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On Choice Principles and Fan Theorems

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Abstract: Veldman proved that the contrapositive of countable binary choice is a theorem of full-fledged intuitionism, to which end he used a principle of continuous choice and the fan theorem. It has turned out that continuous choice is unnecessary in this context, and that a weak form of the fan theorem suffices which holds in the presence of countable choice. In particular, the contrapositive of countable binary choice is valid in Bishop-style constructive mathematics. We further discuss a generalisation of this result and link it to Ishihara's boundedness principle BD-N.
Key Words: constructive mathematics, fan theorem, countable choice
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In this paper, we work in Bishop-style constructive mathematics [Bishop and Bridges 1985], that is informal mathematics using only intuitionistic logic. However, we will state explicitly when choice principles are used. This way our results not only hold in the usual varieties of constructive mathematics such as Brouwer's intuitionism (INT) or Russian recursive mathematics (RUSS) [Bridges and Richman 1987], they can also be interpreted in a wide range of formal systems such as extensions of Heyting arithmetic [Troelstra and van Dalen 1988].

Veldman [Veldman 1982] has proved that the *contrapositive of countable binary choice*

CCC$_2$ $\quad \forall \alpha \, \exists n \, A(n, \alpha(n)) \implies \exists n \, \forall i \, A(n, i)$
for every decidable predicate A *on* $\mathbb{N} \times \{0, 1\}$

is a theorem of intuitionistic mathematics à la Brouwer. To be more precise: Veldman showed that in the presence of a certain (classically false) continuity principle CCC$_2$ follows from the fan theorem for decidable bars.[1] By inspection of Veldman's proof we will see that the bare fan theorem for decidable bars suffices; in particular, to prove CCC$_2$ there is no need to use any continuity principle whatsoever.[2] We will, furthermore, show that it is also possible to

[1] Veldman [Veldman 1982] even claimed that to achieve this implication the predicate A occurring in CCC$_2$ need not be decidable.
[2] We have been informed that this has also been observed by Veldman [Veldman 2005].

derive CCC_2 from an altered version of the fan theorem, which itself can be proved under surprisingly weak choice assumptions.

As usual $\{0,1\}^{\mathbb{N}}$ denotes the set of infinite binary sequences α, β, $\ldots$ and $\{0,1\}^*$ stands for the set of finite binary sequences u, v, w, $\ldots$ The letters k, ℓ, m, n, N, M are understood as variables ranging over the set $\mathbb{N}$ of non-negative integers, whereas i, j are reserved for elements of $\{0,1\}$.

If $u \in \{0,1\}^n$ for some n, then $|u| = n$ is the length of u. The n–th finite initial segment $\overline{\alpha}n = (\alpha(0), \ldots, \alpha(n-1))$ and $\overline{u}n = (u(0), \ldots, u(n-1))$ of α and of u with $|u| \geqslant n$, respectively, has length n. In the case $n = 0$ this yields the empty sequence $()$. Note also that $\overline{u}\,|u| = u$, and that u ends with $u(|u|-1)$ whenever $|u| > 0$. The concatenation of u and v will be denoted by $u * v$.

A predicate B on $\{0,1\}^*$ is a bar, if for every α there is n with $B(\overline{\alpha}n)$, while a bar B is uniform if there exists N such that for every α there is $n \leqslant N$ with $B(\overline{\alpha}n)$—or, equivalently, there is N such that for every $u \in \{0,1\}^N$ there is $n \leqslant N$ with $B(\overline{u}n)$.

The *fan theorem for decidable bars* says that every decidable bar is uniform:

FAN$_\Delta$ $\quad \forall\alpha\,\exists n\, B(\overline{\alpha}n) \implies \exists N\,\forall u \in \{0,1\}^N\,\exists n \leqslant N\, B(\overline{u}n)$
for every decidable predicate B *on* $\{0,1\}^*$.

Recall that an infinite path (in $\{0,1\}^*$ viewed as the complete binary tree) is a function $\gamma : \mathbb{N} \to \{0,1\}^*$ such that $\gamma(0) = ()$ and $\gamma(n+1)$ is an immediate successor of $\gamma(n)$; in particular, $|\gamma(n)| = n$. The infinite paths can be identified with the infinite binary sequences in an obvious way. By an *infinite pseudopath* (in $\{0,1\}^*$) we understand a function $\pi : \mathbb{N} \to \{0,1\}^*$ with $|\pi(n)| = n$; in particular, $\pi(0) = ()$. In the sequel the variable π exclusively stands for infinite pseudopaths; we will often write πn in place of $\pi(n)$.

