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How Incomputable is Finding Nash Equilibria?

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Abstract: We investigate the Weihrauch-degree of several solution concepts from non-cooperative game theory. While the consideration of Nash equilibria forms the core of our work, also pure and correlated equilibria, as well as various concepts of iterated strategy elimination, are dealt with. As a side result, the Weihrauch-degree of solving systems of linear inequalities is settled.

Key Words: Game Theory, Computable Analysis, Nash Equilibrium, Discontinuity, Weihrauch-degree
Category: F.2.0

1 Introduction

The present paper can be understood in two different ways: From one point of view, it contributes to the field of Algorithmic Game Theory by studying the natural algorithmic problems arising from game theory in a framework for real number computation. Alternatively, it classifies the ineffective content of NASH's Theorem asserting the existence of Nash equilibria in a meta-mathematical framework.

While a natural mathematical formulation of game theory uses the real numbers for payoffs and for probabilities constituting mixed strategies, classical models for algorithms require a restriction to countable sets. By imposing certain restrictions and modifications to obtain countable problems, the complexity of computing a Nash equilibrium for a normal form game was proven to be PPAD-complete ([Papadimitriou (1994)], [Chen and Deng (2005)]). In [Gilboa and Zemel (1989)] several decision problems regarding Nash equilibria and correlated equilibria were compared, most of them turned out to be NP-hard for Nash equilibria and to be in P for correlated equilibria. Finding pure equilibria in normal form games where they exist, can be done by a cubic algorithm. However, there are several interesting hardness results for finding pure equilibria in games ([Gottlob et al. (2005)], [Fabrikant et al. (2004)]), originating in other representations or requiring additional properties.

Here we will use another approach: Instead of limiting the problem, we will extend the theory of computation. While the TTE-framework [Weihrauch (2000)] is perfectly capable of formulating the task of computing Nash equilibria from normal form games, we will see that even the most trivial cases are discontinuous, and hence not computable. The relation of relative computability for the

TTE-framework is Weihrauch-reducibility ([Pauly (2009c)]), which goes back to [Weihrauch (1992b)]. So by classifying the Weihrauch-degree of finding Nash equilibria, we identify the smallest extension to the notion of computability sufficient to render finding Nash equilibria effective.

As proposed in [Brattka and Gherardi (2009a)], Weihrauch-reducibility offers an excellent framework for meta-mathematical studies, similar to, but distinct from approaches such as reverse mathematics. In [Pauly (2010)] and [Brattka and Gherardi (2009b)] it was demonstrated that the Weihrauch-degrees form a bounded distributive lattice.

We will show that finding pure equilibria (where they exist) form the simplest Weihrauch-degree of the studied solution concepts. Nash equilibria and correlated equilibria share a strictly harder Weihrauch-degree, while all notions of iterated strategy elimination induce a third degree, strictly harder than the previous one. None of the problems becomes easier by restriction to zero-sum games, in contrast to the classical scenario.

A previous version of the present paper appeared as [Pauly (2009b)], where continuous reductions were used instead of computable ones, thereby yielding weaker results.

2 Preliminaries

2.1 Game Theory

An $n \times m$ bi-matrix game is given by two $n \times m$ real valued matrices A and B. Two players simultaneously pick an index, row player chooses an $i \in \{1, 2, \ldots, n\}$ and column player chooses an $j \in \{1, 2, \ldots, m\}$. Row player gets A_{ij} as a reward, column player gets B_{ij}. We consider several solution concepts defined as equilibria, where no player has an incentive to change her strategy unilaterally. Other types of solution concepts are briefly defined and explored in Subsection 5.2.

Definition 1. A pure equilibrium for a $n \times m$ bi-matrix game (A, B) is a pair $(i, j) \in \{1, \ldots, n\} \times \{1, \ldots, m\}$ satisfying $A_{ij} \geq A_{kj}$ for all $k \in \{1, \ldots, n\}$ and $B_{ij} \geq B_{il}$ for all $l \in \{1, \ldots, m\}$.

As pure equilibria do not exist for all games, a more general notion is introduced. If both players can randomize independently over their actions, one is led to the definition of an m-mixed strategy as an m-dimensional real valued vector s with non-negative coefficients and $\sum_{j=1}^{m} s_j = 1$. The set of m-mixed strategies will be denoted by S^m.

Definition 2. A Nash equilibrium for an $n \times m$ bi-matrix game (A, B) is a pair $(\hat{x}, \hat{y}) \in S^n \times S^m$ satisfying $\hat{x}^T A\hat{y} \geq x^T A\hat{y}$ for all $x \in S^n$ and $\hat{x}^T B\hat{y} \geq \hat{x}^T By$ for all $y \in S^m$.

If $(\hat{x}, \hat{y})$ is a Nash equilibrium, again neither of the players can improve her payoff by changing her mixed strategy unilaterally. A famous result by John Nash ([Nash (1950)]) established that Nash equilibria in bi-matrix games always exist. By identifying a pure strategy with the mixed strategy that puts weight 1 on it, pure equilibria can be considered as a special case of Nash equilibria. An even more general solution concept can be obtained by allowing the individual players' randomization processes to be correlated ([Aumann (1974)]).

Definition 3. A correlated equilibrium for a $n \times m$ bi-matrix game (A, B) is a real valued $n \times m$ matrix C with non-negative entries and $\sum_{i=1}^{n} \sum_{j=1}^{m} C_{ij} = 1$ so that

$$\sum_{j=1}^{m} A_{ij} C_{ij} \geq \sum_{j=1}^{m} A_{lj} C_{ij}$$

holds for all $i, l \in \{1, 2, \ldots, n\}$ and

$$\sum_{i=1}^{n} B_{ij} C_{ij} \geq \sum_{i=1}^{n} B_{ik} C_{ij}$$

holds for all $j, k \in \{1, 2, \ldots, m\}$.

Given a Nash equilibrium (x, y), a correlated equilibrium can be constructed as $C_{ij} = x_i y_j$, while each correlated equilibrium of this form can be obtained from a Nash equilibrium, allowing us to consider Nash equilibria as special cases of correlated equilibria. Thus, finding a correlated equilibrium cannot be harder than finding a Nash equilibrium.

Another way of creating a potentially easier problem consists in a restriction of the games used. A zero-sum game is a bi-matrix game of the form $(A, -A)$.

2.2 Representing Games

In order to consider games as inputs to Type-2-Machines, they have to be coded into infinite sequences. The choice of the countable alphabet used is irrelevant for the theory, to simplify proofs we will use either $\{0, 1\}$ or $\mathbb{N}$, depending on the context.

Definition 4. A *representation* of a set X is a surjective partial function $\delta :\subseteq \mathbb{N}^{\mathbb{N}} \to X$. A set together with a representation of it is called a represented space.

Using representations, we can introduce *realizers*: a realizer will produce a name of a solution given a name of an instance for the associated problem. They also give a straightforward way to consider multi-valued functions, which is necessary, as games can have multiple equilibria. While multi-valuedness will be mentioned explicitly wherever it applies, all (multi-valued) functions are assumed to be partial, as long as not stated otherwise.

Definition 5. Let f be a multi-valued function between two represented spaces (X, δ_X) and (Y, δ_Y). Then we call a function $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ a *realizer* of f $(F \vdash f)$, if $\delta_Y(F(w)) \in f(\delta_X(w))$ holds for all $w \in \operatorname{dom}(f \circ \delta_X)$.

As games in normal form are pairs of real matrices, and (possible) equilibria pairs of real vectors (or again real matrices), one can quickly derive suitable representations by using product and coproduct representations [Weihrauch (2000)], starting from any representation of the real numbers.

Definition 6. Given two represented spaces (X_1, δ_1) and (X_2, δ_2), define the representation $\langle \delta_1, \delta_2 \rangle$ of the set $X_1 \times X_2$ by $\langle \delta_1, \delta_2 \rangle(\langle w_1, w_2 \rangle) = (\delta_1(w_1), \delta_2(w_2))$. Here $\langle w_1, w_2 \rangle$ denotes the usual pairing of sequences.

Definition 7. Given a family $(X_i, \delta_i)_{i \in \mathbb{N}}$ of represented spaces, we define the coproduct representation $\coprod_{i \in \mathbb{N}} \delta_i$ of the disjoint union $\coprod_{i \in \mathbb{N}} X_i$ by $\left[\coprod_{i \in \mathbb{N}} \delta_i\right](nw) = (n, \delta_n(w))$.

The standard representation ρ of the real numbers is chosen for various reasons; it is admissible and provides a convincing class of computable functions, in contrast to some of the alternatives ([Weihrauch (2000)], [Weihrauch (1992a)]). Additionally, as demonstrated in [Pauly (2009d)], the representation ρ is equivalent to the representation naturally arising for the results of repeated physical measurements. For defining ρ, we fix a bijection $\nu : \mathbb{N} \to \mathbb{Q}$ with $\nu(0) = 0$, so that all the usual operations on $\mathbb{Q}$ are computable w.r.t. ν.

Definition 8. Let $\rho(w) = x \in \mathbb{R}$ hold for $w \in \mathbb{N}^{\mathbb{N}}$, if $|\nu(w(i)) - x| \leq 2^{-i}$ holds for all $i \in \mathbb{N}$.

As all the representations used in the present paper are admissible, topological properties carry over from sets of objects to sets of names of said objects. We avoid constructing the needed representations explicitly, up to equivalence it is clear which one is used.

2.3 Weihrauch-degrees

The present form of Weihrauch-reducibility was suggested in [Brattka and Gherardi (2009b)], derived from a concept originally introduced in

[Weihrauch (1992b)]. It is defined based on the sets of realizers of the involved functions, and can be interpreted as relative computability.

Definition 9. Let f and g be multi-valued functions between represented spaces. Define $f \leq_W g$, if there are computable functions F, G satisfying $F \circ \langle \mathrm{id}, r \circ G \rangle \vdash f$ for all $r \vdash g$.

