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Canonical Effective Subalgebras of Classical Algebras as Constructive Metric Completions

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Abstract: We prove general theorems about unique existence of effective subalgebras of classical algebras. The theorems are consequences of standard facts about completions of metric spaces within the framework of constructive mathematics, suitably interpreted in realizability models. We work with general realizability models rather than with a particular model of computation. Consequently, all the results are applicable in various established schools of computability, such as type 1 and type 2 effectivity, domain representations, equilogical spaces, and others.

Key Words: computable and effective algebra, constructive metric spaces, realizability
Category: G.0

1 Introduction

Given an algebra, by which we mean a set with constants and operations, is there a largest subalgebra which carries a computable structure, and is the structure unique up to computable isomorphism? Without further assumptions the answer is in general negative. For example, within the context of Recursive Mathematics every computable subfield of reals may be properly extended to a subfield which is again computable, and this remains true even if we require the subfields to be effectively complete. However, as was proved by Moschovakis [Moschovakis 1965], by requiring also that the strict linear order be semidecidable, we are left with only one choice, namely the recursive reals. An analogous result for type 2 effectivity (TTE), was established by Hertling [Hertling 1999].

We show how these results, as well as others, can be seen as standard facts about completions of metric spaces in the context of constructive mathematics, suitably interpreted in realizability models. We prove two main theorems which together give conditions under which an algebra $\mathcal{A}$, equipped with a complete metric d, has a unique effective subalgebra $\mathcal{B}$ that is effectively complete and for which the relation $d(x, y) < q$ is semidecidable in $x, y \in \mathcal{B}$ and $q \in \mathbb{Q}$.

Rather than choosing a specific model of computation, we work in a general realizability model. Thus our results apply to established schools of computable mathematics, such as type 1 and type 2 effectivity, domain representations, equilogical spaces, and others.

The outline of the paper is as follows. Sections 2–4 introduce the necessary background, namely realizability models, algebras, and premetric spaces. Section 5 states the main theorems, Section 6 extends them to algebras with partial operations, and Section 7 considers extensions to multi-sorted algebras. Finally, in Section 8 we apply the results to specific examples. We conclude with a brief discussion of possible further directions of research.

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2 Assemblies and Realizability

Among the different kinds of realizability the most suitable one for our purposes is *relative realizability*, because it subsumes type 1 and type 2 effectivity, domain representations, equilogical spaces, and other standard models of computation, see [Bauer 2000]. We review the basic definitions here and refer the readers to [van Oosten 2008] for background material on realizability.

A *partial combinatory algebra (PCA)* is a set A with a partial application operation $x \cdot y$, where we write $x\ y$ instead of $x \cdot y$, and associate application to the left, $x\ y\ z = (x\ y)\ z$. Furthermore, there must exist $\mathsf{k}, \mathsf{s} \in A$ satisfying $\mathsf{k}\ x\ y = x$, $\mathsf{s}\ x\ y{\downarrow}$ and $\mathsf{s}\ x\ y\ z \simeq (x\ z)\ (y\ z)$, where the expression $t{\downarrow}$ means "t is defined" and $a \simeq b$ is Kleene's equality which means that if one side is defined then so is the other and they are equal. A PCA is a general model of computation which supports encoding of pairs, natural numbers, recursion, partial recursive functions, etc. An *elementary sub-PCA* is a subset $B \subseteq A$ which is closed under application and contains k and s suitable for A. Thus B is a PCA on its own with the application operation inherited from A.

For the rest of the discussion we fix a PCA $\mathbb{A}$ and an elementary sub-PCA $\mathbb{A}_{\mathrm{eff}} \subseteq \mathbb{A}$. The elements of $\mathbb{A}$ are thought of as "arbitrary" and those of $\mathbb{A}_{\mathrm{eff}}$ as "effective" data or programs, although the exact meaning of these words depends on the particular choice of $\mathbb{A}$ and $\mathbb{A}_{\mathrm{eff}}$.

An *assembly* $\mathbf{S} = (S, \Vdash_S)$ is a set S together with a *realizability relation* $\Vdash_S \subseteq \mathbb{A} \times S$, such that for every $x \in S$ there is at least one $\mathtt{a} \in \mathbb{A}$ for which $\mathtt{a} \Vdash_S x$. We read "$\mathtt{a} \Vdash_S x$" as "$\mathtt{a}$ realizes, or represents, the element $x \in S$". Note that a realizer $\mathtt{a}$ may represent several elements of $x \in S$.

A *realized map* $f : \mathbf{S} \to \mathbf{T}$ between assemblies is a map $f : S \to T$ between the underlying sets which is *tracked* by some $\mathtt{f} \in \mathbb{A}_{\mathrm{eff}}$, which means that whenever

$\mathtt{a} \Vdash_S x$ then $\mathtt{f}\ \mathtt{a}\!\downarrow$ and $\mathtt{f}\ \mathtt{a} \Vdash_T f(x)$. Note that we require maps to be realized by the elements of the *sub*algebra $\mathbb{A}_{\mathrm{eff}}$. Assemblies and realized maps form a category $\mathsf{Asm}(\mathbb{A}, \mathbb{A}_{\mathrm{eff}})$ which we abbreviate as Asm. An assembly $\mathbf{S}$ is *modest*, or a *modest set*, if each realizer realizes at most one element: for all $\mathtt{a} \in A$ and $x, y \in S$, if $\mathtt{a} \Vdash_S x$ and $\mathtt{a} \Vdash_S y$ then $x = y$.

