right. Those inputs are obtained by solving the equation $j - i = -xl + yr$ in integers using, e.g., the extended Euclid algorithm [Knuth 1998]. The particle "shuttles" back and forth in $[L, R]$ and we show that after the required number of moves in each direction are done, its position must be j.

The necessity of condition (ii) is proved in section 6.2 by dividing the graphs for which $1 < P < m$ into three groups and showing that, except for the special cases (a) and (b), in each case there is a position that cannot be reached. Those three groups are:

1) Both boundaries are absorbing;
2) At least one of the boundaries is reflecting and both l and r are greater than one; and
3) At least one of the boundaries is reflecting, and the size of a move in at least one direction is one.

The necessity of condition (i) is proved in section 6.2 assuming that condition (ii) is true, by examining the options $g > 2$ and $g = 2$: First, we show using modular arithmetic that if $g > 2$, then there is at least one position in (L, R) that can be reached neither from L nor from R. Second, we examine the option

$g = 2$ and show that there exists an unreachable position except for the special case (c).

For graphs with absorbing boundaries that fulfil our reachability conditions, we show in section 7 that some of them have a Hamiltonian path or a Hamiltonian circuit and some others not. The non–existence proofs are carried out by contradiction and they are based on the simple facts that in any path there is a single ending node and there can be at most one predecessor of any node.

We also provide general characterizations of cases where non–absorbing part of the restricted walk graph has a Hamiltonian path: The graph is (trivially) Hamiltonian when $l = 1$, or $r = 1$. When $l > 1$ and $r > 1$, the graph is Hamiltonian, if and only if $P = nm + k$, where n is a positive integer and $k = -1, 0, 1$. Also a method to construct Hamiltonian paths, and the number of different paths in those cases are given.

The remaining part of the paper deals with shortest paths in graphs for which conditions (i) and (ii) are true.

Section 8 contains ancillary definitions and methods to estimate and to compute the length of the shortest path.

We derive in section 9 a shortest path algorithm between absorbing boundaries, labeled algorithm B, from algorithm A using the following insight. If $P = m$, then, as proved in section 6, there is a path between any two positions within the interval $[L, R]$. In addition, the assumption $P = m$ does not allow that the particle is simultaneously at least l steps away from L and at least r steps away from R. So, in this case, the particle located within the interval $[L, R]$ can move in *only one way* without leaving that interval: If the particle is less than r steps away from the right boundary, it can move only to the left; otherwise, it can move only to the right. An example of this situation is illustrated in Figure 4.

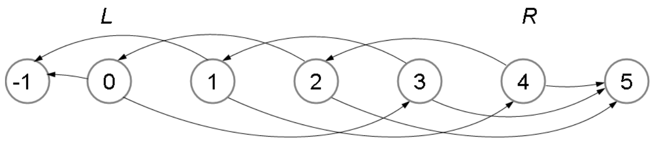

Figure 4: Graph of a walk between absorbing boundaries when $l = 2$, $r = 3$ and $P = 5$.

It follows, that if $P = m$, there is no need to know the required number of moves in each direction. A particle that simply "shuttles" back and forth in

$[L, R]$ will reach every position, and it will do it in the shortest way.

Algorithm B consists of first, moving the particle in the direction of the destination until the distance between the particle and the destination is less than m; second, choosing an interval of size m that includes the current position of the particle and the destination; and third, moving back and forth in that interval until the destination is reached. The correctness of this algorithm is proved by induction.

When one or both of the boundaries are reflecting, the destination may be sometimes reached faster by bouncing off the reflective boundary than by algorithm B: "bouncing" involves first moving in the direction of the reflective boundary until it is reached and then using algorithm B to reach the destination from that boundary.

For that reason, the shortest path algorithm for walks with one or two reflective boundaries, labeled algorithm C in section 9.4, first computes and compares the lengths of the paths with and without bouncing from each reflective boundary, and then moves the particle according to the shortest option.

6 Proof of the reachability result

6.1 Sufficiency

Below we will prove that conditions (i) and (ii) are sufficient for reachability of any position j from any non–absorbing position i in a finite number of moves. Plainly, if j is absorbing and i is not, then the particle can reach j by repeatedly moving toward j. Assume then, that j is non–absorbing: $L \leq j \leq R$.

The proof below is based on a well known fact from elementary number theory (see Theorem 25 in [Hardy & Wright 1979]): the equation

$$d = -xl + yr, \quad \text{where } d = j - i, \tag{1}$$

is solvable in integers x and y if and only if $d \equiv 0$ modulo g.

As a corollary, if $g = 1$, then (1) is always solvable in integers x and y. In this case it is always possible to choose a pair of non–negative integers (x, y) that satisfies (1), because if some (x, y) is a solution of (1), then any pair of the form $(x + nr, y + nl)$, where n is a natural number, is also a solution. Thus, any integer position j can be reached during an unrestricted walk if condition (i) is true. The total amount of moves given by a solution pair $(x + nr, y + nl)$ is $x + y + nm$. It can be verified that the particle needs a multiple of m moves to return into same position during an unrestricted walk.

We define $I_a(u)$ as the range of u consecutive positions that starts at a: $I_a(u)$ is $[a, a + u - 1]$, where $m \leq u \leq P$ and $L \leq a \leq R - u + 1$. Next we will describe a "shuttle" algorithm by which a restricted walk will reach any position within

$I_a(u)$ if both (i) and (ii) are true.

Algorithm A (Shuttle in $I_a(u)$). The algorithm takes three groups of inputs: First, the general parameters of the walk, i. e. l, r such that $g = 1$ by condition (i) and the interval $I_a(u) = [a, a + u - 1]$ whose size u is by condition (ii) at least m: $u \geq m$. Second, the initial position i and the target position j within $I_a(u)$: $a \leq i, j \leq a + u - 1$. Third, a non–negative solution (x, y) of (1).

A1. [Initialize.] The particle starts at $c_0 = i$ and uses two counters: X_t holds the number of moves taken to the left and Y_t the number of the moves taken to the right. The counters are initialized with $X_0 = 0$ and $Y_0 = 0$. The particle walks as follows after initialization:

A2. [Shuttle.] While $X_t \neq x$ or $Y_t \neq y$ do:

A2.1 [Move to the right.] If $c_t \leq a + u - 1 - r$ and $Y_t < y$, then move r steps to the right and increment Y_t: $c_{t+1} = c_t + r$, $Y_{t+1} = Y_t + 1$. The counter X_t is not incremented: $X_{t+1} = X_t$.

A2.2 [Move to the left.] If $c_t \geq a + l$ and $X_t < x$, then move l steps to the left and increment X_t: $c_{t+1} = c_t - l$, $X_{t+1} = X_t + 1$. The counter Y_t is not incremented: $Y_{t+1} = Y_t$.