As demonstrated in [Pauly (2010)] (for suprema) and [Brattka and Gherardi (2009b)] (for infima), $\leq_W$ induces a distributive lattice. In the following, we introduce two basic operations on Weihrauch degrees, products and coproducts. The latter agree with suprema in the finite case. While the definitions make use of representatives of the actual Weihrauch-degrees, we assert that the resulting Weihrauch-degrees do not depend on the choice of the representatives[1].

Definition 10. Let $f :\subseteq (X_1, \alpha_X) \to (Y_1, \alpha_Y)$ and $g :\subseteq (X_2, \beta_X) \to (Y_2, \beta_Y)$ be multi-valued functions between represented spaces. Define

$$\langle f, g \rangle :\subseteq (X_1 \times X_2, \langle \alpha_X, \beta_X \rangle) \to (Y_1 \times Y_2, \langle \alpha_Y, \beta_Y \rangle)$$

by $(y_1, y_2) \in \langle f, g \rangle(x_1, x_2)$, if $y_1 \in f(x_1)$ and $y_2 \in g(x_2)$.

Definition 11. Let $(f_i :\subseteq (X_i, \alpha_i) \to (Y_i, \beta_i))_{i \in \mathbb{N}}$ be a countable family of multi-valued functions between represented spaces. Define

$$\coprod_{i \in \mathbb{N}} f_i :\subseteq (\coprod_{i \in \mathbb{N}} X_i, \coprod_{i \in \mathbb{N}} \alpha_i) \to (\coprod_{i \in \mathbb{N}} Y_i, \coprod_{i \in \mathbb{N}} \beta_i)$$

by $(i, y) \in \left(\coprod_{k \in \mathbb{N}} f_k \right) (j, x)$, if $i = j$ and $y \in f_i(x)$.

The definitions of products can be extended to any finite arity. We will use f^n to denote the product of n copies of f. Restriction of the coproduct to finitely many arguments yields $f \coprod g \leq_W \langle f, g \rangle$, provided that the domains of f and g contain computable points [Brattka et al. (2010)]. Of particular importance is the $\bar{\ }$-operator introduced in [Pauly (2010), Subsection 6.1], as we will classify the Weihrauch-degrees we are interested in in terms of the $\bar{\ }$-operator applied to certain simple problems.

Definition 12. For a multi-valued function f, define $\bar{f}$ by $\bar{f} = \coprod_{n \in \mathbb{N}} f^n$.

The $\bar{\ }$-operator is a closure-operator for Weihrauch-reducibility, as shown in [Pauly (2010), Theorem 6.5]:

[1] For products, this is the statement of [Brattka and Gherardi (2009b), Proposition 3.2].

Theorem 13. $\overline{}$ *satisfies the following properties:*

1. $f \leq_W \overline{f}$
2. $f \leq_W g$ *implies* $\overline{f} \leq_W \overline{g}$
3. $\overline{f} \equiv_W \overline{\overline{f}}$

The same technique used to prove the 3. result above also yields the following lemma, which will be relevant later:

Lemma 14. *Let f_n be a family of multi-valued functions, so that there is a computable function $\lambda : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, so that the reduction $\overline{\langle f_n, f_m \rangle} \leq_W f_{\lambda(n,m)}$ holds uniformly for all $n, m \in \mathbb{N}$. Then $\coprod_{n\in\mathbb{N}} f_n \equiv_W \overline{\left(\coprod_{n\in\mathbb{N}} f_n\right)}$.*

The negative results regarding Weihrauch-reducibility appearing in the present paper rely on continuity arguments. Since this makes them stronger than negative results for computability, we introduce continuous Weihrauch-reducibility:

Definition 15. Let f,g be multi-valued functions between represented spaces. Define $f \leq_W^c g$, if there are continuous functions F, G satisfying $F \circ \langle \mathrm{id}, r \circ G\rangle \vdash f$ for all $r \vdash g$.

Moving from computable to continuous functions essentially is relativizing with respect to an arbitrary classical oracle. We note that the results stated in this subsection for Weihrauch-reducibility carry over to continuous Weihrauch-reducibility. Obviously, Weihrauch-reducibility implies continuous Weihrauch-reducibility, while the converse is false.

Finally, note that coproduct representations and coproducts of multi-valued functions are closely linked. Especially, the dimension-independent versions of the problems studied here are always the coproducts of the dimension-dependent variant over all possible dimensions.

3 Single Player Games and Pure Equilibria

From the perspective of game theory, single player games are trivial: The acting player chooses whatever action is best for her. As a discrete computation problem, this amounts to finding a maximum in a list of integers, a task that can be solved in linear time or logarithmic space. As the problem posed over the reals is discontinuous, we will study the problems $1\textsc{Pure}_n$ and $1\textsc{Pure}$ of finding pure equilibria in single player games with n actions, and without fixed game sizes. It shall be noted that single player games can be identified with $n \times 1$ bi-matrix games, justifying their inclusion.

As every $n \times 1$ bi-matrix game has a pure equilibrium, and $C_{i1} > 0$ can only hold in a correlated equilibrium C, if the entry A_{i1} is maximal in A (and thus $(i, 1)$ is a pure equilibrium), finding pure, Nash and correlated equilibria is equivalent for single player games, so the restriction to pure equilibria does not invoke any loss of generality.

The Weihrauch-degrees of $1\textsc{Pure}_n$ for $n \in \mathbb{N}$ turn out to be equivalent to another family of problems, MLPO_n, introduced in [Weihrauch (1992b)] as generalizations of the *lesser limited principle of omniscience* (LPO) studied in constructive mathematics ([Bishop and Bridges (1985)]). For historical reasons, we will denote MLPO_2 by LLPO in some cases.

Definition 16. A function

$$f : \{(p_1, \dots, p_n) \in (\mathbb{N}^\mathbb{N})^n \mid \exists i \leq n \ p_i = 0^\mathbb{N}\} \to \{1, 2, \dots, n\}$$

is a realizer of MLPO_n, if it fulfills $p_{f(p_1,\dots,p_n)} = 0^\mathbb{N}$ for all $(p_1, \dots, p_n) \in \mathrm{dom}(f)$.

Theorem 17. $MLPO_n \equiv_W 1\textsc{Pure}_n$

Proof. First, we present a reduction from MLPO_n to $1\textsc{Pure}_n$. The n input tapes (or the n projections from the product representation) for MLPO_n can individually be translated to the n relevant input tapes for $1\textsc{Pure}_n$. As long as 0 is read, 0 will be printed. If any other number is read in the ith step for the first time, print the number $\nu^{-1}(-2^{-i-1})$ from now on. All tapes containing $0^\mathbb{N}$ will be translated to a ρ-name of 0, and all other tapes to a ρ-name of some negative number, so a pure equilibrium corresponds to a 0-entry.

For the other direction, all input values have to be compared. As long as no contradiction for the assumption that the ith value is the largest one has been found, 0 will be printed on the ith output tape. If contradiction is found, print 1. Then an output tape contains $0^\mathbb{N}$, if and only if the corresponding input tape contains a ρ-name of a maximal entry.

In the next step, we extend the scope of consideration to finding pure equilibria in arbitrary bi-matrix games. The relevant problems are $\textsc{Pure}_{nm}$, where the size of the game is restricted to $n \times m$, and the general case (i.e. the co-product over all $n, m \in \mathbb{N}$) denoted by $\textsc{Pure}$. For obtaining results, reducibility to MLPO_n shall be expressed by a covering property. It involves the notion of co-r.e.-closed sets. For $A \subseteq B \subseteq \mathbb{N}^\mathbb{N}$, we call A co-r.e.-closed in B, if there is a computable function G with $B \subseteq \mathrm{dom}(G)$ and $A = G^{-1}(\{0^\mathbb{N}\}) \cap B$. Other characterizations of co-r.e.-closed sets can be found in [Weihrauch (2000)].

Lemma 18. *Let $f : (X, \alpha) \to (Y, \beta)$ be a total multi-valued function between represented spaces. Then $f \leq_W MLPO_n$ holds, if and only if there is a covering $(A_i)_{i \leq n}$ of $\alpha^{-1}(X)$, so that for each $i \leq n$ the set A_i is co-r.e.-closed in* $\mathrm{dom}(\alpha)$*, and $f_{|\alpha(A_i)}$ has a computable realizer.*

Proof. Assume $f \leq_W \text{MLPO}_n$, so there are computable F, G with $F \circ \langle \text{id}, M \circ G \rangle \vdash f$ for all $M \vdash \text{MLPO}_n$. Abbreviate $G_i := \text{pr}_i \circ G$, where pr_i is the projection to the ith component. Consider $G_i^{-1}(0^{\mathbb{N}}) \cap \text{dom}(\alpha)$. As G_i is computable, this set is co-r.e.-closed in $\text{dom}(\alpha)$. There is a realizer $M_i \vdash \text{MLPO}_n$, so that for $x \in G_i^{-1}(0^{\mathbb{N}}) \cap \text{dom}(\alpha)$, $M_i(G(x)) = i$ holds. Thus, if $F \circ \langle \text{id}, M_i \circ G \rangle \vdash f$ is restricted to $G_i^{-1}(0^{\mathbb{N}}) \cap \text{dom}(\alpha)$, it is equal to $F \circ \langle \text{id}, i \rangle$, and therefore it is computable. As there is an i with $G_i(x) = 0^{\mathbb{N}}$ for each $x \in \text{dom}(\alpha)$, $\text{dom}(G) \supseteq \text{dom}(\alpha) = \bigcup_{i=1}^{n} G_i^{-1}(0^{\mathbb{N}}) \cap \text{dom}(\alpha)$ holds, completing the first part of the proof.

For the other direction, let d_{A_i} witness that A_i is co-r.e.-closed in $\alpha^{-1}(X)$. Now consider the computable function $D : \alpha^{-1}(X) \to (\{0,1\}^{\mathbb{N}})^n$ defined through $D(x)(i) = d_{A_i}(x)$. Further, define a computable function F on the set $\bigcup_{i=1}^{n} A_i \times \{i\}$ through $F(x, i) = R_i(x)$, where R_i is a computable realizer of $f_{|\alpha(A_i)}$. Then $F \circ \langle \text{id}, M \circ D \rangle \vdash f$ for each $M \vdash \text{MLPO}_n$.