Clearly, the infinite paths are precisely the infinite pseudopaths whose ranges are linearly ordered with respect to the successor relation. In particular, every infinite binary sequence α gives rise to the infinite pseudopath π defined by $\pi n = \overline{\alpha}n$; whence FAN_Δ implies

FAN$^{\mathrm{p}}_\Delta$ $\quad \forall\pi\,\exists n\, B(\pi n) \implies \exists N\,\forall u \in \{0,1\}^N\,\exists n \leqslant N\, B(\overline{u}n)$
for every decidable predicate B *on* $\{0,1\}^*$.

Note that $\mathrm{FAN}^{\mathrm{p}}_\Delta$ differs from FAN_Δ only in $\mathrm{FAN}^{\mathrm{p}}_\Delta$ having a stronger antecedent.

Proposition 1 *FAN^{p}_Δ (and therefore also FAN_Δ) implies CCC_2.*

Proof. Assume $\mathrm{FAN}^{\mathrm{p}}_\Delta$. To prove CCC_2, let A be a decidable predicate on $\mathbb{N} \times \{0,1\}$. We define a decidable predicate B on $\{0,1\}^*$ by

$$B(u) \equiv |u| > 0 \wedge A(|u|-1, u(|u|-1))$$

for every $u \in \{0,1\}^*$.

Suppose that $\forall\alpha\, \exists n\, A\,(n, \alpha\,(n))$. To show that

$$\forall\pi\, \exists n\, B\,(\pi\,(n+1))\,,$$

let π be an infinite pseudopath. Define $\alpha \in \{0,1\}^{\mathbb{N}}$ by

$$\alpha\,(n) = \pi\,(n+1)\,(n)$$

for every n: that is, $\pi\,(n+1)$ ends with $\alpha\,(n)$. By hypothesis there is n with $A\,(n, \alpha\,(n))$: that is, with $B\,(\pi\,(n+1))$ as required.

By $\mathrm{FAN}^{\mathrm{p}}_{\Delta}$ there now is N such that $\forall u \in \{0,1\}^N\, \exists n \leqslant N\, B\,(\overline{u}n)$, for which in particular $N > 0$ and $\forall u \in \{0,1\}^N\, \exists n < N\, A\,(n, u\,(n))$. For this N we claim that $\exists n < N\, \forall i\, A\,(n,i)$. Indeed, if $\forall n < N\, \exists i\, \neg A\,(n,i)$, then there is $u \in \{0,1\}^N$ with $\forall n < N\, \neg A\,(n, u\,(n))$, a contradiction. In all we have $\exists n\, \forall i\, A\,(n,i)$.

In the sequel we will need to invoke the following choice principle:

AC$_\Delta$-NN $\forall n \exists m A(n,m) \implies \exists f : \mathbb{N} \to \mathbb{N}\, \forall n A(n, f(n))$
for every decidable predicate A.

This principle is number-number choice restricted to decidable predicates on $\mathbb{N}\times\mathbb{N}$. In some of the invocations of AC$_\Delta$-NN we tacitly assume a fixed (primitive recursive) bijection between $\{0,1\}^*$ and $\mathbb{N}$. Since countable choice is generally accepted by the practitioners of Bishop-style constructive mathematics and its varieties, so is AC$_\Delta$-NN. It is easy to see that AC$_\Delta$-NN is equivalent to

AC!-NN $\forall n \exists! m A(n,m) \implies \exists f : \mathbb{N} \to \mathbb{N}\, \forall n A(n, f(n))$
for any predicate A whatsoever.

As an instance of unique choice AC$_\Delta$-NN thus also holds in the constructive version of ZF set theory CZF (see for instance [Aczel and Rathjen 2001]), and in certain extensions of Heyting arithmetic [Troelstra and van Dalen 1988].

Proposition 2 *AC$_\Delta$-NN implies FAN$^{\mathrm{p}}_{\Delta}$.*

Proof. Assume that B is a decidable predicate on $\{0,1\}^*$ satisfying the antecedent of $\mathrm{FAN}^{\mathrm{p}}_{\Delta}$. To deduce the consequent of $\mathrm{FAN}^{\mathrm{p}}_{\Delta}$ for B, note first that it is equivalent to $\exists N\, \forall u \in \{0,1\}^N\, B'\,(u)$ where the decidable predicate B' is defined by