An assembly $\mathbf{S}$ is equivalent to a multi-valued representation $\delta_S : \mathbb{A} \to \mathcal{P}(S)$ via the correspondence $\mathtt{a} \Vdash_S x \iff x \in \delta_S(\mathtt{a})$. A modest set is equivalent to a single-valued representation. Traditional schools of computable mathematics typically use (single-valued) representations, see [Bauer 2000] and [van Oosten 2008, Sect 1.4]:

- When $\mathbb{A} = \mathbb{A}_{\mathrm{eff}} = \mathbb{N}$ is the first Kleene algebra whose application is defined by $m \cdot n = \{m\}(n)$, i.e., the m-th partial recursive function applied to n. In this case the modest sets are equivalent to type 1 representations, or *numbered sets*, which are used in the study of recursive mathematics. In this model "effective" means "computable by (type 1) Turing machine".

- Computability with respect to an oracle O is obtained if we take $\mathbb{A} = \mathbb{A}_{\mathrm{eff}} = \mathbb{N}$ with the application $m \cdot n = \{m\}^O(n)$, where $\{m\}^O(n)$ is the result of applying the m-th partial recursive function with access to oracle O to argument n.

- When $\mathbb{A} = \mathbb{N}^{\mathbb{N}}$ is the second Kleene algebra and $\mathbb{A}_{\mathrm{eff}}$ the subalgebra of total computable functions we get *type 2 representations* [Weihrauch 2000]. In this case "effective" means "computable by type 2 Turing machine".

- The case $\mathbb{A} = \mathbb{A}_{\mathrm{eff}} = \mathbb{N}^{\mathbb{N}}$ is the continuous version of type 2 effectivity in which "effective" means "continuously realized".

- When $\mathbb{A}$ is a universal Scott domain and $\mathbb{A}_{\mathrm{eff}}$ its computable analogue, the modest assemblies are equivalent to domain representations and computable maps between them. Of course, "effective" is now interpreted in the sense of domain representations.

- With Scott's graph model $\mathbb{A} = \mathcal{P}\omega$ and its r.e. counterpart $\mathbb{A}_{\mathrm{eff}} = \mathsf{RE}$ we obtain effective equilogical spaces.

Single-valued representations seem to be preferred to general assemblies, perhaps because from a programmer's perspective it makes little sense to use a single realizer for representing several things. We use assemblies because they contain the category of sets, as explained in Section 3.3. This allows us to consider classical and effective algebras in a single framework. Realizability toposes could be used instead, but assemblies are easier to describe and work with.

2.1 The realizability interpretation of first-order logic

Assemblies support an interpretation of first-order intuitionistic logic in which a formula is deemed valid when there is an element $\mathtt{r} \in \mathbb{A}_{\text{eff}}$ witnessing it. The interpretation is given in terms of a *realizability relation* $\mathtt{r} \Vdash \phi$ which is read as "$\mathtt{r}$ realizes ϕ", and is defined inductively on the structure of the sentence ϕ:

- always $\mathtt{r} \Vdash \top$, and never $\mathtt{r} \Vdash \bot$,
- $\langle \mathtt{p}, \mathtt{q} \rangle \Vdash \phi \wedge \psi$ iff $\mathtt{p} \Vdash \phi$ and $\mathtt{q} \Vdash \psi$,[1]
- $\langle \overline{0}, \mathtt{r} \rangle \Vdash \phi \vee \psi$ iff $\mathtt{r} \Vdash \phi$, and $\langle \overline{1}, \mathtt{r} \rangle \Vdash \phi \vee \psi$ iff $\mathtt{r} \Vdash \psi$,[2]
- $\mathtt{r} \Vdash \phi \Rightarrow \psi$ iff for all $\mathtt{q} \in \mathbb{A}$, if $\mathtt{q} \Vdash \phi$ then $\mathtt{r}\,\mathtt{q}\!\downarrow$ and $\mathtt{r}\,\mathtt{q} \Vdash \psi$,
- $\mathtt{r} \Vdash \forall\, x \in \mathbf{S} \,.\, \phi(x)$ iff for all $\mathtt{a} \in \mathbb{A}$, $a \in S$, if $\mathtt{a} \Vdash_S a$ then $\mathtt{r}\,\mathtt{a}\!\downarrow$ and $\mathtt{r}\,\mathtt{a} \Vdash \phi(a)$,
- $\langle \mathtt{a}, \mathtt{r} \rangle \Vdash \exists\, x \in \mathbf{S} \,.\, \phi(x)$ iff for some $a \in S$, $\mathtt{a} \Vdash_S a$ and $\mathtt{r} \Vdash \phi(a)$,
- $\mathtt{r} \Vdash a = b$ iff $a = b$.

A sentence ϕ is *valid* when there exists $\mathtt{r} \in \mathbb{A}_{\text{eff}}$ such that $\mathtt{r} \Vdash \phi$. Note that $\mathtt{r}$ must be an element of the *sub*algebra $\mathbb{A}_{\text{eff}}$. A formula with free variables is valid when its universal closure is valid. Intuitionistic logic is sound with respect to the realizability relation: if intuitionistic logic proves ϕ then ϕ is valid. Realizability also validates additional principles such as Markov's principle and the constructively acceptable dependent choice.

Realizers for formulas reveal their computational content. For example, assuming we have an assembly $\mathbf{C}$ representing the complex numbers, a realizer $\mathtt{r}$ for the formula $\forall\, z \in \mathbf{C} \,.\, \exists\, w \in \mathbf{C} \,.\, z = w^2$ represents a function which accepts a realizer $\mathtt{z}$ of a complex number z and computes a pair $\mathtt{r}\,\mathtt{z} = \langle \mathtt{w}, \mathtt{p} \rangle$ such that $\mathtt{w}$ realizes a number w and $\mathtt{p}$ realizes the equation $z^2 = w$. In other words, $\mathtt{r}$ computes square roots. Note that it does not have to respect equality of complex numbers, i.e., if we give it a different realizer for the same number, it may compute a different square root. In type 2 effectivity such realizers are thought of as realizing multi-valued functions.