(End of algorithm A.)

It follows from (ii) that $c_t \leq a + u - 1 - r$ and $c_t \geq a + l$ cannot be false simultaneously and so each move consists of (full) r or l steps. Therefore, during the walk the following invariant equation holds:

$$c_t = -X_t l + Y_t r + i. \tag{2}$$

What still needs to be shown is that algorithm can proceed until the counters (X_t, Y_t) eventually become equal to (x, y), in which case $c_t = j$ by (1) and (2). Assume the contrary, then there is a point in the execution of the algorithm where either $Y_t = y$, $X_t < x$ and $c_t < a + l$, i. e. the particle should move only to the left but cannot because it is too close to the beginning of the interval, or $Y_t < y$, $X_t = x$ and $c_t > a + u - 1 - r$, i. e. the particle should move only to the right but cannot because it is too close to the end of the interval. Let us look at the first case. Then, by (1) and (2),

$$c_t = -X_t l + yr + i = -xl + yr + i + (x - X_t)l = j + (x - X_t)l.$$

But since $j \geq a$ this means that $c_t > a + l$, which contradicts our assumption of $c_t < a + l$, and therefore the particle can move to the left. Similarly, we find in the latter case that

$$c_t = -xl + Y_t r + i = -xl + yr + i - (y - Y_t)r = j - (y - Y_t)r.$$

Since $j \leq a+u-1$ this means that $c_t \leq a+u-1-(y-Y_t)r \leq a+u-1-r$, which contradicts our assumption of $c_t > a+u-1-r$, and therefore the particle can move to the right.

This concludes our proof of sufficiency of conditions (i) and (ii).

6.2 Necessity of condition (ii)

By rotating a graph of a walk 180° we obtain its dual. In this transformation "left" becomes "right", the values of l and r, as well as the properties of the boundaries are exchanged and it can be verified that position i in the original graph becomes $L+R-i$ in the dual. Note also that the dual of the dual is the graph itself. Since rotating a graph does not affect reachability of its nodes we have the following

Duality lemma. *There is a reachability relation between two positions in a graph of a walk if and only if there is a corresponding relation between their images in the dual graph.*

The necessity of condition (ii) is proved by dividing the graphs for which $1 < P < m$ into three groups and showing that, except for the special cases (a) and (b), in each case there is a position that cannot be reached:

1) Let us first examine the case in which both ends of the interval are absorbing and $1 < P < m$. It is to be observed that in this case the particle cannot move to the right from position $R-r+1$ without being absorbed; and it cannot move to the left from position $L+l-1$ without being absorbed. Suppose now that $l \leq P \leq m-1$. But then $R-L+1 \leq m-1$ and so $R-r+1 \leq L+l-1$, with equality when $P = m-1$, i. e. from position $L+l-1$ the particle can move neither to the left, nor to the right without being absorbed; non–absorbing positions are not reachable from $L+l-1$. Suppose next that $1 \leq P \leq l-1$. Now, from position R the particle can move neither to the left, nor to the right without being absorbed; non–absorbing positions are not reachable from R.

2) Second, let us examine the case in which at least one of the interval ends is reflecting and both l and r are greater than 1.

Let us first consider the case $P = 2$. Now we have only two non-absorbing nodes: L and R. Because $l > 1$ (respectively, $r > 1$) L (resp., R) can only be reached from R (resp., L) if L (resp., R) is reflecting. Thus, there is a non-reachable position unless we have the exceptional case (a).

Let us next assume $2 < P < m$. It is to be observed that position j belonging to the interval (L, R) cannot be reached from the right if $R-l < j$ and it cannot be reached from the left if $L+r > j$.

Assume that $1 < r \leq l$. If $l < P < m$, take $j = R-l+1$. This is a position within the interval, which is not reachable from the right because $R-l <$

$R-l+1$, and it is not reachable from the left because from $P<m$ it follows that $R-l+1<L+r$, i.e. $L+r>j$. If $2<P\leq l$, take $j=L+1$. This position cannot be reached from the left because $r>1$ and it cannot be reached from the right because $R-l<j$ by assumption. And if $1<l\leq r$, then existence of unreachable position follows by the duality lemma.

Notice that the above reasoning is also valid when both boundaries are absorbing, thus giving an alternative proof for case (1) under the assumptions that $P>2$, $l>1$ and $r>1$.

3) Third, suppose that $1<P<m$, and examine the case in which one of the boundaries is reflecting and either l, or r is 1. If $r=1$ and L is reflecting, or if $l=1$ and R is reflecting, then the particle may reach every position. For example, suppose that the left boundary is reflecting. The particle can always reach L by moving only to the left. And if $r=1$, then the particle that starts at L and moves only to the right will reach every position between the boundaries. Thus, every position is reachable in the special case (b).

Suppose, on the other hand, that L is reflecting, $R+1$ absorbing, $l=1$ and $l<r$. In this case position R is unreachable from the right because $R+1$ is absorbing and it is unreachable from the left because from $1<P<m$ and $l=1$ it follows that $L+r>R$. And if $1<P<m$, R is reflecting, $L-1$ absorbing, $r=1$ and $r<l$, then position L is unreachable by the duality lemma.

This concludes our proof of necessity of condition (ii).

6.3 Necessity of condition (i)

Reachability in the special case (a) is obvious. In the rest of this section we will assume that condition (ii) is true.

First, notice that $g\leq l$ and $g\leq r$ because the greatest common divisor function cannot exceed any of its arguments. Therefore, $g\geq 2$ together with $P\geq l+r$ implies that $P\geq g+2$.

Second, let us denote by v the value of the distance $d(L,R)$ modulo g, by w_j the value of the distance $d(L,j)$ modulo g and by z_j the value of the distance $d(j,R)$ modulo g. Clearly, the equality

$$v=w_j+z_j \pmod g \tag{3}$$

is true for any j in the interval (L,R). On the one hand, j can be reached from L (or, respectively, from R) if $w_j=0$, (or, respectively, $z_j=0$) by solving (1) and following algorithm A. Moreover, j can be reached from L (or from R) *only* if $w_j=0$ (or $z_j=0$); otherwise we could solve (1) in integers when d is not a multiple of g, which is impossible.

On the other hand, if $P\geq g+2$, then there are at least g possible values of j such that $j\neq L$ and $j\neq R$ and it can be verified that there are exactly g ways to form the right hand side of (3) when the left hand side v is fixed. But at most two out of those g ways, namely $0+v$ and $v+0$, are such that j is reachable from L or from R.

Therefore, if $g > 2$, then for any v taken from $0, 1, \ldots, g-1$ there exists at least one pair of non–zero w_j and z_j values that satisfy (3); there will be at least one position j in (L, R) that can be reached neither from L nor from R.