$$B'(u) \Leftrightarrow \exists m \leqslant |u|.\, B(\overline{u}m)\,.$$

Clearly, $B(u)$ implies $B'(u)$. We next define the decidable predicate D by

$$\begin{aligned} D(u) \Leftrightarrow & \Big(u = 0^{|u|} \wedge \forall v \in \{0,1\}^{|u|}.\, B'(v)\Big) \\ & \vee \Big(\neg B'(u) \wedge \forall v \in \{0,1\}^{|u|}.\, \neg B'(v) \to u \preccurlyeq v\Big)\,, \end{aligned}$$

where $\preccurlyeq$ is the lexicographic (or any other decidable) order on $\{0,1\}^n$ and 0^n is the finite sequence of n zeroes. It is easy to see that for a given $n \in \mathbb{N}$, there is a unique $u \in \{0,1\}^n$ such that $D(u)$, and if there is any $v \in \{0,1\}^n$ with $\neg B'(v)$, then $\neg B'(u)$ for this u. By AC_Δ-NN, there is a (unique) pseudopath π such that

$$\forall n \in \mathbb{N}.\ D(\pi n)\ .$$

The antecedent of $\mathrm{FAN}^{\mathrm{p}}_\Delta$ applied to this π yields the existence of $N \in \mathbb{N}$ such that $B(\pi N)$ and therefore $B'(\pi N)$ holds. Now if there exists $v \in \{0,1\}^N$ such that $\neg B'(v)$, then, because $D(\pi N)$, also $\neg B'(\pi N)$; a contradiction. Hence, because of the decidability of B', $B'(v)$ holds for all $v \in \{0,1\}^N$.

Corollary 3 *AC_Δ-NN implies CCC_2.*

There are two more versions of the fan theorem that have been investigated in constructive (reverse) mathematics: The fan theorem for Π_1-bars (FAN_{Π_1}) and for c-bars (FAN_c). In this context a predicate B on $\{0,1\}^*$ is called Π_1, if there exists a decidable predicate D on $\{0,1\}^* \times \mathbb{N}$ such that

$$B(u) \Leftrightarrow \forall i \in \mathbb{N}. D(u,i)\ ,$$

and it is called a c-predicate if there exists a decidable predicate D on $\{0,1\}^*$ such that

$$B(u) \Leftrightarrow \forall v \in \{0,1\}^*. D(u * v)\ .$$

Naturally FAN_c and FAN_{Π_1} are just FAN_Δ ranging over c-predicates and Π_1-predicates respectively. It is easy to see that the following implications hold

$$\mathrm{FAN}_\Delta \Leftarrow \mathrm{FAN}_c \Leftarrow \mathrm{FAN}_{\Pi_1}.$$

It is furthermore known that the left implication is actually strict [Berger 2009], and that the right one can be reversed under the assumption of the principle BD-N, which will be stated below.

Next we will consider the following pseudo-fan principle.

$\mathbf{FAN}^{\mathrm{p}}_{c-\Pi_1}$ $\quad \forall \pi \exists n \forall m \geqslant n B(\pi m) \implies \exists N\, \forall u \in \{0,1\}^N\, \exists n \leqslant N.\ B(\overline{u}n)$
for every Π_1-predicate B on $\{0,1\}^*$.

Since $\mathrm{FAN}^{\mathrm{p}}_{c-\Pi_1}$ differs from FAN_{Π_1} only in having a stronger antecedent:

$$\mathrm{FAN}_{\Pi_1} \implies \mathrm{FAN}^{\mathrm{p}}_{c-\Pi_1}.$$

However, neither of the implications

$$\mathrm{FAN}_c \implies \mathrm{FAN}^{\mathrm{p}}_{c-\Pi_1} \implies \mathrm{FAN}^{\mathrm{p}}_\Delta$$

seems provable.

The question we consider next is, whether $\mathrm{FAN}^{\mathrm{p}}_{c-\Pi_1}$ can also be proved, solely with the help of AC_{Δ}-NN. To answer this question, we need to recall Ishihara's boundedness principle BD-N. A countable[3] subset S of $\mathbb{N}$ is pseudobounded if for every sequence (a_m) in S there is M such that $a_m \leqslant n$ whenever $m \geqslant M$. The principle reads as follows:

BD-N *Every countable, pseudobounded subset of* $\mathbb{N}$ *is bounded.*

Apart from being valid with classical logic, BD-N holds both in INT and in RUSS. More information on BD-N can be found in [Richman 2009, Ishihara 1992]. Even though BD-N is a very weak principle it is strongly reminiscent of all the other principles discussed in this note; since they all are about concluding from bounds for every function of a certain type to a uniform bound for all such functions. This might motivate the following result.