2.2 The role of double negation

Negation $\neg\phi$ is defined as $\phi \Rightarrow \bot$. This gives us

$$\mathtt{r} \Vdash \neg\phi \text{ iff for all } \mathtt{q} \in \mathbb{A}, \text{ not } \mathtt{q} \Vdash \phi,$$
$$\mathtt{r} \Vdash \neg\neg\phi \text{ iff there is } \mathtt{q} \in \mathbb{A} \text{ such that } \mathtt{q} \Vdash \phi.$$

[1] $\langle \mathtt{p}, \mathtt{q} \rangle$ is the encoding of the pair whose components are $\mathtt{p}$ and $\mathtt{q}$.
[2] $\overline{n}$ is the encoding of the natural number n.

A realizer $\mathtt{r}$ of a doubly negated formula $\neg\neg\phi$ does not carry any information about the computational content of ϕ, because it is as good as any other realizer. Thus double negation is a way of erasing the constructive or computational meaning of a formula. To illustrate this, consider a morphism $f : \mathbf{S} \to \mathbf{T}$ and a realizer $\mathtt{r}$ of the formula

$$\forall\, y \in \mathbf{T} \,.\, \exists\, x \in \mathbf{S} \,.\, f(x) = y. \tag{1}$$

Whenever $\mathtt{b} \Vdash_T y$ then $\mathtt{r}\ \mathtt{b}\downarrow$ and $\mathtt{r}\ \mathtt{b} \Vdash_S x$ for some $x \in S$ such that $f(x) = y$. Thus $\mathtt{r}$ is like a right inverse of f, except that it is a realizer, not a morphism, and it need not respect equality on $\mathbf{T}$. In the terminology of type 2 effectivity, $\mathtt{r}$ would be a realizer for a multi-valued right inverse of f. Now, if we put a double negation in front of (1) we obtain a formula that is intuitionistically equivalent to

$$\forall\, y \in \mathbf{T} \,.\, \neg\forall\, x \in \mathbf{S} \,.\, f(x) \neq y. \tag{2}$$

A realizer of (2) does not compute anything useful. Indeed, it accepts a realizer $\mathtt{b}$ and outputs whatever it wants because a negation is realized either by everything or nothing. Thus (2) is realized when f is onto and (1) when it is "effectively" onto.

A formula which is equivalent to its double negation is called $\neg\neg$-*stable*. Since $\phi \Rightarrow \neg\neg\phi$ is always intuitionistically provable, only the direction $\neg\neg\phi \Rightarrow \phi$ is relevant. An important family of stable formulas are the *negative* ones, which are those built from $\bot$, $\top$, $=$, $\neg$, $\wedge$, $\Rightarrow$, $\forall$, and possibly other $\neg\neg$-stable primitive relations. The realizers of a $\neg\neg$-stable formula ϕ are computationally irrelevant in the sense that any information that can be computed with the help of a realizer $\mathtt{r} \Vdash \phi$ can be computed without $\mathtt{r}$. Ultimately, this implies that $\mathtt{r}$ itself can be computed from nothing, as long as it exists.

In Asm the morphism $f : \mathbf{S} \to \mathbf{T}$ is mono precisely when $f : S \to T$ is injective as a map between underlying sets. The formula expressing injectivity of f,

$$\forall\, x, y \in \mathbf{S} \,.\, f(x) = f(y) \Rightarrow x = y, \tag{3}$$

is negative. Thus it has computationally irrelevant realizers and its "classical" and "effective" readings are the same.

A mono $f : \mathbf{S} \to \mathbf{T}$ which also satisfies (1) is an isomorphism because in this case a realizer $\mathtt{r}$ for (1) respects equality on $\mathbf{T}$. If a mono $f : \mathbf{S} \to \mathbf{T}$ satisfies (2) then f is a bijection but $f^{-1} : T \to S$ need not be realized. Such a mono is called $\neg\neg$-*dense*, and is always isomorphic to a mono $i : \mathbf{S} \rightarrowtail \mathbf{T}$ such that $S = T$ and i is the identity map. A realizer of the identity on S, considered as a morphism $(S, \Vdash_S) \to (S, \Vdash_T)$, is precisely a many-one reduction between representations $\Vdash_S$ and $\Vdash_T$, thus the $\neg\neg$-dense monos play in Asm the role of reductions between representations.

A mono $i : \mathbf{S} \rightarrowtail \mathbf{T}$ is called $\neg\neg$-*stable* when

$$\forall\, x \in \mathbf{T} \,.\, (\neg\neg(x \in \mathbf{S}) \Rightarrow x \in \mathbf{S})$$

is realized, where "$x \in \mathbf{S}$" is a shorthand for $\exists\, y \in \mathbf{S} \,.\, i(y) = x$. Up to isomorphism, such a mono is a restriction of $\mathbf{T}$ to a subset $S \subseteq T$, and the realizability relation $\Vdash_S$ is $\Vdash_T$ restricted to S. Thus the $\neg\neg$-stable monos into $\mathbf{T}$ correspond to subsets of T (with the induced realizability relations).