When $g = 2$, the variables in (3) are either 0 or 1. If $v = 0$, then $w_j = z_j = 1$ satisfy (3), from which follows existence of position j that is reachable neither from L, nor from R. But if $v = 1$, which is the case when P is even, then one of w_j, or z_j must be 0; and any position within (L, R) is reachable either from L, or from R but not from both. This does not help the particle to reach other positions when both boundaries are absorbing: for a given position j any position i such that $d(i, j) \neq 0$ modulo g is still unreachable from j. But if $v = 1$ and both boundaries are reflecting, then L and R can be reached from any position and reachability of any position from any other in the special case (c) follows.

Suppose next, that $v = 1$, the left boundary L is reflecting, the right boundary is absorbing and take j within (L, R) such that $w_j = 0$. But then position $j-1$ within (L, R) is reachable neither from j (without going via the reflecting boundary L), because $d(i, j) \neq 0$ modulo g, nor from L, because $w_{j-1} = 1$. And if $v = 1$, the left boundary is absorbing and the right boundary R is reflecting, then unreachable position exists in (L, R) by the duality lemma.

We have thus proved that condition (i) is necessary, with the exception of special cases (a) and (c).

This concludes our proof. In the rest of this paper we will assume that conditions (i) and (ii) are true.

7 Hamiltonian circuits and paths

In this section we concentrate only on cases where both end points are absorbing. Furthermore, we exclude the absorbing end points completely because it is clear that an absorbing node can only be an end point of a Hamiltonian path. Therefore, we say that the graph is Hamiltonian if there is a Hamiltonian path through all non–absorbing nodes, i. e. a path that visits each (non–absorbing) node exactly once. If the beginning node of the Hamiltonian path can be reached from its end node in one move, we say that the graph has also a Hamiltonian circuit.

It is clear that the graph is Hamiltonian if either $r = 1$ or $l = 1$. Thus, let us assume in the following that $r > 1$ and $l > 1$. We will also assume in this section that $L = 1$, without loss of generality.

7.1 Hamiltonian circuits

We will prove below that the graph of restricted walk is a Hamiltonian circuit if, and only if, $P = m$.

Sufficiency: This part follows from our reachability results in the preceding section. Indeed, inside the block of $m = r+l$ nodes there must be a path traveling through all nodes (because both conditions (i) and (ii) are fulfilled). On the other hand, by Eq. (1) in section 6.1 the particle in unrestricted walk can return to the same place only after a multiple of m moves. Because we have only m nodes, the walk has to return to the same place after m moves and we have a Hamiltonian circuit.

In addition the Hamiltonian circuit is unique (apart from the fact that starting point can be chosen freely) because for each node (inside the block of m nodes) there is only one direction where to continue while staying inside the block: from the leftmost l nodes we can only move to the right while from the rightmost r nodes we can only move to the left. Remember here, again, our assumption about absorbing boundaries in the beginning of this section.

Necessity: As already mentioned above, the particle in unrestricted walk can return to the same place only after a multiple of $m = l + r$ moves. Therefore, a graph of restricted walk between absorbing boundaries may be a Hamiltonian circuit only when its number of non–absorbing nodes P is a multiple of m. It follows that the graph does not have a Hamiltonian circuit when $1 < P < m$.

We will prove next that there is no Hamiltonian circuit also for $P > m$, by showing that if a Hamiltonian circuit exists, then its length h is at most m. Since the cases of $l < r$ and of $l > r$ can be transformed into each other by the Duality lemma, it is enough to prove that $h \le m$ in only one of those cases; we will assume that $l > r$ in the proof. (See the illustration in Figure 5.)

Let λ be the set of nodes $\{ 1, 2, \ldots, l \}$, and ρ the set of nodes $\{ l+1, l+2, \ldots, l+r\}$. The sizes of those sets are: $|\lambda| = l$ and $|\rho| = r$.

It can be observed that:

i. any node $i \in \lambda$ has a single outgoing edge to the non–absorbing node $i + r$;

ii. each of the r nodes $i = 1, 2, \ldots, r$ in λ, have a single incoming edge from the node $j = i+l$ in ρ. (In Figure 5, the nodes 1, 2 and 3 have a single incoming edge.) So, if Hamiltonian circuit exists, then it must enter node $i \in \{1, 2, \ldots, r\}$ from node $j = i + l$ in ρ.

Now, let us assume that the graph is a Hamiltonian circuit of length h. By (i), the circuit can proceed only to the right from any node $i \in \lambda$ until it leaves λ and reaches one of the nodes in ρ. By (ii), the circuit can proceed only to the left from any node $j \in \rho$, because otherwise the Hamiltonian circuit cannot include the node $i = j - l \in \lambda$.

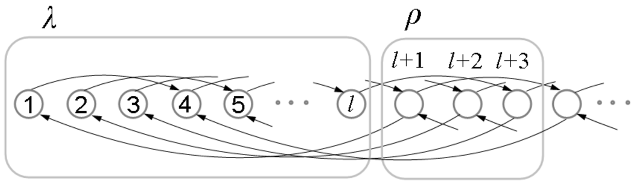

Figure 5: Graph of a walk with absorbing boundary on the left when $l > r$ and $r = 3$. The edges to the absorbing boundary on the left and the boundary itself are not shown.

It follows that $h \leq |\lambda| + |\rho| = m$.

This completes our proof.

7.2 Hamiltonian paths

7.2.1 Non–existence

Let us continue with an example of non–existence result, illustrated in Figure 6 (b).

If $P = r + l + 2$ then the graph is not Hamiltonian. This can be seen based on the fact that a path must have a single end, as follows.

Let us assume the contrary, i. e. that there is a Hamiltonian path in the graph. Similarly as above, from nodes 1 and $r+l+1$, the path can continue only to one node: $r+1$. Indeed, from node $r+l+1$ we cannot go to the right because $r > 1$. This implies that either 1 or $r + l + 1$ has to be at the end of the Hamiltonian path.

Respectively, from both nodes 2 and $r + l + 2$ the path can only continue to node $r + 2$ (remember here that $l > 1$). Thus, either 2 or $r + l + 2$ must be also at end of the Hamiltonian path. But since a path must have a single end, we have a contradiction. In Figure 6 (b) we have an example of this case. There is no Hamiltonian path because both 1 and 6 point only to 4 while both 2 and 7 point only to 5.

We can extend the above argument, and show that the graph is not Hamiltonian when $P = r+l+3$: The assumptions of $\gcd(r, l) = 1$ and of l and r being greater than one, imply that either $r > 2$ or $l > 2$. In each case we can extend the graph from one of the ends without losing the argument used to prove that $P = r+l+2$ is not Hamiltonian. If $r > 2$ we can simply add a new node $r+l+3$. If, on the other hand, $l > 2$, then we can add a new node labeled by 0 into the left–hand side of the graph.