Lemma 18 relativizes, that means that it remains true, if Weihrauch-reducibility is replaced by continuous Weihrauch-reducibility, co-r.e.-closed is replaced by closed and computable by continuous.

Theorem 19. $\textsc{Pure}_{nm} \leq_W \mathit{MLPO}_{n*m}$.

Proof. Given an $n \times m$ bi-matrix game (A, B), the condition for the pair (i, j) to be a pure equilibrium is $A_{ij} \geq A_{kj}$ and $B_{ij} \geq B_{il}$ for all $k \leq n$, $l \leq m$. This implies that the set

$$P_{nm}^{ij} = \{(A, B) \mid (i, j) \text{ is an equilibrium of } (A, B)\} \subseteq \mathbb{R}^{n \times m} \times \mathbb{R}^{n \times m}$$

is closed. As the condition can be effectively rejected, if false, the set $(\rho^{n \times m})^{-1}(P_{nm}^{ij})$ is co-r.e.-closed (in $\text{dom}(\rho^{n \times m})$)[2]. As the set of $n \times m$ bi-matrix games which have some pure equilibrium is the union $\bigcup_{i \leq n, j \leq m} P_{nm}^{ij}$, and the restriction of $\textsc{Pure}_{nm}$ to any fixed set P_{nm}^{ij} is trivially computable, an application of Lemma 18 yields the claim.

Corollary 20. $1\textsc{Pure} \equiv_W \textsc{Pure} \equiv_W \coprod_{n \in \mathbb{N}} \mathit{MLPO}_n$

Proof. The reductions given in the proofs of Theorems 17 and 19 are uniform in the dimensions, and so are the trivial reductions involved.

The same reasoning used to establish the equivalence of finding pure strategies in 1 player games and in 2 player games can directly be extended to any finite number of players. While Nash and correlated equilibria have the same Weihrauch-degree as pure equilibria in single player games, we will continue to show that a higher Weihrauch-degree emerges in the two player case.

[2] As $\text{dom}(\rho^{n \times m})$ itself is co-r.e. closed in $\mathbb{N}^{\mathbb{N}}$, the relativized formulation is not necessary here. However, it is sufficient to invoke Lemma 18.

4 Nash and Correlated Equilibria in Bi-matrix Games

We will now consider Nash and correlated equilibria in bi-matrix games. The problems $\textsc{Corr}_{nm}$ and $\textsc{Nash}_{nm}$ are the fixed size versions, $\textsc{Corr}$ and $\textsc{Nash}$ the general problems. Restriction to zero-sum games yields the problems $\mathrm{Z}\textsc{Corr}_{nm}$, $\mathrm{Z}\textsc{Nash}_{nm}$ and the corresponding general problems. Straightforward reasoning yields the reductions:

$$\mathrm{Z}\textsc{Corr}_{nm} \leq_W \textsc{Corr}_{nm} \leq_W \textsc{Nash}_{nm}\ ,\ \mathrm{Z}\textsc{Corr}_{nm} \leq_W \mathrm{Z}\textsc{Nash}_{nm} \leq_W \textsc{Nash}_{nm}$$

4.1 The Weihrauch-degree of Robust Division

Similar to LLPO (or MLPO_n) being representative of the kind of incomputability we face when searching for pure equilibria, we will start with considering division, which will turn out to be typical for correlated and Nash equilibria. Computing $\frac{a}{b}$ given two real numbers a, $b \neq 0$ is possible, of course. However, testing whether $b \neq 0$ is not. A robust variant of division, which accepts division by zero and returns an arbitrary value, is not computable anymore. Alternatively, this problem can be phrased as finding a zero of a linear function that admits at least one.

Definition 21. Given two real numbers x, y with $0 \leq x \leq y$, $\textsc{rDiv}$ returns $\frac{x}{y}$, if that number exists, and any real number z with $0 \leq z \leq 1$ otherwise.

While robust division is only slightly incomputable, as it is reducible to deciding whether a real number is 0 or not, the following theorem shows that robust division introduces a new kind of incomputability not present in finding pure equilibria. Before this, we present a technical detail used in the proof which also is of independent interest.

Proposition 22. *The restriction $\textsc{rDiv}'$ of $\textsc{rDiv}$ to those (x, y) additionally satisfying $x, y \leq 1$ is equivalent to $\textsc{rDiv}$.*

Proof. Given a real number y, we can compute some natural upper bound $k > 0$, and subsequently also x/k and y/k. If x, y are in the domain of $\textsc{rDiv}$, then $x/k, y/k$ are in the domain of $\textsc{rDiv}'$, and we have $\textsc{rDiv}(x, y) = \textsc{rDiv}'(x/k, y/k)$.

Theorem 23. $\textsc{rDiv} \not\leq_W^c \textsc{Pure}$.

Proof. The proof proceeds by reductio ad absurdum. To this end, we can assume $\textsc{rDiv}' \leq_W^c 1\textsc{Pure}$, due to Corollary 20 and Proposition 22. This implies the existence of continuous functions F, G, L so that for each $E \vdash 1\textsc{Pure}$ the function d_E defined through $d_E(u, v) = F(u, v, E(\langle L(u, v), G(u, v)\rangle))$ is a realizer of $\textsc{rDiv}'$. Here L chooses the size of the game, G gives the game and F uses a maximal value of the game to derive the result.

We consider $n = L(0^{\mathbb{N}}, 0^{\mathbb{N}})$. As L is continuous, the set $L^{-1}(\{n\})$ is open and closed in $\mathrm{dom}(L)$, so it contains an open neighbourhood of $(0^{\mathbb{N}}, 0^{\mathbb{N}})$. Especially there is a $k \in \mathbb{N}$ with $\mathrm{dom}(L) \cap (0^k\mathbb{N}^{\mathbb{N}} \times 0^k\mathbb{N}^{\mathbb{N}}) \subseteq L^{-1}(\{n\})$. We note $\frac{\rho(u)}{\rho(v)} = \frac{\rho(0^k\overline{u})}{\rho(0^k\overline{v})}$ where $\nu(\overline{u}(i)) = \nu(u(i)) * 2^{-k-1}$ and $\nu(\overline{v}(i)) = \nu(v(i)) * 2^{-k-1}$. Thus we obtain $\textsc{rDiv}' \leq_W^c 1\textsc{Pure}_n$.

According to the topological version of Lemma 18, $\textsc{rDiv}' \leq_W^c 1\textsc{Pure}_n$ implies the existence of n closed[3] sets A_i so that for each i there is an $f_i \vdash \textsc{rDiv}'$ so that f_i restricted to A_i is continuous. If there is an l with $(0^{\mathbb{N}}, 0^{\mathbb{N}}) \notin A_l$, then there is a $k \in \mathbb{N}$ with $(0^k\mathbb{N}^{\mathbb{N}} \times 0^k\mathbb{N}^{\mathbb{N}}) \cap A_l = \emptyset$, so with a repetition of the argument used above we can conclude $\textsc{rDiv}' \leq_W^c 1\textsc{Pure}_{n-1}$. Thus we can assume $(0^{\mathbb{N}}, 0^{\mathbb{N}}) \in A_l$ for all $l \leq n$.

For $l \leq n+1$ we define a sequence $(w_k^l)_{k\in\mathbb{N}}$ of sequences through $w_k^l(i) = 0$ for $i \leq k$ and $\nu(w_k^l(i)) = (l2^k)^{-1}$ for $i > k$. Furthermore, define the sequence $(v_k)_{k\in\mathbb{N}}$ of sequences through $v_k(i) = 0$ for $i \leq k$ and $\nu(v_k(i)) = 2^{-k}$ for $i > k$. For each sequence (w_k^l, v_k) there must be an l' so that $A_{l'}$ contains an infinite subsequence $(\overline{w}_k^l, \overline{v}_k)$ of (w_k^l, v_k). As there are $n+1$ sequences and n sets, the pigeonhole principle ensures that there is a set A_i containing the sequences $(\overline{w}_k^{l_1}, \overline{v}_k)$ and $(\overline{w}_k^{l_2}, \overline{v}_k)$.

Now observe $\lim\limits_{k\to\infty}(\overline{w}_k^{l_1}, \overline{v}_k) = \lim\limits_{k\to\infty}(\overline{w}_k^{l_2}, \overline{v}_k) = (0^{\mathbb{N}}, 0^{\mathbb{N}})$, but $\rho(f_i(\overline{w}_k^{l_1}, \overline{v}_k)) = l_1^{-1} \neq l_2^{-1} = \rho(f_i(\overline{w}_k^{l_2}, \overline{v}_k))$. Thus, the restriction of f_i to A_i is not continuous in $(0^{\mathbb{N}}, 0^{\mathbb{N}})$, yielding a contradiction to the assumption.

We will now use modifications of the game *matching pennies* as a gadget to implement divisions in a game. For real numbers a, b, the game $MP(a, b)$ is specified via the following payoff matrices:

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} , \quad B = -A , \quad MP(a, b) = (A, B)$$

Theorem 24. $\textsc{rDiv} \leq_W \mathrm{Z}\textsc{Corr}_{22}$

Proof. Given a pair of ρ-names for real numbers a, b with $0 \leq a \leq b$, a name for the game $MP(a, b-a)$ can be computed. Let C be correlated equilibrium of $MP(a, b-a)$. We claim $c_{12} + c_{22} = \frac{a}{b}$ for $b > 0$. The 4 inequalities from the definition of a correlated equilibrium are:

$$\begin{array}{ll} (1)\ a(c_{11} + c_{12}) \geq bc_{12} & (2)\ bc_{22} \geq a(c_{21} + c_{22}) \\ (3)\ a(c_{12} + c_{22}) \geq bc_{22} & (4)\ bc_{21} \geq a(c_{11} + c_{21}) \end{array}$$

$0 = a < b$ If $a = 0$, then (1) and (3) together with $b > 0$ state $c_{12} = c_{22} = 0 = \frac{a}{b}$.