Knowing that a formula is $\neg\neg$-stable may be quite useful because it allows us to ignore its realizers. *Markov's principle*

$$\forall\, f \in \{0,1\}^{\mathbf{N}} \,.\, (\neg\neg\exists\, n \in \mathbf{N} \,.\, f(n) = 1) \Rightarrow \exists\, n \in \mathbf{N} \,.\, f(n) = 1. \tag{4}$$

states that a formula of the form $\exists\, n \in \mathbf{N} \,.\, f(n) = 1$ is $\neg\neg$-stable, uniformly in f. Here $\mathbf{N}$ is the modest set of natural numbers, cf. Section 3.2, and the exponential $\{0,1\}^{\mathbf{N}}$ is the modest set of those maps $\mathbb{N} \to \{0,1\}$ which are tracked by elements of $\mathbb{A}$. A realizer $\mathtt{mp}$ for (4) accepts $\mathtt{f}$ that tracks f and an (irrelevant) realizer $\mathtt{r}$ for the antecedent of the implication. It then computes a pair $\langle \overline{n}, \mathtt{s} \rangle$ such that $f(n) = 1$, and $\mathtt{s}$ is irrelevant. The realizer $\mathtt{mp}$ may accomplish this by searching for the smallest $n \in \mathbb{N}$ that yields $\mathtt{f}\ \overline{n} = \overline{1}$. The search terminates because $\neg\neg\exists\, n \in \mathbf{N} \,.\, f(n) = 1$ is realized by $\mathtt{r}$ so that $f(n) = 0$ cannot be the case for all $n \in \mathbb{N}$.

2.3 Semidecidable predicates

To illustrate how the realizability interpretation is used, and for later use, we explain how to treat semidecidability in Asm. We say that a mono $i : \mathbf{S} \rightarrowtail \mathbf{T}$, seen as a predicate on $\mathbf{T}$, is *semidecidable* when

$$\forall\, x \in \mathbf{T} \,.\, \exists\, f \in \{0,1\}^{\mathbf{N}} \,.\, (x \in \mathbf{S} \iff \exists\, n \in \mathbf{N} \,.\, f(n) = 1) \tag{5}$$

is realized. By Markov's Principle the formula $\exists\, n \in \mathbf{N} \,.\, f(n) = 1$ is $\neg\neg$-stable. Hence, without loss of generality we may restrict attention to those $i : \mathbf{S} \rightarrowtail \mathbf{T}$ that are $\neg\neg$-stable and for which i is a subset inclusion. Validity of (5) is then equivalent to there being $\mathtt{r} \in \mathbb{A}_{\mathrm{eff}}$ which works as follows: if $\mathtt{a} \Vdash_T x$ then, for all $n \in \mathbb{N}$, $\mathtt{r}\ \mathtt{a}\ \overline{n}{\downarrow}$ and $\mathtt{r}\ \mathtt{a}\ \overline{n} \in \{\overline{0}, \overline{1}\}$, and furthermore, $x \in S$ if, and only if, $\mathtt{r}\ \mathtt{a}\ \overline{n} = \overline{1}$ for some $n \in \mathbb{N}$.

The semidecidable monos have the expected properties: decidable monos are semidecidable, and the semidecidable monos are closed under conjunctions and existential quantification over $\mathbf{N}$, by which we mean that $\exists\, n \in \mathbf{N} \,.\, (n, x) \in \mathbf{S}$ interpreted in the internal language denotes a semidecidable mono, provided that $\mathbf{S} \rightarrowtail \mathbf{N} \times \mathbf{T}$ is semidecidable. Consequently, a mono is semidecidable if it can be expressed in the internal language as a formula built from decidable predicates, conjunctions, disjunctions, and existential quantifications over $\mathbb{N}$.

In type 1 effectivity our notion of semidecidability coincides with the usual one, while in type 2 effectivity the notion is known as "r.e. open subset". In a purely topological model, such as the continuous version of type 2 effectivity "semidecidable" means "topologically open". The interpretation in Set is trivial because there every subset is semidecidable (even decidable) thanks to the law of excluded middle.

3 Algebras

A *signature* Σ for an algebra is given by a list of *function symbols* $\mathsf{f}_1, \dots, \mathsf{f}_k$. Each f_i has an *arity*, which is a non-negative integer. The set $\mathsf{Term}(\Sigma)$ of *terms over* Σ is built inductively from variables $x, y, z, \dots$, and terms $\mathsf{f}(t_1, \dots, t_n)$, where f is a function symbol with arity n and $t_1, \dots, t_n$ are terms.

We assume that a standard numbering $\nu(-) : \mathbb{N} \to \{\star\} \cup \mathsf{Term}(\Sigma)$ of terms is given. The qualifier "standard" means that the syntax of the terms can be manipulated in a computable way. The special value $\nu(n) = \star$ signifies that n is not a valid code. This is needed for enumerating the set of closed terms, which is an empty set when Σ contains no constant symbols, as well as in Section 6 where we consider partial operations.

The numbering induces the structure of a modest set on $\mathsf{Term}(\Sigma)$ with the realizability relation

$$\overline{n} \Vdash_{\mathsf{Term}(\Sigma)} t \iff \nu(n) = t.$$

We may similarly form the modest set of all closed terms over Σ.

A *Σ-algebra* $\mathcal{A}$ in a category C with finite products is given by an object $|\mathcal{A}|$ called the *carrier* of $\mathcal{A}$, and for each function symbol f with arity n a morphism $\mathsf{f}^{\mathcal{A}} : |\mathcal{A}|^n \to |\mathcal{A}|$, called an *operation*. Each term $t \in \mathsf{Term}(\Sigma)$ whose free variables are among $x_1, \dots, x_k$ determines a morphism $|\mathcal{A}|^k \to |\mathcal{A}|$: a variable x_i is the i-th projection, while a term $\mathsf{f}(t_1, \dots, t_n)$ is the composition of $\mathsf{f}^{\mathcal{A}}$ with the morphisms determined by $t_1, \dots, t_n$. A *subalgebra* of $\mathcal{A}$ is a Σ-algebra $\mathcal{B}$ with a mono $\mathcal{B} \rightarrowtail \mathcal{A}$ such that the operations in $\mathcal{A}$ restrict to operations in $\mathcal{B}$. We write $\mathcal{B} \leq \mathcal{A}$ when $\mathcal{B}$ is a subalgebra of $\mathcal{A}$.