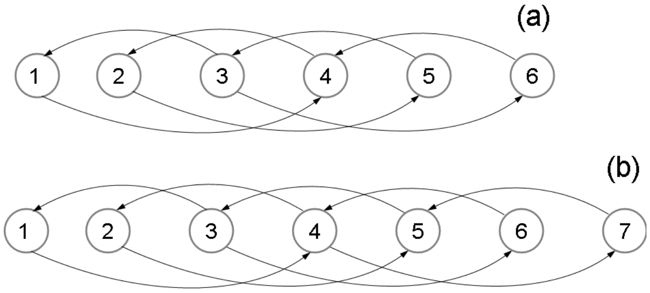

Figure 6: Graphs of a walk between absorbing boundaries when $l = 2$ and $r = 3$. In (a) $P = 6$ and in (b) $P = 7$. The edges to the absorbing boundaries and the boundaries themselves are not shown.

7.2.2 Existence

Next we give some existence results. Let us begin with a couple of examples of graphs that are Hamiltonian. The first example is the graph of Figure 3 (b). Moreover, it contains a Hamiltonian circuit: 1, 4, 2, 5, 3, 1.

Our second example is the graph of Figure 6 (a), where $1, 4, 2, 5, 3, 6$ is a Hamiltonian path. The third example is a little bit more complex and it paves the way for our more general existence results.

Let $r = 5$, $l = 2$ and $P = 21$. Now the following path is Hamiltonian: 1, 6, 4, 2, 7, 5, 3, 8, 13, 11, 9, 14, 12, 10, 15, 20, 18, 16, 21, 19, 17. There is certain periodic structure in the path: first we travel through 7 first nodes, then we travel through another block of 7 nodes and finally through the 7 last nodes.

Now we are ready to prove the following result. Let $\gcd(l, r) = 1$. Then graphs with $P = n(r + l) = m$ are Hamiltonian.

The proof follows the structure of the preceding example. We construct a Hamiltonian path that begins from node 1 and travels through all of the first $m = r + l$ nodes, continues then to the leftmost of the remaining nodes, i. e. to the node $m + 1$. What follows is an exact copy of the set of movements in the first part of the path. After that all nodes on the first two blocks of m nodes have been covered and we continue from the first of the remaining nodes, i. e. from the node $2m + 1$. Next we travel through the third block of nodes, and so on. It would be a straight–forward task to create an exact proof by an induction on the multiplier n once the first part of the path (covering first $m + 1$ nodes) has been constructed.

This first part can be constructed from our results in the preceding subsection: there is even a Hamiltonian circuit that travels through all of the $m = r + l$

nodes in the first block and we may start this circuit from any node, in particular, we may start from the node 1.

What still has to be shown is how to "glue" different blocks (of $r + l$ nodes) together. When starting from node 1, the last node in the path has to be $l + 1$, since node 1 can be reached only from $l + 1$. Instead of completing the circuit, we could now take a move to the right from $l + 1$, thus moving to the first node of the next block: the node $r + l + 1 = m + 1$.

Starting from the node $m + 1$ the same movement pattern is repeated in the second block after which the path continues to the first node of the third block and so on.

It can be easily seen from the construction that the Hamiltonian path can be extended from the end by one node, i. e. a node $nm + 1$ can be appended to the path. This shows that also graphs with $P = nm + 1$ are Hamiltonian (of course, still under the assumption $\gcd(l, r) = 1$).

Alternatively, because our path begins from the very leftmost node 1 that node can also be cut out from the graph, and the remaining path would still be Hamiltonian. Therefore, also graphs with $P = nm - 1$ are Hamiltonian.[4]

For the simplest non–trivial case where $r = 2$ and $l = 3$ we now know, on the one hand, that the following values of P give a Hamiltonian path:

$$4, 5, 6, 9, 10, 11, 14, 15, 16, \ldots.$$

On the other hand, from our non–existence result we know that the cases $P = 7$ and $P = 8$ are not Hamiltonian when $r = 2$ and $l = 3$.

But there are still values of P that are open cases; we resolve these in the next section.

7.2.3 Complete characterization

In this section we extend our non–existence results and show that all cases that were left open above are actually not Hamiltonian. The argumentation follows along the lines that were used above to prove that graphs with $P = m + 2$ and $P = m + 3$ are not Hamiltonian.

Let us begin with a graph with $P = nm + 2$ and show that there cannot be a Hamiltonian path. To prove this, we start by assuming, once again, the contrary.

Consider, first, the leftmost node 1 and the node $r + l + 1 = m + 1$. Both nodes point to the same node: $r + 1$. Because from 1 there is no other edge to a non–absorbing node, except the edge to $r + 1$, one of the following cases must be true: either 1 or $m + 1$ is the end of the Hamiltonian path, or alternatively, the Hamiltonian path continues from $m + 1$ to the right, i. e. to the node $m + r + 1$.

[4] Notice that the existence of Hamiltonian path in the case of $P = m - 1$ does not contradict reachability condition (ii): $P \geq m$; the Hamiltonian path starts at a specific node r, while (ii) is required for reachability from any starting node.

We are going to show that the end point of the Hamiltonian path must be one of the nodes we take into closer consideration. Therefore, we assume now the last option: the path continues to the right from $m + 1$. But then we can consider the node $m + r + l + 1 = 2m + 1$. Either $2m + 1$ is the end point, or the path has to continue to the right from it. Assuming the latter, we follow a similar argument for the node $3m+1$, then $4m+1$, and so on, until we reach the node $nm + 1$. Because from $nm + 1$ we cannot go anymore to the right without being absorbed, there is only one option for it: left, i. e. it must be the end point of the path.

Altogether, we know now that the end point of the Hamiltonian path must be in the set of nodes of the form $km + 1$ where $k = 0, 1, \ldots, n$.

Next we repeat exactly the same argumentation starting from the node 2 and ending in the (rightmost) node $nm + 2$. Thus, the end point of the Hamiltonian path has to be in the set of nodes of the form $km + 2$ where $k = 0, 1, \ldots, n$. Because the path cannot have more than a single end and the two sets are disjoint, we have a contradiction.

Now we know that the case of $P = nm + 2$ is not Hamiltonian. We follow a method similar to that in the earlier subsection for $P = m + 3$ to extend this non–existence result to other cases.

Indeed, what is essential in the above argumentation for the case $P = nm+2$, is that the path cannot continue to the left from the two leftmost nodes 1 and 2, and it cannot continue to the right from the two rightmost nodes $nm + 1$ and $nm + 2$. As long as these conditions remain true, we may add more nodes to the left and to the right.