Proposition 4 *BD-N together with AC_{Δ}-NN implies $FAN^{\mathrm{p}}_{c-\Pi_1}$.*

Proof. Assume that B is a Π_1-predicate satisfying the antecedent of $\mathrm{FAN}^{\mathrm{p}}_{c-\Pi_1}$. Since B is Π_1, there exists a decidable predicate D on $\{0,1\}^* \times \mathbb{N}$ with

$$B(u) \Leftrightarrow \forall i \in \mathbb{N}.\, D(u,i).$$

We will show below that

$$S = \{0\} \cup \{n \in \mathbb{N} \;:\; \exists u \in \{0,1\}^n \exists i \in \mathbb{N}.\, \neg D(u,i)\}$$

is pseudobounded. Since it is also, as a simply existential and inhabited subset of $\mathbb{N}$, countable, an application of BD-N yields a bound $N \in \mathbb{N}$ of S; that is $n < N$ for all $n \in S$. This N satisfies the consequent of $\mathrm{FAN}^{\mathrm{p}}_{c-\Pi_1}$, since for $u \in \{0,1\}^*$ with $|u| \geqslant N$, the assumption that there exists $i \in \mathbb{N}$ such that $\neg D(u,i)$ holds implies that $|u| \in S$. But this contradicts S being bounded by N and thus, since D is decidable, we have $D(u,i)$ for all $i \in \mathbb{N}$; whence $B(u)$ holds.

It remains to show that S is pseudobounded. So let $(a_n)_{n\geqslant 1}$ be a sequence in S. For each n there exists $u \in \{0,1\}^*$ and $i \in \mathbb{N}$ with $|u| = a_n$ and $\neg D(u,i)$. By AC_{Δ}-NN, there exists a function $p : \mathbb{N} \to \{0,1\}^* \times \mathbb{N}$ such that $|P_1(p(n))| = a_n$ and $\neg D(p(n))$. To get a pseudopath out of p, using AC_{Δ}-NN again, we define $\pi : \mathbb{N} \to \{0,1\}^*$ the following way: For every $k \in \mathbb{N}$ it is decidable if there exists $l \leqslant k$ such that $k = a_l$, or if $k \neq a_l$ for all $l \leqslant k$. In the first case we set $\pi(k) = P_1(p(l))$ for the smallest $l \leqslant k$ with $a_l = k$. In the second case we set $\pi(k) = 0^k$. This way, we ensure that π is a pseudopath. By the antecedent of $\mathrm{FAN}^{\mathrm{p}}_{c-\Pi_1}$ there exists M such that $B(\pi(m))$ for all $m \geqslant M$. Now assume that $a_m > m$ for a $m \geqslant M$, and let $l \leqslant a_m$ be the smallest natural number such that

[3] We call a set S countable if there exists a surjection $\varphi : \mathbb{N} \to S$.

$a_l = a_m$. (Note that m is a candidate for this l.) Then $\pi(a_m) = P_1(p(l))$, which implies, in particular, that $\neg B(\pi(a_m))$ holds. This would be a contradiction and therefore $a_m \leqslant m$ for all $m \geqslant M$. Thus S is pseudobounded and we are done.

It might be worth pointing out that the proof of the above proposition actually shows that BD-N together with AC_Δ-NN implies the stronger

$$\forall \pi \exists n \forall m \geqslant n B(\pi m) \implies \exists N \forall u. \ |u| \geqslant N \Rightarrow B(u).$$

Proposition 4 may be compared with a result in [Diener 2008], where it was shown—with the use of countable choice—that under the assumption of BD-N

$$\mathrm{FAN}_c \implies \mathrm{FAN}_{\Pi_1} \ .$$

Corollary 5 *Since FAN_Δ is provably wrong in RUSS we cannot expect to find a proof of*

$$FAN^{\mathrm{p}}_{c-\Pi_1} \implies FAN_\Delta \ .$$

Overview

To summarise the results of and conclude the paper:

$$\begin{tikzcd}
\mathrm{AC}_\Delta\text{-NN} \arrow[r, "\text{BD-N}"] \arrow[dr] & \mathrm{FAN}^{\mathrm{p}}_{c-\Pi_1} \arrow[dr, "/" marking] & \mathrm{FAN}_{\Pi_1} \arrow[l] \arrow[d] \\
 & \mathrm{FAN}^{\mathrm{p}}_\Delta \arrow[d] & \mathrm{FAN}_\Delta \arrow[l] \\
 & \mathrm{CCC}_2 &
\end{tikzcd}$$

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