[3] To be precise, also the fact that $(\rho^2)^{-1}(\mathrm{dom}(\textsc{rDiv}))$ is closed is relevant to obtain that the sets A_i are closed in $\mathbb{N}^{\mathbb{N}}$.

$0 < a < b$

Combination of the inequalities (1) and (4) yields $a(c_{21} - c_{12}) \leq b(c_{21} - c_{12})$, together with $a < b$ this implies $c_{21} - c_{12} \geq 0$. Combination of (2) and (3) yields $a(c_{12} - c_{21}) \geq 0$. Hence, since $a > 0$, the correlated equilibrium is symmetric.

Then addition of (1) and (3) yields $a \geq b(c_{12} + c_{22})$, and addition of (2) and (4) yields $b(c_{12} + c_{22}) \geq a$, hence the claim follows.

$0 < a = b$ [4]

Here, (2) and $c_{21} \geq 0$ imply $c_{21} = 0$, in the same way (4) and $c_{11} \geq 0$ imply $c_{11} = 0$. Thus, we have:

$$c_{12} + c_{22} = c_{11} + c_{12} + c_{21} + c_{22} = 1 = \frac{a}{b}$$

In the case $a = b = 0$, $0 \leq c_{12} + c_{22} \leq 1$ holds due to the normalization requirement for correlated equilibria, hence $c_{12} + c_{22}$ is a valid answer to $\textsc{rDiv}(0, 0)$.

Theorem 24 in conjunction with Theorem 23 implies $\textrm{Z}\textsc{Corr}_{22} \not\leq^c_W \textsc{Pure}$, so even the simplest case of finding mixed strategies is not reducible to finding pure strategies. The problem $\textsc{rDiv}$ itself does not capture the discontinuity of finding Nash equilibria[5], compelling us to move to suitable products, and eventually $\overline{\textsc{rDiv}}$.

4.2 Products of Games

The product of functions can be considered as computing all of them in parallel. This allows us to specify exactly the Weihrauch-degree of problems solvable by multiple robust divisions. For games, our notion of a product will be inspired by the model of playing two independent games at once. This will allow us to establish a link between products of functions and products of games. We will use $[\,]$ to denote a bijection between $\{1, 2, \dots, n\} \times \{1, 2, \dots, m\}$ and $\{1, 2, \dots, nm\}$ for suitable n, m.

Definition 25. Given an $n_1 \times m_1$ bi-matrix game (A^1, B^1) and an $n_2 \times m_2$ bi-matrix game (A^2, B^2), we define the $(n_1 n_2) \times (m_1 m_2)$ product game $(A^1, B^1) \times (A^2, B^2)$ as (A, B) with $A_{[i_1,i_2][j_1,j_2]} = A^1_{i_1 j_1} + A^2_{i_2 j_2}$ and $B_{[i_1,i_2][j_1,j_2]} = B^1_{i_1 j_1} + B^2_{i_2 j_2}$.

[4] The author is grateful to a referee for pointing out the necessity and details of this case.

[5] This claim could be shown using the Level introduced in [Hertling (1996)].

The product of games nicely commutes with the notions from game theory used in this paper, as will be established by the following theorems. A slight exception holds for the zero-sum property: A zero-sum game can always be expressed as the product of two constant-sum games which are not zero-sum. However, as a constant-sum game can always be normalized to an equivalent zero-sum game[6], this is not problematic for our purposes.

For simplifying notation, in the following theorems and their proofs, (A, B) always abbreviates $(A^1, B^1) \times (A^2, B^2)$.

Theorem 26. *(A, B) is constant-sum, if and only if both (A^1, B^1) and (A^2, B^2) are constant-sum.*

Proof. Assume that (A^1, B^1) and (A^2, B^2) are constant-sum, that is $A^k_{ij} + B^k_{ij} = c^k$ for $k \in \{1, 2\}$ and all i, j. Then we have

$$A_{[i_1,i_2][j_1,j_2]} + B_{[i_1,i_2][j_1,j_2]} = A^1_{i_1j_1} + A^2_{i_2j_2} + B^1_{i_1j_1} + B^2_{i_2j_2} = c^1 + c^2$$

for all i_1, i_2, j_1, j_2, so (A, B) is also a constant-sum game.

For the other direction, we assume w.l.o.g. that (A^1, B^1) is not constant-sum, so there are i_1, j_1, k_1, l_1 with $A^1_{i_1,j_1} + B^1_{i_1,j_1} \neq A^1_{k_1,l_1} + B^1_{k_1,l_1}$. Then we have:

$$\begin{aligned} A_{[i_1,1],[j_1,1]} + B_{[i_1,1],[j_1,1]} &= A^1_{i_1,j_1} + B^1_{i_1,j_1} + A^2_{1,1} + B^2_{1,1} \\ \neq A^1_{k_1,l_1} + B^1_{k_1,l_1} + A^2_{1,1} + B^2_{1,1} &= A_{[k_1,1],[l_1,1]} + B_{[k_1,1],[l_1,1]} \end{aligned}$$

Thus, the product (A, B) is not constant-sum.

Theorem 27. *(i_k, j_k) is a pure equilibrium of (A^k, B^k) for both $k \in \{0, 1\}$, if and only if $([i_1, i_2], [j_1, j_2])$ is a pure equilibrium of (A, B).*

Proof. The proof is done by contraposition. Assume w.l.o.g. that $\hat{i}_1$ is a better response to j_1 than i_1, that is $A^1_{\hat{i}_1,j_1} > A^1_{i_1,j_1}$. Then we also have $A_{[\hat{i}_1,i_2],[j_1,j_2]} > A_{[i_1,i_2],[j_1,j_2]}$, so if (i_1, j_1) is not a pure equilibrium, then $([i_1, i_2], [j_1, j_2])$ cannot be one either.

If, on the other hand, $[\hat{i}_1, \hat{i}_2]$ is a better response against $[j_1, j_2]$ than $[i_1, i_2]$, then we have $A^1_{\hat{i}_1,j_1} + A^2_{\hat{i}_2,j_2} > A^1_{i_1,j_1} + A^2_{i_2,j_2}$. Obviously, this contradicts the conjunction of $A^1_{i_1,j_1} \geq A^1_{\hat{i}_1,j_1}$ and $A^2_{i_2,j_2} \geq A^2_{\hat{i}_2,j_2}$.

For our next result, observe that $[\ ,\ ] : S^n \times S^m \to S^{n*m}$ defined by $[x^1, x^2]_{[i,j]} = x^1_i x^2_j$ is a computable bijection, the (also computable) inverse is obtained via $x^1_i = \sum_{j=1}^m [x^1, x^2]_{[i,j]}$ and $x^2_j = \sum_{i=1}^n [x^1, x^2]_{[i,j]}$.

6 Given a game (A, B), one can compute the game (A', B') with $A'_{ij} = A_{ij} - A_{11}$ and $B'_{ij} = B_{ij} - B_{11}$. Due to the linearity of all our solution concepts, (A, B) and (A', B') have exactly the same pure, Nash and correlated equilibria. Furthermore, if (A, B) was constant-sum, then (A', B') is zero-sum.

Theorem 28. *$([x^1,x^2],[y^1,y^2])$ is a Nash equilibrium of (A,B), if and only if (x^k,y^k) is a Nash equilibrium of (A^k,B^k) for both $k\in\{0,1\}$.*

Proof. Central to the proof is the following equality:

$$\begin{aligned}
&\sum_{i=1}^{(n_1n_2)}\sum_{j=1}^{(m_1m_2)}[x^1,x^2]_iA_{i,j}[y^1,y^2]_j\\
&=\sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2}\sum_{j_1=1}^{m_1}\sum_{j_2=1}^{m_2}[x^1,x^2]_{[i_1,i_2]}A_{[i_1,i_2],[j_1,j_2]}[y^1,y^2]_{[j_1,j_2]}\\
&=\sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2}\sum_{j_1=1}^{m_1}\sum_{j_2=1}^{m_2}x^1_{i_1}x^2_{i_2}(A^1_{i_1,j_1}+A^2_{i_2,j_2})y^1_{j_1}y^2_{j_2}\\
&=\left[\sum_{i_1=1}^{n_1}\sum_{j_1=1}^{m_1}x^1_{i_1}A^1_{i_1,j_1}y^1_{j_1}\left(\sum_{i_2=1}^{n_2}x^2_{i_2}\right)\left(\sum_{j_2=1}^{m_2}y^2_{j_2}\right)\right]\\
&\quad+\left[\sum_{i_2=1}^{n_2}\sum_{j_2=1}^{m_2}x^2_{i_2}A^2_{i_2,j_2}y^2_{j_2}\left(\sum_{i_1=1}^{n_1}x^1_{i_1}\right)\left(\sum_{j_1=1}^{m_1}y^1_{j_1}\right)\right]\\
&=\left[\sum_{i_1=1}^{n_1}\sum_{j_1=1}^{m_1}x^1_{i_1}A^1_{i_1,j_1}y^1_{j_1}\right]+\left[\sum_{i_2=1}^{n_2}\sum_{j_2=1}^{m_2}x^2_{i_2}A^2_{i_2,j_2}y^2_{j_2}\right]
\end{aligned}$$

Using it, the structure of this proof is identical to that of Theorem 27. Assume that $\hat{x}^1$ is a better response against y^1 than x^1, that is:

$$\sum_{i=1}^{n_1}\sum_{j=1}^{m_1}\hat{x}^1_iA^1_{i,j}y^1_j>\sum_{i=1}^{n_1}\sum_{j=1}^{m_1}x^1_iA^1_{i,j}y^1_j$$

Given any strategy profile (x^2,y^2) in the second game, we can add $\sum_{i=1}^{n_2}\sum_{j=1}^{m_2}x^2_iA^2_{i,j}y^2_j$, and apply the transformation above on both sides to obtain:

$$\sum_{i=1}^{n_1n_2}\sum_{j=1}^{m_1m_2}[\hat{x}^1,x^2]_iA_{i,j}[y^1,y^2]_j>\sum_{i=1}^{n_1n_2}\sum_{j=1}^{m_1m_2}[x^1,x^2]_iA_{i,j}[y^1,y^2]_j$$

Thus, $[\hat{x}^1,x^2]$ is a better response against $[y^1,y^2]$ than $[x^1,x^2]$. By repeating the argument with the roles of the players exchanged, and then with the order of the games exchanged, it is demonstrated that if either (x^1,y^1) or (x^2,y^2) is not a Nash equilibrium of the respective game, then $([x^1,x^2],[y^1,y^2])$ cannot be a Nash equilibrium of the product game.