The amount of nodes that may be added without losing the argument, depends on values l and r: on the left-hand side, we are able to add $l-2$ nodes (and still, from our original two leftmost nodes, the path can only go to the right); on the right-hand side, respectively, we may add $r - 2$ nodes (and still, from our original two rightmost nodes, the path cannot continue to the right). Now the largest number of nodes for which the argument still holds is

$$P = nm + 2 + l - 2 + r - 2 = nm + m - 2 = (n + 1)m - 2.$$

But this implies that all the open cases have been covered.

The resulting complete characterization is: In restricted walk between absorbing boundaries, the graph of the non-absorbing middle part is Hamiltonian when $l = 1$, or $r = 1$. When $l > 1$ and $r > 1$, the graph is Hamiltonian, if and only if $P = nm + k$, where n is a positive integer and $k = -1, 0, 1$.

7.2.4 Number of Hamiltonian paths

In this section we prove that when Hamiltonian paths exist, then the number of different Hamiltonian paths (denoted by η) is

$$\eta = \begin{cases} 1 & \text{if } P = m - 1, \\ m & \text{if } P = nm, \\ 2 & \text{if } P = nm - 1 \text{ and } n > 1, \\ 2 & \text{if } P = nm + 1, \end{cases} \tag{4}$$

where n is a positive integer.

We assume still here that both $l > 1$ and $r > 1$. If, e.g., $r = 1$ then the graph has a Hamiltonian path $1, 2, 3, \ldots, P$. This is also the only Hamiltonian path (HP for short) unless $P = m = l + 1$ in which case there is a Hamiltonian circuit (and m different HPs).

The proof is as follows.

1) *Case of* $P = m - 1$: We have shown in section 7.1, that the graph contains a single Hamiltonian circuit when $P = m$. Removing from that graph either the first node: 1, or the last node: m, we obtain a Hamiltonian path for a graph with $P = m - 1$.

The node r has to be the starting point of any HP because it cannot be reached from either right or from left. As discussed earlier, there is always only one possible direction where to continue from any node. Therefore, we have a unique Hamiltonian path in the graph and $\eta = 1$ in this case.

2) *Case of* $P = nm$: We have seen in section 7.1 that the graph has a Hamiltonian circuit if $n = 1$. This implies $\eta = m$: the path may begin in any one of the m nodes and follow the circuit until the starting point is reached again.

If $n > 1$ we assume without loss of generality that $r < l$, and divide the graph of the walk into n consecutive blocks B_1, B_2, $\ldots$, B_n with m nodes in each, as shown in Figure 7.

As illustrated with the three darker edges in Figure 7, nodes 1, 2, $\ldots$, r in the first block B_1 are reachable only from the right, via the edges $(l + 1, 1)$, $(l + 2, 2)$, $\ldots (l + r, r)$, respectively. Because a path has only one starting node it follows that either,

i. all those r edges are in the Hamiltonian path, or

ii. exactly one of them: $(l + i, i)$, where $1 \leq i \leq r$, is not in the Hamiltonian path.

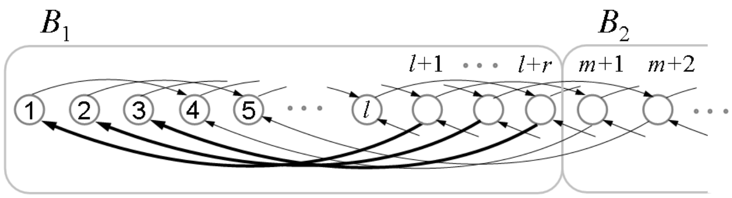

Figure 7: The graph from Figure 5 with the leftmost blocks of m nodes marked.

In case of (i) the Hamiltonian path cannot leave B_1; hence, it must end in that first block. We deal with such paths in the second part of the proof and assume—for the moment—that (ii) is true.

The Hamiltonian path must start at node i, where $1 \leq i \leq r$, and it can leave B_1 *only* through node $l + i$, from which it proceeds to node $l + i + r = m + i$ in B_2.

So, first, there are exactly r paths starting in B_1. Those paths can be constructed following the example for $i = 1$ in section 7.2.2. Second, it is important to notice, once again, that inside the first block B_1 there is always just one possible option to continue the path and, when starting from the node i, the path travels through all nodes of B_1 ending in the node $l+i$. In other words, the path has to go through all nodes in B_1 before it is able to reach the only way out of B_1 (i.e. the node $l + i$).

This implies that, in the case (ii), the Hamiltonian path of the whole graph must have a prefix path that is a Hamiltonian path inside the first block. Furthermore, it follows that the rest of the path must be a Hamiltonian path inside the rest of the graph. This suffix HP has its starting point in the node $m + i$.

Now we can use similar reasoning for the second block B_2 that we applied above for the first block: the only way out of B_2 is the node $m + l + i$ and the Hamiltonian path visits all other nodes of B_2 before reaching $m+l+i$. Next we enter the third block B_3, starting from the node $2m + i$ and so on. Eventually we reach the last block B_n and the whole path ends in the node $mn - r + i$.

Because proceeding of the Hamiltonian path is fully determined after the starting point has been chosen from r different options in the first block B_1, we have exactly r different Hamiltonian paths in the case where (ii) is true.

Let us now consider the other alternative (i). As already mentioned above, the Hamiltonian path cannot leave the first block B_1 after it has been entered. This implies that the Hamiltonian path for the whole graph must have a suffix path that is a Hamiltonian path inside the first block. Furthermore, the rest of the path must be a Hamiltonian path inside the rest of the graph (i.e. blocks

$B_2, \ldots, B_n$) and the end point of the prefix HP must be one of the first l nodes of B_2: $m+1$, $m+2$, $\ldots$, $m+l$ (because otherwise the next node could not be in the first block B_1).

Now similar analysis to that done so far in this proof for the whole graph, may be applied to the prefix Hamiltonian path in the smaller graph of blocks $B_2, \ldots, B_n$. Either the starting node for the whole prefix HP is among the first r nodes of B_2 or the prefix HP contains a suffix path that is a Hamiltonian path inside the block B_2. The first option leads to a contradiction unless $n = 2$ because it would eventually imply that the prefix HP would end in the last block B_n. If $n = 2$ then the whole prefix HP is inside the (last) block B_2.

Repeating the same reasoning for the rest of the blocks results in concluding that the Hamiltonian path must first cover nodes in the last block B_n after which it travels all nodes of the penultimate block B_{n-1} and so on.