If C and D are categories with finite products and $F : \mathsf{C} \to \mathsf{D}$ a functor which preserves finite products then a Σ-algebra $\mathcal{A}$ in C is mapped by F to a Σ-algebra $F(\mathcal{A})$ in $\mathcal{D}$, where $|F(\mathcal{A})| = F(|\mathcal{A}|)$ and $\mathsf{f}^{F(\mathcal{A})} = F(\mathsf{f}^{\mathcal{A}})$. The mapping preserves valid equations in $\mathcal{A}$, and also reflects them if F is faithful.

A *(first-order) formula over* Σ is a formula ϕ in first-order logic with equality and terms over Σ. If $\mathcal{A}$ is a Σ-algebra in C, where C is either Set or Asm, then we may interpret such a ϕ as a statement about $\mathcal{A}$: the terms are interpreted according to $\mathcal{A}$, while the logic is interpreted either in the standard set-theoretic way, as given by Tarski, or using the realizability interpretation from Section 2.1.

We write $\mathcal{A} \models_{\mathsf{C}} \phi$ when ϕ is valid when so interpreted. We refer to interpretations in Set as "classical" and those in Asm as "effective". More generally the adjectives "classical" and "effective" are used to distinguish between the two settings. For example, a "classical algebra" is an algebra in Set, while an "effective algebra" is one in Asm. Similarly, a (classical) space is "classically complete" if the formula expressing completeness is valid in Set, and an (effective) space is "effectively complete" if the same formula is valid in Asm. Note however that the exact interpretation of "effective" depends on the choice of the PCA $\mathbb{A}$ and sub-PCA $\mathbb{A}_{\mathrm{eff}}$.

3.1 Subalgebras generated by subassemblies

Suppose $\mathcal{A}$ is classical Σ-algebra, and consider a subset $C \subseteq |\mathcal{A}|$ of the carrier. Then there exists the smallest subalgebra $\mathcal{I} \leq \mathcal{A}$ such that $C \subseteq |\mathcal{I}|$, namely the intersection of all subalgebras of $\mathcal{A}$ that contain C. We say that $\mathcal{I}$ is *generated by* C and denote it by $\langle C \rangle_{\mathcal{A}}$.

Now let $\mathcal{A}$ be an effective Σ-algebra and $\mathbf{C} \rightarrowtail |\mathcal{A}|$ a subassembly of $|\mathcal{A}|$. There exists the smallest effective subalgebra $\langle \mathbf{C} \rangle_{\mathcal{A}} \leq \mathcal{A}$ containing $\mathbf{C}$ as a subassembly. One way of proving this is to work in the internal language of the realizability topos $\mathsf{RT}(\mathbb{A}, \mathbb{A}_{\mathrm{eff}})$, where $\langle \mathbf{C} \rangle_{\mathcal{A}}$ is the intersection of all subalgebras of $\mathcal{A}$ that contain the assembly $\mathbf{C}$, just like in Set. A special case is the *initial subalgebra* $\mathbf{I} = \langle \emptyset \rangle_{\mathcal{A}}$ which is generated by the empty subassembly. It is always modest, even if $\mathcal{A}$ is not. Specifically, the underlying set of $\mathbf{I}$ is

$$I = \{t^{\mathcal{A}} \mid t \text{ is a closed } \Sigma\text{-term}\}$$

and the realizability relation is given by

$$\overline{n} \Vdash_I x \iff \text{ for some closed } \Sigma\text{-term } t,\ \nu(n) = t \text{ and } t^{\mathcal{A}} = x\ .$$

The initial algebra is effectively enumerated by the morphism $e : \mathbf{N} \to \{\star\} + \mathbf{I}$ defined by

$$e(n) = \begin{cases} t^{\mathcal{A}} & \text{if } \nu(n) = t \text{ is a closed term} \\ \star & \text{otherwise.} \end{cases}$$

The special value $\star$ is needed because there may be no closed terms.

3.2 Algebras characterized by their universal properties

When a classical algebra is characterized up to isomorphism by a universal property, we may use the property to identify the corresponding effective algebra. It turns out that we usually get the generally accepted "correct" computability structure:

- The natural numbers $\mathbb{N}$ are the initial commutative semiring with unit. In Asm this is the modest set $\mathbf{N} = (\mathbb{N}, \Vdash_{\mathbb{N}})$ where $\overline{n} \Vdash_{\mathbb{N}} n$ for each $n \in \mathbb{N}$.
- The initial commutative ring with unit in Set are the integers $\mathbb{Z}$, while in Asm it is the modest set $\mathbf{Z} = (\mathbb{Z}, \Vdash_{\mathbb{Z}})$ where, for each $m, n \in \mathbb{N}$ and $k \in \mathbb{Z}$, $\langle \overline{m}, \overline{n} \rangle \Vdash_{\mathbb{Z}} k$ when $k = m - n$.
- The field of fractions over the integers in Set are the rationals $\mathbb{Q}$. In Asm it is the modest set $\mathbf{Q} = (\mathbb{Q}, \Vdash_{\mathbb{Q}})$ where, for all $k, m, n \in \mathbb{N}$ and $q \in \mathbb{Q}$, $\langle \overline{k}, \overline{m}, \overline{n} \rangle \Vdash_{\mathbb{Q}} q$ when $q = (k - m)/n$.
- The reals $\mathbb{R}$ are the Cauchy-complete archimedean ordered field.[3] The counterpart in assemblies is the modest set $\mathbf{R} = (R, \Vdash_R)$ where $\mathsf{a} \Vdash_R x$ when $\mathsf{a} \in \mathbb{A}$ represents a fast Cauchy sequence[4] of rational numbers converging to x, and $R = \{x \in \mathbb{R} \mid \exists \mathsf{a} \in A \,.\, \mathsf{a} \Vdash_R x\}$. Depending on the choice of $\mathbb{A}$ the set R could consist just of the computable reals, or all reals, or all reals computable with respect to an oracle, etc.