If, on the other hand, x^k is a best response against y^k for both $k\in\{0,1\}$, i.e. provides greater-or-equal payoff than all alternatives $\hat{x}^k$, the respective inequalities can be added to yield:

$$\begin{gathered}
\left[\sum_{i_1=1}^{n_1}\sum_{j_1=1}^{m_1}x^1_{i_1}A^1_{i_1,j_1}y^1_{j_1}\right]+\left[\sum_{i_2=1}^{n_2}\sum_{j_2=1}^{m_2}x^2_{i_2}A^2_{i_2,j_2}y^2_{j_2}\right]\\
\geq\\
\left[\sum_{i_1=1}^{n_1}\sum_{j_1=1}^{m_1}\hat{x}^1_{i_1}A^1_{i_1,j_1}y^1_{j_1}\right]+\left[\sum_{i_2=1}^{n_2}\sum_{j_2=1}^{m_2}\hat{x}^2_{i_2}A^2_{i_2,j_2}y^2_{j_2}\right]
\end{gathered}$$

Using again the equivalence transformation outline in the beginning of this proof, this inequality can be brought into the form:

$$\sum_{i=1}^{n_1 n_2} \sum_{j=1}^{m_1 m_2} [x^1, x^2]_i A_{i,j} [y^1, y^2]_j \geq \sum_{i=1}^{n_1 n_2} \sum_{j=1}^{m_1 m_2} [\hat{x}^1, \hat{x}^2]_i A_{i,j} [y^1, y^2]_j$$

The latter inequality is just the condition for $[x^1, x^2]$ to be a best response against $[y^1, y^2]$. By exchanging the roles of the player, the remaining implication is shown.

In order to derive an analogous result for correlated equilibria, again a suitable homeomorphism is needed. Let $\mathfrak{C}^{n,m}$ denote the set of real $n \times m$-matrices C with non-negative entries and $\sum_{i=1}^{n} \sum_{j=1}^{m} C_{ij} = 1$. Then a computable homeomorphism $[\ ,\ ] : \mathfrak{C}^{n_1,m_1} \times \mathfrak{C}^{n_2,m_2} \to \mathfrak{C}^{n_1 n_2, m_1 m_2}$ can be defined via $[C^1, C^2]_{[i_1,i_2],[j_1,j_2]} := C^1_{i_1,j_1} C^2_{i_2,j_2}$. The existence and computability of the inverse follows from $C^1_{i_1,j_1} = \sum_{i_2=1}^{n_2} \sum_{j_2=1}^{m_2} [C^1, C^2]_{[i_1,i_2],[j_1,j_2]}$ and $C^2_{i_2,j_2} = \sum_{i_1=1}^{n_1} \sum_{j_1=1}^{m_1} [C^1, C^2]_{[i_1,i_2],[j_1,j_2]}$.

Theorem 29. *If $[C^1, C^2]$ is a correlated equilibrium of (A, B), then C^k is a correlated equilibrium of (A^k, B^k) for both $k \in \{0, 1\}$.*

Proof. Without limitation of generality, we assume that the first condition in Definition 3 is violated in the first game, and show that this implies a violation in the composed game. Hence, we start with:

$$\sum_{j_1=1}^{m_1} A^1_{i_1,j_1} C^1_{i_1,j_1} < \sum_{j_1=1}^{m_1} A^1_{l_1,j_1} C^1_{i_1,j_1}$$

for some i_1, l_1. Due to $\sum_{i_2=1}^{n_2} \sum_{j_2=1}^{m_2} C^2_{i_2,j_2} = 1$, this is equivalent to:

$$\sum_{j_1=1}^{m_1} \sum_{i_2=1}^{n_2} \sum_{j_2=1}^{m_2} A^1_{i_1,j_1} C^1_{i_1,j_1} C^2_{i_2,j_2} < \sum_{j_1=1}^{m_1} \sum_{i_2=1}^{n_2} \sum_{j_2=1}^{m_2} A^1_{l_1,j_1} C^1_{i_1,j_1} C^2_{i_2,j_2}$$

Using the definitions of A and $[C^1, C^2]$, and adding identical terms to both sides for any i_2, we arrive at:

$$\begin{aligned} &\textstyle\sum_{j_1=1}^{m_1} \sum_{i_2=1}^{n_2} \sum_{j_2=1}^{m_2} A_{[i_1,i_2],[j_1,j_2]} [C^1, C^2]_{[i_1,i_2],[j_1,j_2]} \\ < &\textstyle\sum_{j_1=1}^{m_1} \sum_{i_2=1}^{n_2} \sum_{j_2=1}^{m_2} A_{[l_1,i_2],[j_1,j_2]} [C^1, C^2]_{[i_1,i_2],[j_1,j_2]} \end{aligned}$$

As the inequality holds for the sum over all i_2, there has to be some value $\hat{i}_2$, so that the following holds:

$$\sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} A_{[i_1,\hat{i}_2],[j_1,j_2]} [C^1, C^2]_{[i_1,\hat{i}_2],[j_1,j_2]} < \sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} A_{[l_1,\hat{i}_2],[j_1,j_2]} [C^1, C^2]_{[i_1,\hat{i}_2],[j_1,j_2]}$$

This is the desired contradiction.

As the product game can be computed from the constituent games, we can use the properties of the products of games to obtain the following results regarding the problem of finding equilibria:

Theorem 30. *Let* $\textsc{Game} \in \{\textsc{Pure}, \textsc{ZCorr}, \textsc{ZNash}, \textsc{Corr}, \textsc{Nash}\}$. *Then* $\langle \textsc{Game}_{nm}, \textsc{Game}_{kl} \rangle \leq_W \textsc{Game}_{(nk),(ml)}$.

Corollary 31. *Let* $\textsc{Game} \in \{\textsc{Pure}, \textsc{ZCorr}, \textsc{ZNash}, \textsc{Corr}, \textsc{Nash}\}$. *Then* $\textsc{Game} \equiv_W \overline{\textsc{Game}}$.

Proof. This is a consequence of applying Lemma 14 to Theorem 30.

The present paper contains two results interpretable as counterparts to Theorem 30, as they allow to reduce finding equilibria for a large game to finding equilibria in several smaller games; for mixed strategies, this will be a consequence of the main result presented in Subsection 4.3, the corresponding statement for pure strategies is given in the next theorem:

Theorem 32. $MLPO_{n+1} \leq_W LLPO^n$ *for* $n > 1$.

Proof. Let $\preceq$ denote the quasi-ordering on $\mathbb{N}^{\mathbb{N}}$ defined by $p \preceq q$, if 0^M is a prefix of q whenever it is a prefix of p for all $M \in \mathbb{N}$. On its domain, MLPO_{n+1} is the problem of picking an index of a maximal element from $n+1$ elements with respect to $\preceq$. Computing a maximum $\max_{\preceq}\{p_1, \dots, p_{n+1}\}$ is computable. The order $\preceq$ is total, and the task of picking a true statement from $p \preceq q$ and $q \preceq p$ is reducible to LLPO: As long as 0s are read from both input tapes, write 0s on both output tapes. If the first 1 is encountered on an input tape, disregard the rest of the input and write 1s on the corresponding output tape, and 0s on the other.

Now the preliminaries are in place to describe the reduction from MLPO_{n+1} to LLPO^n. Let $p_1, \dots, p_{n+1}$ denote the input of MLPO_{n+1}. For each k with $1 \leq k \leq n$, a copy of LLPO is used to pick a true statement out of $\max_{\preceq}\{p_1, \dots, p_k\} \preceq p_{k+1}$ and $p_{k+1} \preceq \max_{\preceq}\{p_1, \dots, p_k\}$.

Let K be the largest number, so that $\max_{\preceq}\{p_1, \dots, p_K\} \preceq p_{K+1}$ was determined to be true, and 0 if none of these statements was chosen by LLPO^n. We claim that p_{K+1} is a maximal element, hence $K+1$ a correct solution to MLPO_{n+1}. Assume there was a l with $p_{K+1} \preceq p_l$, but $p_l \not\preceq p_{K+1}$. The assumption $l \leq K$ contradicts $\max_{\preceq}\{p_1, \dots, p_K\} \preceq p_{K+1}$.

So $K + 1 < l$ has to hold. With out limitation of generality, let l be the smallest counterexample satisfying the constraints. $p_l \preceq \max_{\preceq}\{p_1, \dots, p_{l-1}\}$ has to be true, otherwise l would have been picked in place of K. Due to minimality of l, this implies $p_l \preceq \max_{\preceq}\{p_1, \dots, p_{K+1}\}$. Together with $\max_{\preceq}\{p_1, \dots, p_K\} \preceq p_{K+1}$, we arrive at $p_l \preceq p_{K+1}$, a contradiction. Due to exhaustion of alternatives, p_{K+1} is maximal.

Corollary 33. $\textsc{Pure} \equiv_W \overline{\mathit{LLPO}}$.

Proof. Theorem 17 has LLPO $\leq_W$ $\textsc{Pure}$ as a special case. $\overline{\text{LLPO}} \leq_W \overline{\textsc{Pure}}$ follows by Theorem 13 (2). Considering Corollary 31 yields $\overline{\text{LLPO}} \leq_W \textsc{Pure}$.

For the other direction, start with $\textsc{Pure}_{nm} \leq_W \text{MLPO}_{n*m}$ due to Theorem 19. Theorem 32 then implies $\textsc{Pure}_{nm} \leq_W \text{LLPO}^{n*m}$. As the reductions are uniform, application of the coproduct on both sides yields $\textsc{Pure} \leq_W \overline{\text{LLPO}}$.