Similarly as in the case (ii), there is always only one possible way to continue the Hamiltonian path after the initial point has been chosen. In each block the last node has to be one of the first l nodes because otherwise it is not possible to jump into the preceding block. This corresponds to choosing one of the last l nodes in the last block B_n: mn, $mn-1$, $\ldots$, $mn-l+1$, as the starting point of the path.

By applying the Duality lemma to the construction of ("left-to-right") Hamiltonian paths presented earlier in this proof, it is easy to see that it is indeed possible to construct a Hamiltonian path that starts from any of the last l nodes and covers the blocks in the reverse order, i.e. B_n first and B_1 last. Thus, we have l different HPs in the case where (i) is true.

Altogether, we have $m = r + l$ different Hamiltonian paths.

This completes the proof of the case $P = nm$.

3) *Cases of $P = nm - 1$, where $n > 1$, and of $P = nm + 1$*: Let us try to repeat the arguments used for the case $P = nm$ as far as possible. In order to do that we divide these graphs also into blocks of m nodes, starting from node 1. This means that the number of nodes in the last block is not m: in the case $P = nm - 1$, where $n > 1$, the last block B_n contains $m - 1$ nodes and in the case of $P = nm + 1$ the last block B_{n+1} contains only one node.

Because the division to the cases (i) and (ii) was based only on the first block B_1, the same division applies here. The further analysis of case (ii) applies as well until we reach the last block. Both in the case where the last block contains $m - 1$ nodes and in the (trivial) case where the last block contains only one node, there is a unique Hamiltonian path in the block. To be able to append this unique HP as a suffix to the Hamiltonian path covering the earlier blocks, the starting point of the whole path must be chosen as follows: it has to be the node r in the case of $P = nm - 1$ and the node 1 in the case of $P = nm + 1$. In both cases there is exactly one Hamiltonian path.

Let us next see how to adapt the analysis of the case (i). Again, the last block appears fairly late in the reasoning and the conclusion still holds that the Hamiltonian path must cover all blocks, one by one, in reverse order starting from the last block. Similarly as for (ii), there is exactly one Hamiltonian path in the last block in either case. In the case of trivial last block B_{n+1} containing a single node, it is clear that the HP can be continued in the penultimate block B_n. In the case of last block B_n containing $m-1$ nodes, the Hamiltonian path must start from the r^{th} node of B_n: $m(n-1)+r$, as shown in case (1) of this proof. The HP inside the last block then ends in the l^{th} node of B_n: $m(n-1)+l$, and therefore the path can be continued to the penultimate block B_{n-1}. Thus, in both cases there is exactly one Hamiltonian path.

Altogether, we have two different Hamiltonian paths and $\eta = 2$ in each of these cases.

8 Length of the shortest path

We say that the path is "internal" if boundaries have no effect on it. As above, let c_t be the position of a particle at certain point of time t. A particle on internal path will not move to the left if $c_t - l$ is less than the leftmost non–absorbing position L and it will not move to the right if $c_t + r$ is greater than the rightmost non–absorbing position R. We say that paths not having this property are "external". For instance, any path to an absorbing boundary is external by definition.

An external path may be shorter than internal when the target position is sufficiently close to a reflecting boundary. For example, in a walk between reflecting boundaries with parameters $l = 2$, $r = 3$, $L = 0$ and $R = 8$ the shortest internal path from 7 to 6 is 7, 5, 3, 6, while the shortest external path is 7, 8, 6.

8.1 Property of shortest internal paths

Every internal path of a particle from i to j is characterized by a non–negative solution pair (x, y) of (1). The length of the path is $x + y$. Conversely, every non–negative solution pair (x, y) has a set of internal paths associated with it. The set of shortest internal paths is characterized by the *smallest* non–negative solution pair (x, y), i. e. (x, y) is such that

$$\begin{aligned} &d = -lx + ry, \\ &\text{where } x, y \geq 0 \text{ and } (x < r \text{ or } y < l). \end{aligned} \tag{5}$$

This pair is unique for the following reasons. First, assume (x, y) is a non-negative solution of (1). (We already know that such a solution exists). Any other solution of (1) must be of the form $(x+nr, y+nl)$, where n is an integer. If both $x \geq r$ and $y \geq l$ then we can find an appropriate negative integer n such that at least one of the conditions $x < r$ and $y < l$ becomes true. We also know that at this point we have found the smallest solution pair because subtracting once more r from x and l from y would make either x or y negative (or both).

We will use this property in computing the length of the shortest path below, and again in proving correctness of Algorithm B in section 9.

8.2 Definitions

We define $D(i,j)$ as the length of the shortest path from i to j, $D_I(i,j)$ as the length of the shortest internal path, $D_L(i,j)$ as the length of the shortest external path that contains, or jumps over L, and $D_R(i,j)$ as the length of the shortest external path that contains, or jumps over R. $D(i,j)$ is the minimum among $D_I(i,j)$, $D_L(i,j)$ and $D_R(i,j)$, by which $D(i,j) \leq D_I(i,j)$.

Clearly, $D(i,i) = 0$. If $j \neq i$ and i is an absorbing position, then $D(i,j) = \infty$. If j is an absorbing position at $L-1$ and i is non–absorbing, then

$$D(i,j) = D_L(i,j) = \lceil (i-L+1)/l \rceil, \tag{6}$$

where $\lceil (i-L+1)/l \rceil$ is the smallest integer greater than or equal to $(i-L+1)/l$. Similarly, if j is an absorbing position at $R+1$ and i is non–absorbing, then

$$D(i,j) = D_R(i,j) = \lceil (R+1-i)/r \rceil. \tag{7}$$

Let us assume that both j and i are non–absorbing. In this case, if the left boundary is reflecting, it can be verified that

$$D_L(i,j) = \lceil (i-L)/l \rceil + D_I(L,j). \tag{8}$$

Similarly, if the right boundary is reflecting, then

$$D_R(i,j) = \lceil (R-i)/r \rceil + D_I(R,j). \tag{9}$$

8.3 Range of $D_I(i,j)$

Let us denote by $b_I(i,j)$ and $B_I(i,j)$ (any) lower and upper bounds on $D_I(i,j)$

$$b_I(i,j) \leq D_I(i,j) \leq B_I(i,j). \tag{10}$$

When i is a non–absorbing position we can estimate those bounds by using the following argument. A distance d between two positions in interval $I_L(P)$ belongs to the set $\{-P+1, -P+2, \ldots, P-1\}$. If $0 < |d| < m$, then the shortest path of a particle that is in the range $I_a(m)$ of m positions one of which is the target and which, as we have proved in section 6, can reach the target position without leaving $I_a(m)$, is between 0 and m moves: if more than m moves are done without leaving $I_a(m)$, then the particle must cycle through some of the m positions in $I_a(m)$, in which case the path is not the shortest one. If $P-1 \geq |d| \geq m$, then the target position j is reachable by first moving the particle in the direction of j until it is less than m steps from j, choosing a such that $I_a(m)$ contains both the current and the target positions of the particle, and then reaching the target without leaving $I_a(m)$. In summary, the lower and upper bounds are as follows.