Unfortunately, such universal characterizations are not always available or practical. Apart from first-order formulas over a signature Σ we shall also consider more general first-order formulas which additionally refer to the natural numbers $\mathbb{N}$, the integers $\mathbb{Z}$, and the rationals $\mathbb{Q}$. We call them *extended formulas over the signature Σ*. When they are interpreted in Set, the symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ receive their usual meaning, whereas in Asm we interpret them as the corresponding assemblies $\mathbf{N}$, $\mathbf{Z}$, and $\mathbf{Q}$, as described above. We do *not* allow an extended formula to refer directly to the real numbers because Proposition 1 below fails for formulas that refer to the reals. An extended formula over Σ which is also negative is called *extended negative formula over Σ*.

3.3 Transfer of algebras between sets and assemblies

An effective Σ-algebra $\mathcal{A}$ becomes a classical Σ-algebra when the effective structure is removed. More precisely, there is a functor $\Gamma : \mathsf{Asm} \to \mathsf{Set}$ which forgets the effective structure: it maps an assembly $\mathbf{S} = (S, \Vdash_S)$ to its underlying set $\Gamma \mathbf{S} = S$ and a morphism $f : \mathbf{S} \to \mathbf{T}$ to the underlying map $\Gamma f = f : S \to T$. Because Γ preserves finite limits, and finite products in particular, it maps a Σ-algebra $\mathcal{A}$ in Asm to the Σ-algebra $\Gamma\mathcal{A}$ in Set.

Proposition 1. *The Σ-algebra $\mathcal{A}$ in* Asm *satisfies the same extended negative formulas as $\Gamma\mathcal{A}$ does in* Set.

[3] Strictly speaking, this is not a universal property, but it still fixes the reals up to isomorphism both in Set and Asm.

[4] A sequence $(a_n)_n$ is fast Cauchy if $|a_m - a_n| \leq 2^{-\min(m,n)}$ for all $m, n \in \mathbb{N}$.

Proof. The proof proceeds by induction on the structure of an extended negative formulas over Σ. The base cases $\perp$ and $\top$ are trivial. The remaining base case is an equation $t_1 = t_2$, for which

$$\mathcal{A} \models_{\mathsf{Asm}} t_1 = t_2 \qquad \text{iff} \qquad \Gamma\mathcal{A} \models_{\mathsf{Set}} t_1 = t_2,$$

holds because Γ is faithful. For the induction step, suppose ϕ_1 and ϕ_2 are extended negative formulas over Σ. The realizability interpretation of implication gives

$$\mathcal{A} \models_{\mathsf{Asm}} \phi_1 \Rightarrow \phi_2 \qquad \text{iff} \qquad \text{if } \mathcal{A} \models_{\mathsf{Asm}} \phi_1 \text{ then } \mathcal{A} \models_{\mathsf{Asm}} \phi_2.$$

By the induction hypotheses for ϕ_1 and ϕ_2 we obtain a further equivalence with

$$\text{if } \Gamma\mathcal{A} \models_{\mathsf{Set}} \phi_1 \text{ then } \Gamma\mathcal{A} \models_{\mathsf{Set}} \phi_2$$

which is just the definition of $\Gamma\mathcal{A} \models_{\mathsf{Set}} \phi_1 \Rightarrow \phi_2$. Negation and conjunction are treated similarly. For a universal quantification $\forall\, x \in \mathbf{S} \,.\, \phi(x)$, where $\phi(x)$ is an extended negative formula over Σ and $\mathbf{S} = (S, \Vdash_S)$ is one of $|\mathcal{A}|$, $\mathbf{N}$, $\mathbf{Z}$ or $\mathbf{Q}$, we have

$$\mathcal{A} \models_{\mathsf{Asm}} \forall\, x \in \mathbf{S} \,.\, \phi(x) \qquad \text{iff} \qquad \text{for all } a \in S,\ \mathcal{A} \models_{\mathsf{Asm}} \phi(a).$$

By induction hypothesis this is equivalent to

$$\text{for all } a \in S,\ \Gamma\mathcal{A} \models_{\mathsf{Set}} \phi(a),$$

which is the definition of

$$\Gamma\mathcal{A} \models_{\mathsf{Set}} \forall\, x \in \Gamma S \,.\, \phi(x).$$

The proof is finished because $\Gamma|\mathcal{A}| = |\Gamma\mathcal{A}|$, $\Gamma\mathbf{N} = \mathbb{N}$, $\Gamma\mathbf{Z} = \mathbb{Z}$, and $\Gamma\mathbf{Q} = \mathbb{Q}$. $\square$

A $\neg\neg$-*dense* subalgebra $\mathcal{B} \leq \mathcal{A}$ in Asm is a subalgebra for which the mono $|\mathcal{B}| \rightarrowtail |\mathcal{A}|$ is $\neg\neg$-dense.

Corollary 2. *An effective Σ-algebra and a $\neg\neg$-dense subalgebra satisfy the same extended negative formulas over Σ.*

Proof. Let $\mathcal{A}$ be an effective Σ-algebra and $\mathcal{B} \leq \mathcal{A}$ a $\neg\neg$-dense subalgebra. Recall that we may assume without loss of generality that $|\mathcal{B}| = |\mathcal{A}|$ and that the mono $|\mathcal{B}| \rightarrowtail |\mathcal{A}|$ is the identity map. The corollary follows from Proposition 1 and the fact that $\Gamma\mathcal{A} = \Gamma\mathcal{B}$. $\square$

It is possible to equip every set S with an effective structure, albeit a completely trivial one: define the *constant assembly* $\nabla S = (S, \Vdash_{\nabla S})$ where $\mathtt{r} \Vdash_S x$ holds for all $\mathtt{r} \in A$ and $x \in S$. In other words, in ∇S every realizer realizes every element. Every function $f : S \to T$ between sets S and T is realized as a map $\nabla f : \nabla S \to \nabla T$, for example by the realizer $\mathsf{s\,k\,k}$. This gives us a full and faithful embedding $\nabla : \mathsf{Set} \to \mathsf{Asm}$.