As we have identified LLPO (or MLPO_2 or $1\textsc{Pure}_2$) as the basic building stone in the Weihrauch-degree of finding pure strategies, the following theorem will establish the missing link in the relationship between finding pure strategies and multiple robust divisions:

Theorem 34. *LLPO* $<_W$ $\textsc{rDiv}$.

Proof. $\textsc{rDiv} \nleq_W$ LLPO has already been proven. To see LLPO $\leq_W$ $\textsc{rDiv}$, note that there is a computable function turning arbitrary sequences of natural numbers into ρ-names, so that a sequence is mapped to a ρ-name of 0 if and only if it is $0^{\mathbb{N}}$. Thus we can assume that the input of LLPO is given as two ρ-names a, b of real numbers, with at least one of them being 0. Consider $\textsc{rDiv}(|a|, |a|+|b|)$. If this is not a ρ-name of 1, then a must be a ρ-name of 0. If the output is not a ρ-name of 0, then b must be a ρ-name of 0.

To sum up the results established sofar, we have:

$$\overline{\text{LLPO}} \equiv_W 1\textsc{Pure} \equiv_W \textsc{Pure} <_W \overline{\textsc{rDiv}} \leq_W \textsc{ZCorr}$$

4.3 Problems Reducible to Robust Divisions

The goal of this subsection is to present a way of designing reductions to $\overline{\textsc{rDiv}}$, and, in particular, to present a reduction from $\textsc{Nash}$. This equivalently can be considered as the task to design algorithms for a Type-2-Machine capable of making a finite number of independent queries to an oracle for $\textsc{rDiv}$. Due to Theorems 32, 34 also oracle calls to MLPO_n are permitted.

We will start by providing a technical lemma similar to Lemma 18. Using the lemma, we can prove that the Fourier-Motzkin-algorithm ([Keler (1996)]) for solving systems of linear inequalities can be executed using computable operations and oracle calls to $\textsc{rDiv}$.

Lemma 35. *Let $f : (X, \alpha) \to (Y, \beta)$ be a total multi-valued function between represented spaces. Then $f \leq_W \overline{\textsc{rDiv}}$ holds, if and only if there is a covering $(A_i)_{i \leq n}$ of $\alpha^{-1}(X)$ with some $n \in \mathbb{N}$, such that for each $i \leq n$ the set A_i is co-r.e.-closed in $\alpha^{-1}(X)$, and $f_{|\alpha(A_i)}$ is Weihrauch-reducible to $\overline{\textsc{rDiv}}$.*

Proof. One direction of the implication is trivial. For the other direction, assume that there are computable functions R_i, Q_i for each $i \leq n$ witnessing $f_{|\alpha(A_i)} \leq_W \overline{\text{rDiv}}$. Let $\hat{Q}_i$ be a computable extension of Q_i to $\text{dom}(\alpha)$; such an extension exists for co-r.e. closed sets. Further let D be the computable function defined for the sets A_i as in the proof to Lemma 18. Define the function $Q = (D, \hat{Q}_1, \ldots, \hat{Q}_n)$ and $R(x, i, y_1, \ldots, y_n) = R_i(x, y_i)$. Both Q and R are computable, and satisfy $R \circ \langle \text{id}_X, \hat{q} \circ Q\rangle \vdash f$ for $\hat{q} \vdash \langle \text{MLPO}_n, \overline{\text{rDiv}}^n\rangle$, so we have $f \leq_W \langle \text{MLPO}_n, \overline{\text{rDiv}}^n\rangle \equiv_W \overline{\text{rDiv}}$.

Definition 36. The problem BLinIneq_{nm} asks for a vector $v \in \mathbb{R}^m$, so that $Av \leq b$ holds in addition to $0 \leq v \leq 1$ (each inequality is to be considered componentwise), given a matrix $A \in \mathbb{R}^{n \times m}$ and a vector $b \in \mathbb{R}^n$, provided that a solution exists. For simplicity, we assume that $Av \leq b$ always contains $0 \leq v \leq 1$. The coproduct over all values of n, m is denoted by BLinIneq.

Theorem 37. $\text{BLinIneq} \leq_W \overline{\text{rDiv}}$.

Proof. [7] As BLinIneq is a coproduct, it suffices to prove $\text{BLinIneq}_{nm} \leq_W \overline{\text{rDiv}}$ uniformly for all $n, m \in \mathbb{N}$. We begin by describing our version of the Fourier Motzkin algorithm, while pointing out all the non-computable parts. Then it is demonstrated how using MLPO_n and rDiv is sufficient to solve the non-computable aspects of the algorithm.

We rewrite the given inequalities as $a_{k1}v_1 \leq b_k - \sum_{i=2}^{m} a_{ki}v_i$ for $a_{k1} \geq 0$ and $-b_j + \sum_{i=2}^{m} a_{ji}v_i \leq -a_{j1}v_1$ for $a_{j1} \leq 0$, using **sign tests**[8]. From this we can obtain a system of linear inequalities expressing consistency of partial solutions:

For each pair k, j with $a_{k1} \geq 0$ and $a_{j1} \leq 0$, the corresponding inequalities can be multiplied by $-a_{j1}$ respective a_{k1}, and then contracted to:

$$a_{k1}(-b_j + \sum_{i=2}^{m} a_{ji}v_i) \leq -a_{j1}(b_k - \sum_{i=2}^{m} a_{ki}v_i)$$

In the second step we treat the variable v_2 in the same way, that is we rewrite the new inequalities as

$$(a_{k1}a_{j2} + a_{j1}a_{k2})v_2 \leq (a_{k1}b_j - a_{j1}b_k) - \sum_{i=3}^{m}(a_{k1}a_{ji} - a_{j1}a_{ki})v_i$$

for $(a_{k1}a_{j2} + a_{j1}a_{k2}) \geq 0$ and

$$-(a_{k1}b_j - a_{j1}b_k) + \sum_{i=3}^{m}(a_{k1}a_{ji} - a_{j1}a_{ki})v_i \leq (a_{k1}a_{j2} + a_{j1}a_{k2})v_2$$

[7] Recently Vasco Brattka suggested to the author that results from [Brattka et al. (2010)] might be employed to simplify this proof.

[8] Mirroring the definition of LLPO, the sign of 0 shall be non-deterministically either 0 or 1.

for $(a_{k1}a_{j2} + a_{j1}a_{k2}) \leq 0$. Here we encounter additional **sign-tests**. As before, a system of linear inequalities for the consistency of partial assignments to v_i for $i > 2$ is obtained.

This process is iterated (which includes more **sign-tests**), until v_m is reached. Hence, we have inequalities $\hat{a}_k v_m \leq \hat{b}_k$ and $\hat{c}_l \leq \hat{d}_l v_m$ with $\hat{a}_k, \hat{d}_l \geq 0$. The values $\hat{a}_k$, $\hat{b}_k$, $\hat{c}_l$, $\hat{d}_l$ are all obtained as constant-free polynomial expressions with degree m from the values a_{ij} and b_j.

In order to pick a suitable value for v_m, we assume we knew the **order** of the values $\hat{a}_k$ and $\hat{d}_l$. We want to assume $0 \leq \hat{a}_1 \leq \hat{d}_1 \leq \hat{a}_2 \leq \hat{d}_2 \leq \ldots$. The internal order in each sequence can be obtained by renumbering the elements. In order to assure that $\hat{a}_k$'s and $\hat{d}_l$'s alternate, additional inequalities of the form $\hat{a}_k v_m \leq \hat{a}_k$ and $0 \leq \hat{d}_l v_m$ may be inserted. Additionally, we assume $0 \leq \hat{b}_k \leq \hat{a}_k$ and $0 \leq \hat{c}_l \leq \hat{d}_l$. As we require $0 \leq v_m \leq 1$, this is unproblematic and can be ensured by replacing $\hat{b}_k$ by $\max(0, \min(\hat{a}_k, \hat{b}_k))$ and $\hat{c}_l$ by $\max(\min(\hat{c}_l, \hat{d}_l), 0)$.

Now we iteratively pick values v_m^τ satisfying the first 2τ inequalities each, starting with $v_m^0 = 1$ and continuing with:

$$v_m^\tau = \max\left(\textsc{rDiv}(\hat{c}_\tau, \hat{d}_\tau), \min\left(\textsc{rDiv}(\hat{b}_\tau, \hat{a}_\tau), v_m^{\tau-1}\right)\right)$$

As long as $\hat{a}_\tau > 0$, $\hat{d}_\tau > 0$ holds, it can be easily verified that this assignment is valid, provided that all the inequalities indeed have a common solution. Due to the order imposed, $\hat{a}_\tau = 0$ implies $\hat{a}_\sigma = \hat{d}_\sigma = 0$ for all $\sigma < \tau$, hence any value would be a valid assignment to the first $2\tau - 2$ inequalities; so the arbitrary value returned by $\textsc{rDiv}(\hat{b}_\tau, \hat{a}_\tau)$ is unproblematic. The same argument works for $\textsc{rDiv}(\hat{c}_\tau, \hat{d}_\tau)$.

In the standard Fourier Motzkin algorithm, the value obtained for v_m would now be inserted in the inequalities containing only v_{m-1} and v_m, allowing to use the same technique to determine a value for v_{m-1}, by iterating, this solves the initial system of linear inequalities. This way is not available for our purpose, though, because the input in any call of rDiv must never depend on the result of another call of rDiv.

Instead, sort all the inequalities containing only the variables v_{m-1} and v_m in order of increasing absolute value of the larger coefficient of a variable (**ordering**, again). Consider the inequality $av_{m-1} \leq bv_m + c$ with $a, b \geq 0$. If $a \geq b$, it can be turned into a constraint for v_{m-1}:

$$v_{m-1} \leq \textsc{rDiv}(b, a)v_m + \mathrm{sg}(c)\textsc{rDiv}(\min(a, |c|), a)$$

Here sg denotes the sign-operation. The cutoff for c is valid due to the restriction $0 \leq v_{m-1} \leq 1$. The variable v_m does not appear inside rDiv, so all the oracle calls can be processed. As before, the constraints for v_{m-1} are built up to the final expression using appropriately nested max and min operators. The other assignment of signs for a and b are dealt with accordingly (**sign-tests** are necessary).