Distance	$b_I(i,j)$	$B_I(i,j)$
$d=0$	0	0
$0<\vert d\vert<m$	0	m
$m \leq d \leq P-1$	$\lceil (d-m)/r \rceil + 1$	$b_I(i,j)+m$
$-P+1 \leq d \leq -m$	$\lceil \vert d+m\vert/l \rceil + 1$	$b_I(i,j)+m$

(11)

Notice that the highest value of $B_I(i,j)$, which is $B_I(L,R)$ if $l>r$ and $B_I(R,L)$ otherwise, is also the upper bound on the amount of moves that would be necessary to transfer the particle from a non–absorbing position to any other position. In the rest of this section we describe a method to compute $D(i,j)$.

8.4 Computing $D(i,j)$

If $d=0$, then $D(i,j)=0$. If $d \neq 0$ and i is an absorbing position, then $D(i,j)=\infty$. If $d \neq 0$ and $j=L$, then $D(i,j)=D_L(i,j)=\lceil (i-L)/l \rceil$. Similarly, if $d \neq 0$ and $j=R$, then $D(i,j)=D_R(i,j)=\lceil (R-i)/r \rceil$. Otherwise, to compute $D(i,j)$ we need to compare $D_I(i,j)$ with $D_L(i,j)$ (if the left boundary is reflecting) and with $D_R(i,j)$ (if the right boundary is reflecting), for which we need to find the lengths of internal paths from the reflecting boundaries to j and from i to j. This computation requires us to solve (5) and one of the ways to do this is by using algorithm B in section 9 below.[5]

To find the minimum of the distances $D_I(i,j)$, $D_L(i,j)$ and $D_R(i,j)$ it appears that we would need to solve (5) three times, once for each distance. Note, however, that when i is a non–absorbing position and the left boundary is reflecting it is not necessary to compute $D_L(i,j)$ if $D_I(i,j) < \lceil (i-L)/l \rceil + b_I(L,j)$, because in this case $D_I(i,j) < D_L(i,j)$ by (8) and (10). Similarly, when i is a non–absorbing position and the right boundary is reflecting it is not necessary to compute $D_R(i,j)$ if $D_I(i,j) < \lceil (R-i)/r \rceil + b_I(R,j)$, because in this case $D_I(i,j) < D_R(i,j)$ by (9) and (10).

[5] Another one is to first solve $1 = lx + ry$, e.g., using extended Euclid algorithm [Knuth 1998], and then modify the solution so that it satisfies (5).

9 Shortest path algorithm

We will first describe a shortest internal path algorithm and then a general shortest path algorithm for graphs of restricted walk that satisfy reachability conditions (i) and (ii).

9.1 Shortest internal path algorithm

When both boundaries are absorbing there is no bouncing and the particle that moves from i to j, both of which are non–absorbing, must follow an internal path. And if a particle follows a shortest internal path, then by counting the number of moves taken in each direction we will obtain the solution of (5) when the particle reaches j.

First, let us examine algorithm A when $u = m$. On the one hand, we have proved in section 6 that any position in the range $I_a(m)$ is reachable from within and without leaving that range. On the other hand, when $u = m$ the conditions $c_t \leq a + m - 1 - r$ and $c_t \geq a + l$ in steps A2.1 and A2.2 of algorithm A are mutually exclusive for any c_t within $I_a(m)$: If both conditions would be true then c_t would be simultaneously at most $a + l - 1$ and at least $a + l$; if both conditions would be false, then c_t would be simultaneously less than $a + l$ and more than $a + l - 1$.

So, moving l steps to the left if the second condition is true and r steps to the right otherwise, is the *only* way in which the particle can move within and without leaving $I_a(m)$. Therefore in this case a particle that shuttles in $I_a(m)$, will reach j in the shortest way.

Second, we can move the particle into any non–absorbing position by following an internal path as we outlined when deriving the relations (11) in section 8: Move the particle toward j until it is less than m steps from j, choose an interval $I_a(m)$ that includes both the current position of the particle and j and shuttle in $I_a(m)$ until j is reached.

Algorithm B (Shortest internal path). The inputs of this algorithm are l and r that satisfy condition (i); the interval $I_a(u)$ the ends of which, a and $a+u-1$, are non–absorbing, and $u \geq m$ by (ii); the initial position i and the target position j within $I_a(u)$: $a \leq i, j \leq a + u - 1$.

B1. [Initialize.] $c_0 = i$, $a' = a$, $X_0 = 0$, $Y_0 = 0$.

B2. [Move into the range $I_{a'}(m)$.] While $|j - c_t| \geq m$ do

B2.1 If $j - c_t < 0$, then $c_{t+1} = c_t - l$, $X_{t+1} = X_t + 1$.

B2.2 If $j - c_t > 0$, then $c_{t+1} = c_t + r$, $Y_{t+1} = Y_t + 1$.

B3. [Fix $I_{a'}(m)$.] Set $a' = c_t$. If $a' + m > a + u$, then $a' = a + u - m$. If $j < a'$, then $a' = j$.

B4. [Shuttle in $I_{a'}(m)$.] While $c_t \neq j$ do as follows: If $c_t \geq a' + l$, then move l steps to the left and increment X_t: $c_{t+1} = c_t - l$, $X_{t+1} = X_t + 1$; otherwise, move r steps to the right and increment Y_t: $c_{t+1} = c_t + r$, $Y_{t+1} = Y_t + 1$.

(End of algorithm B.)

Note to stage B2: When algorithm B is used solely to compute a solution of (5) the sequence of substractions, or additions in stage B2 can be replaced with a single division in an obvious way:

B2. [Move into the range $I_{a'}(m)$.]

B2.1 If $j - c_t \leq -m$, then $k = \lfloor (c_t - j)/l \rfloor$, $c_{t+k} = c_t - lk$, $X_{t+k} = X_t + k$.

B2.2 If $j - c_t \geq m$, then $k = \lfloor (j - c_t)/r \rfloor$, $c_{t+k} = c_t + rk$, $Y_{t+k} = Y_t + k$.

Here, $\lfloor (c_t - j)/l \rfloor$ is the greatest integer not exceeding $(c_t - j)/l$. A similar adjustment can be made in stage B4 when $r > l/2$, or $l > r/2$.