The functor ∇ preserves limits, and finite products in particular. Therefore, it maps a Σ-algebra $\mathcal{A}$ in Set to a Σ-algebra $\nabla\mathcal{A}$ in Asm.

Proposition 3. *The Σ-algebra $\mathcal{A}$ in* Set *satisfies the same extended negative formulas as $\nabla\mathcal{A}$ does in* Asm.

Proof. By Proposition 1 $\nabla\mathcal{A}$ satisfies the same extended negative formulas as $\Gamma(\nabla\mathcal{A})$, but $\Gamma(\nabla\mathcal{A}) = \mathcal{A}$. □

We mention that for any assembly $\mathbf{S}$ and set T the maps $\Gamma\mathbf{S} \to T$ correspond bijectively and naturally to the maps $\mathbf{S} \to \nabla T$, which means that Γ is left adjoint to ∇. In fact, it is well known that Γ and ∇ are part of a geometric inclusion of toposes $\mathsf{Set} \to \mathsf{RT}(\mathbb{A}, \mathbb{A}_{\text{eff}})$, and that Set is equivalent to the sheaves for the $\neg\neg$-topology in $\mathsf{RT}(\mathbb{A}, \mathbb{A}_{\text{eff}})$. This implies that Proposition 1, Corollary 2, and Proposition 3 have their analogues in the realizability topos $\mathsf{RT}(\mathbb{A}, \mathbb{A}_{\text{eff}})$.

4 Premetric spaces

A *metric algebra* is a Σ-algebra in the category of metric spaces and continuous maps, i.e., its carrier is a metric space and the operations are continuous maps. A metric algebra is *complete* if its carrier is a complete metric space. We face two difficulties when we try to transfer metric algebras from sets to assemblies. The first difficulty is that ∇ maps a metric $d : S \times S \to \mathbb{R}$ to the realized map $\nabla d : \nabla S \times \nabla S \to \nabla\mathbb{R}$, which is not a metric anymore because its codomain $\nabla\mathbb{R}$ is not the object $\mathbf{R}$ of real numbers in Asm. The second difficulty is that ∇ need not preserve continuity of operations.

To overcome the first difficulty we use a formulation of metric spaces which does not directly refer to real numbers, is classically equivalent to the usual metric spaces,[5] and is constructively acceptable. Such a notion, namely *premetric spaces*, was defined by Fred Richman [Richman 2008]. We use a slight variation:

Definition 4. A *premetric space* (X, d) is a set X with a ternary relation $d \subseteq X \times X \times \mathbb{Q}$ satisfying the following conditions, where we write $d(x, y) \leq q$ instead of $(x, y, q) \in d$:

1. if $q < 0$ then not $d(x, y) \leq q$,
2. $d(x, y) \leq 0$ if, and only if, $x = y$,
3. if $d(x, y) \leq q$ then $d(y, x) \leq q$,

[5] We allow infinite distances but that is inessential.

4. if $d(x, y) \leq q$ and $d(y, z) \leq r$ then $d(x, z) \leq q + r$,
5. $d(x, y) \leq q$ if, and only if, $d(x, y) \leq r$ for all $r > q$.

Richman's definition also requires that for all $x, y \in X$ there is a rational $q \geq 0$ such that $d(x, y) \leq q$. We omit the requirement as it would prevent us from proving Proposition 6 below, because the classical existence of such a q in Set cannot be transfered to effective existence in Asm.

The strict version $d(x, y) < q$ of the premetric relation is defined by

$$d(x, y) < q \iff \exists\, r \in \mathbb{Q}\,.\, d(x, y) \leq r \wedge r < q.$$

An open ball centred at x with radius r is defined as usual, $B_X(x, r) = \{y \in X \mid d(x, y) < q\}$.

Every metric space (M, d) is a premetric space (M, d') with $d' = \{(x, y, q) \in X \times X \times \mathbb{Q} \mid d(x, y) \leq q\}$. Classically, the converse holds if we allow infinite distances[6] because the metric d may be recovered from the premetric d' as $d(x, y) = \inf \{q \in \mathbb{Q} \mid d'(x, y) \leq q\}$. Constructively however the infimum need not exist.

Requirement 4 in the definition of premetric spaces corresponds to the triangle inequality. We often use it to show that $d(x, z) \leq q + r$ because $d(x, y) \leq q$ and $d(y, z) \leq r$. In such cases we abuse notation and express the argument by writing $d(x, z) \leq d(x, y) + d(y, z) \leq q + r$. This is how the familiar triangle inequality is used in the case of metric spaces. Of course, we need to keep in mind the fact that there is no distance function d to speak of.

The basic theory of premetric spaces parallels that of metric spaces. The notions of completeness, continuity, density, etc., are all easily expressed in terms of the premetric. In fact, the whole theory is constructively valid (even without choice), as was shown by Richman [Richman 2008]. Despite our allowing infinite distances, Proposition 5 below still holds constructively, and is therefore valid both in Set and Asm. We say that a map $f : X \to Y$ is *locally uniformly continuous* if its domain can be written as a union of open balls such that f is uniformly continuous on every one of them.