However, in the case $b \geq a$ (**sign-test** for $b-a$), we cannot obtain a constraint for v_{m-1}, as division of b by a is no longer permitted. We, however, introduce the inequality as another constraint to v_m in form of:

$$\textsc{rDiv}(a, b)v_{m-1} - \mathrm{sg}(c)\textsc{rDiv}(\min(b, |c|), b) \leq v_m$$

The problematic aspect entailed by this is that now the expressions for v_{m-1} and v_m depend on each other. To the extent that the values of the variables are determined by the system of inequalities this is unproblematic, but the non-determinism introduced by division by 0 has to be contained. For this, each occurrence of v_m in the constraints for v_{m-1} is replaced by a pointer to those constraints for v_m with smaller (**ordering**) denominators, and vice versa.

In both cases it is sufficient to insert the max-min-expression for the other variable obtained from the constraints with smaller denominators, rather than the final expression (which would lead to a circular definition anyway). If the denominator in the inequality under consideration is 0, then the result of the transformation is meaningless, and due to the nested max and min operators, can be ignored anyway. If the denominator is positive, then all the influence of arbitrary results of division by 0 on the other variable are taken into consideration through this constraint. The non-arbitrary influence of the other inequalities containing both variables is already included in the contracted set of inequalities including only one variable.

When extending the procedure outlined above to three and more variables (which includes more **sign-tests** and more **ordering**), two aspects have to be considered. First, if a new constraint is obtained for a variable which has a smaller denominator than a constraint from a previous computation where this variable occurs, the corresponding pointer will include the new constraint as well. Second, it is no longer safe to assume that $|c/a| \leq 1$ for $a > 0$, where a is the largest coefficient of a variable, and c is the additive term in the inequality. If there are $k+1$ variables in the inequality, it is safe to assume $|c/a| \leq k$, though, as a larger value could not be balanced by the remaining terms. Thus, the call $\textsc{rDiv}(\min(a, |c|), a)$ is replaced by $k * \textsc{rDiv}(\min(a, |c/k|), a)$.

Once this procedure is completed, all the calls to $\textsc{rDiv}$ can be evaluated, and the results are then combined to yield the final values for the variables; which by construction fulfill the initial set of inequalities, if these admit a solution.

It remains only to discuss the occurring problems:

Sign-test In all occurrences of this problem we need to know the sign of a value that is obtained as a constant-free polynomial with degree bounded by m applied to the original input values. For each input size there are only finitely many of these polynomial expressions. Provided that all other problems are resolved, for any vector of signs for these expressions, the restriction of $\mathrm{BL{\scriptstyle IN}I{\scriptstyle NEQ}}_{nm}$ to those inputs inducing these signs (which is a closed set) is reducible to $\overline{\mathrm{rDiv}}$. The corresponding name sets form a co-r.e.-closed cover as in Lemma 35, which in turn implies the reducibility to $\overline{\mathrm{rDiv}}$.

Informally, the main algorithm is executed with all potential results for the signs, which are obtained in parallel by an oracle call to LLPO^k. In the end, the signs are used to determine the run of the algorithm operating with correct data.

Ordering The situation here is similar to the one for sign-tests: All involved values are obtained as constant-free polynomials with degrees bounded by m applied to the original input values, hence there is only a finite number of potential orderings. Each fixed ordering induced a co-r.e.-closed subset of the domain of $\mathrm{BLinIneq}_{nm}$, which all together satisfy the requirements of Lemma 35.

Robust Division Obviously oracle calls to rDiv are the solution here.

As the problem BLinIneq is of considerable interest on its own, we shall note that the converse statement to Theorem 37 is also true:

Theorem 38. $\overline{\mathrm{rDiv}} \leq_W \mathrm{BLinIneq}$.

Proof. We have to show $\mathrm{rDiv}^n \leq_W \mathrm{BLinIneq}$ uniformly for $n \in \mathbb{N}$. Given n pairs of reals (p_i, q_i), consider the system of linear equalities given by $A_{ii} = q_i$, $A_{ij} = 0$ for $i \neq j$ and $b_i = p_i$. The only solutions are given by $v_i = \frac{p_i}{q_i}$ for $q_i > 0$, and arbitrary values for those v_j with $q_j = 0$. By replacing every equality with two inequalities, the needed reduction is found.

By adapting [von Stengel (2007), Algorithm 3.4] and applying Lemma 35 and Theorem 37 we proceed to prove the main theorem of this subsection.

Theorem 39. $\mathrm{Nash} \leq_W \overline{\mathrm{rDiv}}$.

Proof. It suffices to show $\mathrm{Nash}_{nm} \leq_W \overline{\mathrm{rDiv}}$ uniformly in $n, m \in \mathbb{N}$.

By the best response condition [von Stengel (2007), Proposition 3.1], a pair of mixed strategies (x, y) is a Nash equilibrium of a game if each pure strategy played with positive probability in x (in y) is a best response against y (against x). This condition can be formalized by noting that the following set is the set of games and their Nash equilibria with support in I, J:

$$\hat{G}_{I,J} = \begin{array}{r} \{(A, B, x, y) \in \mathbb{R}^{n\times m} \times \mathbb{R}^{n\times m} \times S^n \times S^m \mid \forall j, k \in J \ \forall l \notin J \\ (x^T B)_j = (x^T B)_k \geq (x^T B)_l \ \wedge \ y_l = 0 \\ \forall i, p \in I \ \forall q \notin I \ (Ay)_i = (Ay)_p \geq (Ay)_q \ \wedge \ x_q = 0\} \end{array}$$

The set of names of elements in $\hat{G}_{I,J}$ is co-r.e.-closed, and so is the set of names of the projection

$$G_{I,J} = \{(A, B) \mid \exists x, y \in S^n \times S^m \ (A, B, x, y) \in \hat{G}_{I,J}\}$$

since $S^n \times S^m$ is computably compact. Now consider Nash_{nm} restricted to $G_{I,J}$. Given $(A, B) \in G_{I,J}$, solving a system of linear inequalities is sufficient to obtain (x, y) with $(A, B, x, y) \in \hat{G}_{I,J}$; hence this restriction is reducible to BLinIneq, and by Theorem 37 reducible to $\overline{\text{rDiv}}$.

This establishes that the conditions of Lemma 35 are fulfilled. Its application yields $\text{Nash}_{nm} \leq_W \overline{\text{rDiv}}$.

Corollary 40. ZCorr $\equiv_W$ Corr $\equiv_W$ ZNash $\equiv_W$ Nash $\equiv_W$ $\overline{\text{rDiv}}$.

The same technique applied in the proof of Theorem 37 could also be used to show that Gaussian Elimination can be reduced to $\overline{\text{rDiv}}$. This would show that the reduction of Gaussian Elimination to the rank of a matrix given in [Ziegler and Brattka (2004)] is strict, taking into consideration the results of Subsection 5.1.

5 Additional Results

5.1 Placing Robust Division in the Weihrauch-hierarchy

We proceed to place rDiv (and subsequently $\overline{\text{rDiv}}$) in the fragment of the Weihrauch-hierarchy outlined in [Brattka and Gherardi (2009a), Figure 1]. This will allow us to derive some interesting properties of Nash as consequences of general results. One problem relevant for the comparison is B_I, which has an increasing sequence $(l_n)_{n\in\mathbb{N}}$ and a decreasing sequence $(r_m)_{m\in\mathbb{N}}$ of rational numbers with $\lim_{n\to\infty} l_n \leq \lim_{m\to\infty} r_m$ as input, and real numbers x with $\lim_{n\to\infty} l_n \leq x \leq \lim_{m\to\infty} r_m$ as solutions. The other one is B_I^-, the restriction of B_I to the sequences with $\lim_{n\to\infty} l_n < \lim_{m\to\infty} r_m$.

Theorem 41. rDiv $\leq_W B_I$.

Proof. The reduction from rDiv to B_I is straightforward: While the machine searches for some $n \in \mathbb{N}$ with $\frac{1}{n} \leq y$ on input x, y, it prints $(0, 1)$. If such a lower bound is found, $\frac{x}{y}$ can be computed, that is we can compute an increasing sequence $(l_i)_{i\in\mathbb{N}} \in \mathbb{Q}$ and an decreasing sequence $(r_i)_{i\in\mathbb{N}} \in \mathbb{Q}$ with $\lim\limits_{i\to\infty} l_i = \frac{x}{y} = \lim\limits_{i\to\infty} r_i$. In this case, said sequences serve as input to B_I.

Theorem 42. $B_I^- \not\leq_W^c \overline{\text{rDiv}}$.

Proof. The separation follows from considerations of the Level of the involved problems. Following [Hertling (1996)], we introduce the following sets for a function F on Baire space:

$$\mathcal{L}_0(F) = \mathrm{dom}(F)$$

$$\mathcal{L}_{\alpha+1}(F) = \{x \in \mathcal{L}_\alpha(F) \mid F_{|\mathcal{L}_\alpha(F)} \text{ is discontinuous in } x\}$$

$$\mathcal{L}_\gamma(F) = \bigcap_{\alpha<\gamma} \mathcal{L}_\alpha(F) \text{ for limit ordinals } \gamma$$

The Level of a multi-valued function f between represented spaces is the smallest ordinal α, so that there is a realizer $F \vdash f$ with $\mathcal{L}_\alpha(F) = \emptyset$. As shown in [Hertling (1996), Korollar 2.4.3], the Level is non-increasing under Weihrauch-reducibility (this result was extended to multi-valued functions in [Pauly (2010)]), including the case of non-existent Level.