In stage B3 one of the ways to choose the interval $I_{a'}(m)$ is shown. It can be verified that at stage B3 there are at most $m - |j - c_t|$ ways to choose $I_{a'}(m)$ so that it contains both c_t and j. Those choices may result in different paths from c_t to j. However, since the distance between c_t and j is the same in those paths and the length of the enclosing interval is m, they all solve (5) with $d = j - c_t$. This equation has a unique solution, so those paths will be of same length.

It follows from (ii) that each move consists of (full) r or l steps and so during the walk the invariant (2) holds. Thus, the counters (X_t, Y_t) will hold a solution of (1) after the walk has completed and $c_t = j$.

It still remains to be proved that the length of an internal path taken by a particle that walks according to this algorithm is the shortest possible, in which case the counters (X_t, Y_t) will hold the solution of (5) after the walk has completed and $c_t = j$.

9.2 Proof

The proof is by induction on u. We have shown above that the particle that shuttles in an interval of m positions, $I_a(m)$, as prescribed in stage B4, will reach the destination in a shortest way.

Assume that algorithm B produces shortest paths in interval of u positions, where $u \geq m$. The set of distances covered by algorithm B in $I_a(u)$ is $\{-(u-1), -(u-2), \ldots, u-2, u-1\}$. Increasing the size of the interval by one position, to $u + 1$, will add $-u$ and u to that set. We will prove next that in the case of

$d = -u$ algorithm B produces a shortest path. In the case of $d = u$ the proof is symmetrical and will be omitted.

To cover the negative distance $d = -u$ a particle that follows algorithm B must first move l steps to the left (see stage B2.1). After that move its distance from the target would be $-u + l$. By the induction hypothesis

$$\begin{aligned} &-u + l = -lx_1 + ry_1, \\ &\text{where } x_1, y_1 \geq 0 \text{ and } (x_1 < r \text{ or } y_1 < l). \end{aligned} \tag{12}$$

Therefore, the solution of equation (1) that corresponds to the particle's path is $(x_1 + 1, y_1)$:

$$-u = -l(x_1 + 1) + ry_1, \quad \text{where } x_1, y_1 \geq 0. \tag{13}$$

We still need to show that the pair $(x_1 + 1, y_1)$ is the smallest non-negative solution of (1), i.e. that $x_1 + 1 < r$, or $y_1 < l$. This condition is true if $y_1 < l$ in (12). So, suppose that in (12) $y_1 \geq l$ and $x_1 < r$. In this case $x_1 + 1 \leq r$ because $r \geq 1$. If $x_1 + 1 = r$, then by (13)

$$-u = -lr + ry_1 = r(y_1 - l) \geq 0,$$

which contradicts our assumption of $d < 0$. Therefore $x_1 + 1 < r$.

This concludes our proof.

9.3 Run-time

We will estimate the run–time of algorithm B as the upper bound on the number of comparisons and assignments it will use. At most seven such operations are used in stages B1 and B3. The algorithm uses three arithmetic operations in each move of stage B2 and two arithmetic operations in each move of stage B4. Finally, it can be verified using (11) that the length of the shortest path is at most $\lceil (R - L)/\min(l, r) \rceil + 1 + m$ moves. It follows from the above that the run–time of algorithm B grows linearly with P.

When algorithm B is used solely to solve (5), the interval $I_a(m)$ can be reached in one "move", as described in the note to stage B2, and so algorithm B will finish using one division and at most $2m + 3$ arithmetic operations. In this case the run–time of algorithm B does not grow with P.

9.4 Shortest path algorithm

The following algorithm computes $D(i, j)$ in stage C1 and uses algorithm A to move the particle over internal part of the shortest path in stage C4. In stage C1 algorithm B is used to solve (5). In stages C2, and C3 the particle is moved to the left boundary if $D(i, j) = D_L(i, j)$, or to the right boundary if $D(i, j) = D_R(i, j)$.

Algorithm C (Shortest path). The algorithm takes two groups of inputs: First, the general parameters of the walk: the type of the boundaries, l, r, L, and R that fulfill the reachability conditions. Second, the initial position i and the target position j: $L-1 \leq i, j \leq R+1$.

- **C1.** [Find shortest path parameters.] Set $c_0 = i$ and compute $D(i,j)$. If the shortest path length is ∞, exit the algorithm with ∞ as its result. At this stage we know the shortest path type, its length, as well as the associated pair (x,y) if part, or all of the path is internal.
- **C2.** [Absorbing j.] If j is at the absorbing boundary on the left: $j = L-1$, then while $c_t > L-1$ do $c_{t+1} = c_t - l$. Set $c_t = L-1$ and exit the algorithm. If $j = R+1$, then while $c_t < R+1$ do $c_{t+1} = c_t + r$. Set $c_t = R+1$ and exit the algorithm.
- **C3.** [Bounce, if needed.] If the shortest path is of $i \to L \to j$ type, then while $c_t > L$ do $c_{t+1} = c_t - l$. Set $c_t = L$. Similarly, if the shortest path is of $i \to R \to j$ type, then while $c_t < R$ do $c_{t+1} = c_t + r$. Set $c_t = R$.
- **C4.** [Shuttle in $I_L(P)$.] Invoke algorithm A with the first input being $I_L(P)$, second input $i = c_t$ and j, and the third input being the pair (x,y) that was found in stage C1.

(End of algorithm C.)

The correctness of this algorithm follows from the fact that the parameters computed in stage C1 prescribe the shortest path.

9.5 Run–time

As we have done with algorithm B, we will estimate the run–time of algorithm C as the upper bound on the number of comparisons and assignments it will use. Stage C1, in which the shortest path is computed, will call algorithm B to solve (5) at most three times. We have seen above that when algorithm B is used solely to solve (5) its run–time does not depend on P. Consequently, the run–time of stage C1 does not grow with P. Stages C2 and C3 require at most two operations per move and the amount of moves in each is bounded from above by $\lceil (R-L)/\min(l,r) \rceil + 1$. The run–time of stage C4 is the same as that of algorithm B, i. e. it grows linearly with P. It follows from the above that the run–time of algorithm C grows linearly with the size of the graph.

10 Summary and conclusion

We have proved that two conditions are sufficient, and with three exceptions also necessary, for reachability of any position in restricted walk on integers.

For restricted walk between absorbing boundaries we gave a complete characterization of cases in which there is a Hamiltonian path or a Hamiltonian circuit through the non-absorbing nodes. For each case, we have also found all different Hamiltonian paths.

A property of shortest internal paths was given in equation (5). For a graph of restricted walk that fulfills reachability conditions, we have described a method to compute the length of the shortest path and a shortest path algorithm that uses this property. The algorithm does not need ancillary data structures and the upper bound on its the run–time grows linearly with the number of graph's vertices.

Many physical and engineering systems can be modeled with restricted walk. We hope that the results of this paper will facilitate better a understanding of these systems.

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