The Level of $\overline{\text{rDiv}}$ is ω_0, the smallest infinite ordinal, as can be verified by straightforward calculation. The Level of B_I^- does not exist, as every realizer of B_I^- is discontinuous in every point of its domain: Assume that the realizer returns a ρ-name of x given the input $(l_i)_{i\in\mathbb{N}}$, $(r_i)_{i\in\mathbb{N}}$. Either $\lim_{i\to\infty} l_i < x$ or $x < \lim_{i\to\infty} r_i$ holds, w.l.o.g. assume the former. Then there is a rational q with $\lim_{i\to\infty} l_i < q < x$. For any $K \in \mathbb{N}$, define the sequence r^K by $r_k^K = r_k$ for $k \leq K$, and $r_k^K = q$ otherwise. Each neighbourhood of (l, r) contains some (l, r^K), at which no value y with $d(x, y) < d(x, q)$ is a valid solution.

Corollary 43. *Every computable bi-matrix game has a computable Nash equilibrium.*

Proof. It is easy to verify that the existence of a computable solution for each computable instance, i.e. non-uniform computability of the problem, is downwards-preserved by Weihrauch-reducibility. The non-uniform computability of B_I was observed in [Brattka and Gherardi (2009a)]; the claim then follows via Theorem 41.

Corollary 44. *If the task of finding Nash equilibria is restricted to games with a unique equilibrium, it becomes computable.*

Proof. As shown in [Brattka and Gherardi (2009a)], B_I is weakly computable (in the sense of [Brattka and Gherardi (2009b)]), so by Theorem 41, rDiv is weakly computable. A straightforward proof shows that $\overline{\text{rDiv}}$, Nash and the restriction of Nash to games with unique equilibria inherit the weak computability. The latter is a single-valued function into a computable metric space, so due to [Brattka and Gherardi (2009b), Corollary 8.8], weak computability already implies computability here.

There is a result closely related to the special case of the last corollary for 2×2-games in Constructive Mathematics. In [Bridges (2004)] it is shown that if a 2×2-bimatrix game has at most one Nash equilibrium, then it constructively has a Nash equilibrium.

5.2 Elimination of Dominated Strategies

A conceptually simpler (partial) solution concept for games is given by (iterated) elimination of dominated strategies. Following [Kalai and Zemel (1988)], three concepts of dominance[9] in bi-matrix games are distinguished.

Definition 45.

- A strategy i *weakly dominates* a strategy j, if for all k $A_{ik} \geq A_{jk}$.
- A strategy i *dominates* a strategy j, if for all k $A_{ik} \geq A_{jk}$ and there is a k_0 satisfying $A_{ik_0} > A_{jk_0}$.
- A strategy i *strictly dominates* a strategy j, if for all k $A_{ik} > A_{jk}$.

Analogous definitions are assumed for column-player's strategies. As removing a strategy can induce new dominations, it makes sense to define *iterated elimination*, where dominated strategies are removed until no dominations remain. For more detailed definitions and the computational complexity of the associated problems in the classical model, we refer to [Pauly (2009a)].

The basic building block we will use to classify the Weihrauch-degrees occurring here is the problem LPO [Weihrauch (1992b)] defined as follows:

Definition 46. LPO is the problem of deciding whether a real number is 0 or not.

Considering games where the second player has only one action available already allows to state some basic results about the occurring kinds of incomputability. In this case, dominance and strict dominance coincide, and one round of elimination is already sufficient to obtain a completely reduced subgame. Iterated elimination of weakly dominated strategies simply requires to determine an index corresponding to a maximal entry, so it turns out to be the same problem as finding Nash or pure equilibria.

Elimination of (strictly) dominated strategies, however, means that instead of returning *an* index of a maximal value, we have to return *all* indices corresponding to maximal values. This problem is equivalent to LPO^n, where n is the

[9] Unfortunately, there does not seem to be a consensus in the literature how to name the different concepts of dominance. The concept of *dominance* is sometimes referred to as *weak dominance*, while our *weak dominance* sometimes is called *very weak dominance*.

size of the game, or, if any size is allowed, to $\overline{\mathrm{LPO}}$. Thus, in the special case of just one action available for the second player, elimination of weakly dominated strategies is strictly less incomputable as elimination of (strictly) dominated strategies.

Given access to a $\overline{\mathrm{LPO}}$-oracle, one can ask whether any pair of payoff values is equal, or which one is bigger. With this information, it is clearly possible to compute the remaining game after repeated elimination of (strictly) dominated strategies. Therefore, considering any bi-matrix game does not increase the Weihrauch-degree for dominance or strict dominance.

Extending iterated elimination of weakly dominated strategies to non-trivial bi-matrix games, however, increases the incomputability, as can be demonstrated by the following gadget:

Example 1. Given a real number x, let E_x be the following game:

$$A = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \quad B = -A$$

Clearly, for $x \neq 0$, E_x is a case of the game matching pennies, and already completely reduced under elimination of weakly dominated strategies. For $x = 0$, however, the reduced form of E_0 is $A = (0)$, $B = (0)$.

With this example, we have a reduction from LPO to iterated elimination of weakly dominated strategies. By using the product structure of games, one can prove that also iterated elimination of weakly dominated strategies is equivalent to $\overline{\mathrm{LPO}}$.

As we have demonstrated that the iterated elimination of (weakly/strictly) dominated strategies is even less computable than finding Nash equilibria, none of the three variants of dominance is suitable for e.g. pre-processing from this point of view.

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References

[Aumann (1974)] Robert Aumann. Subjectivity and correlation in randomized strategies. *Journal of Mathematical Economics*, 1:67–96, 1974.

[Bishop and Bridges (1985)] E. Bishop and D.S. Bridges. *Constructive Analysis*. Springer Verlag, Berlin, Heidelberg, 1985.

[Brattka and Gherardi (2009a)] Vasco Brattka and Guido Gherardi. Effective choice and boundedness principles in computable analysis. *Bulletin of Symbolic Logic* (to appear), arXiv:0905.4685, 2009.

[Brattka and Gherardi (2009b)] Vasco Brattka and Guido Gherardi. Weihrauch degrees, omniscience principles and weak computability. *Journal of Symbolic Logic* (to appear), arXiv:0905.4679, 2009.

[Brattka et al. (2010)] Vasco Brattka, Matthew de Brecht and Arno Pauly. Closed choice and a uniform low basis theorem. arXiv:1002.2800, 2010.

[Bridges (2004)] Douglas Bridges. First steps in constructive game theory. *Mathematical Logic Quaterly*, 50:501–506, 2004.

[Chen and Deng (2005)] Xi Chen and Xiaotie Deng. Settling the complexity of 2-player Nash-equilibrium. Technical Report 134, Electronic Colloquium on Computational Complexity, 2005.

[Fabrikant et al. (2004)] Alex Fabrikant, Christos Papadimitriou, and Kunal Talwar. The complexity of pure Nash equilibria. In *STOC '04: Proceedings of the 36th ACM symposium on Theory of computing*, pages 604–612, 2004. ACM.

[Gilboa and Zemel (1989)] Itzhak Gilboa and Eitan Zemel. Nash and correlated equilibria: Some complexity considerations. *Games and Economic Behavior*, 1989.

[Gottlob et al. (2005)] Georg Gottlob, Gianluigi Greco, and Francesco Scarcello. Pure Nash equilibria: Hard and easy games. *Journal of Artificial Intelligence Research*, 24:357–406, 2005.

[Hertling (1996)] Peter Hertling. Unstetigkeitsgrade von Funktionen in der effektiven Analysis. Dissertation, Fernuniversität Hagen, 1996.

[Kalai and Zemel (1988)] Ehud Kalai and E. Zemel. On the order of eliminating dominated strategies. Discussion Papers 789, Northwestern University, Center for Mathematical Studies in Economics and Management Science, 1988.

[Keler (1996)] Christoph W. Keler. Parallel Fourier–Motzkin Elimination. In *In Proc. 2nd Int. Euro-Par Conference*, pages 66–71. Springer, 1996.

[Nash (1950)] John F. Nash. Equilibrium Points in N-Person Games. *Proceedings of the National Academy of Sciences*, 36(1):48–49, 1950.

[Papadimitriou (1994)] Christos H. Papadimitriou. On the complexity of the parity argument and other inefficient proofs of existence. *Journal of Computer and Systems Science*, 48(3):498–532, 1994.

[Pauly (2009a)] Arno Pauly. The complexity of iterated strategy elimination. arXiv:0910.5107, October 2009.

[Pauly (2009b)] Arno Pauly. How discontinuous is computing Nash equilibria? 6th Conf. on Computability and Complexity in Analysis. Schloss Dagstuhl, 2009.

[Pauly (2009c)] Arno Pauly. Infinite oracle queries in type-2 machines (extended abstract). arXiv:0907.3230v1, July 2009.

[Pauly (2009d)] Arno Pauly. Representing Measurement Results. *Journal of Universal Computer Science*, 15(6):1280–1300, 2009.

[Pauly (2010)] Arno Pauly. On the (semi)lattices induced by continuous reducibilities. *Mathematical Logic Quarterly*, 56(5): 488–502, 2010.

[von Stengel (2007)] Bernhard von Stengel. Equilibrium computation for two-player games in strategic and extensive form. In Noam Nisan, Tim Roughgarden, Éva Tardos, and Vijay Vazirani, editors, *Algorithmic Game Theory*, pages 53–78. Cambridge University Press, 2007.

[Weihrauch (1992a)] Klaus Weihrauch. The degrees of discontinuity of some translators between representations of the real numbers. Informatik Berichte 129, FernUniversität Hagen, Hagen, July 1992.

[Weihrauch (1992b)] Klaus Weihrauch. The TTE-interpretation of three hierarchies of omniscience principles. Informatik Berichte 130, FernUniversität Hagen, Hagen, September 1992.

[Weihrauch (2000)] Klaus Weihrauch. *Computable Analysis*. Springer-Verlag, 2000.

[Ziegler and Brattka (2004)] Martin Ziegler and Vasco Brattka. Computability in Linear Algebra. *Theoretical Computer Science*, 326:187–211, 2004.