Proposition 5. *Let $X \subseteq Y$ be a dense subset of a complete premetric space Y. Every locally uniformly continuous $f : X \to Z$ to a complete premetric space Z has a unique locally uniformly continuous extension $\overline{f} : Y \to Z$.*

Proof. See [Richman 2008, Theorems 2.2 and 2.3]. □

[6] With Richman's extra axiom the correspondence between metric and premetric spaces is exact, classically. At any rate, every metric space (with infinite distances) is *topologically* equivalent to one with a bounded metric.

An easy consequence of the theorem is that any two completions of a premetric space, i.e., isometric embeddings with dense image into a complete metric space, are isometrically isomorphic.

Let us see how a premetric space (X, d) is transferred from Set to Asm by ∇. The relation $d \subseteq X \times X \times \mathbb{Q}$ is mapped to a $\neg\neg$-stable mono $\nabla d \rightarrowtail \nabla X \times \nabla X \times \nabla\mathbb{Q}$. This is not quite what we want because in Asm the premetric structure on ∇X should be a relation on $\nabla X \times \nabla X \times \mathbf{Q}$. We restrict ∇d to the desired domain via the mono $\mathrm{id}_{\nabla X} \times \mathrm{id}_{\nabla X} \times i$, where $i : \mathbf{Q} \rightarrowtail \nabla\mathbb{Q}$ is the morphism represented by the identity on $\mathbb{Q}$. As far as extended negative formulas are concerned, this does not make a difference because $i : \mathbf{Q} \rightarrowtail \nabla\mathbb{Q}$ is $\neg\neg$-dense. We shall not bother writing the inclusion i explicitly.

Proposition 6. *If (X, d) is a classical premetric space then $(\nabla X, \nabla d)$ is an effective premetric space. Furthermore, (X, d) and $(\nabla X, \nabla d)$ satisfy the same extended negative formulas.*

Proof. The proposition follows from Proposition 3 because ∇d is a $\neg\neg$-stable mono and the axioms for premetric spaces are extended negative formulas. $\square$

A premetric space (X, d) is complete when there is an operator

$$\lim_X : \mathsf{Cauchy}(X) \to X$$

that computes limits of Cauchy sequences. More precisely, let

$$\mathsf{Cauchy}(X) = \{a \in X^{\mathbb{N}} \mid \forall\, m, n \in \mathbb{N} \,.\, d(a_m, a_n) \leq 2^{-\min(m,n)}\}.$$

The operator $\lim_X$ has to satisfy

$$\forall\, a \in \mathsf{Cauchy}(X) \,.\, \forall\, n \in \mathbb{N} \,.\, d(a_n, \lim_X a) \leq 2^{-n}. \tag{6}$$

Proposition 7. *If (X, d) is a classical complete premetric space then $(\nabla X, \nabla d)$ is an effective complete premetric space.*

Proof. Suppose (X, d) is a complete premetric space in Set with a limit operator $\lim_X$. We need to show that $(\nabla X, \nabla d)$ has a limit operator $\lim_{\nabla X}$ in Asm for which (6) is realized. Observe that $(\nabla X)^{\mathbf{N}} \cong \nabla(X^{\mathbb{N}})$ because every map $\mathbb{N} \to X$ is realized as a morphism $\mathbf{N} \to \nabla X$, say by the realizer $\mathsf{s\,k\,k}$. This implies that $\mathsf{Cauchy}(\nabla X) \cong \nabla\mathsf{Cauchy}(X)$. By harmlessly pretending that the isomorphism is actually an equality, we obtain

$$\nabla \lim_X : \mathsf{Cauchy}(\nabla X) \to \nabla X.$$

Because (6) is a negative formula satisfied by $\lim_X$ in Set, $\nabla \lim_X$ satisfies it in Asm. Therefore, we may take $\lim_{\nabla X} = \nabla \lim_X$. $\square$

We now address the second difficulty, namely that ∇ need not preserve continuity of maps. More precisely, ∇ does not preserve *pointwise* continuity of $f : X \to Y$ at $x \in X$ because it is expressed by a formula which is not negative:

$$\forall k \in \mathbb{N} . \exists m \in \mathbb{N} . \forall y \in X . d(x, y) \leq 2^{-m} \Rightarrow d(f(x), f(y)) \leq 2^{-k}. \tag{7}$$

To circumvent this problem, we consider instead *sequentially continuous* maps, i.e., those that preserve limits of Cauchy sequences. Classically pointwise and sequential continuity are equivalent, but constructively the latter is weaker than the former.[7]

Lemma 8. *If $f : X \to Y$ is a sequentially continuous map between classical complete premetric spaces then $\nabla f : \nabla X \to \nabla Y$ is effectively sequentially continuous.*

Proof. For $\nabla f : \nabla X \to \nabla Y$ to be effectively sequentially continuous it has to satisfy $\nabla f \circ \lim_{\nabla X} = \lim_{\nabla Y} \circ \nabla f$, which is equivalent to

$$\Gamma(\nabla f \circ \lim_{\nabla X}) = \Gamma(\lim_{\nabla Y} \circ \nabla f)$$

because Γ is faithful. The equality holds because $\lim_{\nabla X} = \nabla \lim_X$, $\lim_{\nabla Y} = \nabla \lim_Y$, and f is sequentially continuous:

$$\Gamma(\nabla f \circ \lim_{\nabla X}) = \Gamma(\nabla(f \circ \lim_X)) = \Gamma(\nabla(\lim_Y \circ f)) = \Gamma(\lim_{\nabla Y} \circ \nabla f).$$

□

In the context of Σ-algebras Lemma 8 tells us that ∇ transfers a classical complete premetric algebra $\mathcal{A}$ to an effective complete premetric algebra $\nabla\mathcal{A}$, with the caveat that the operations of $\nabla\mathcal{A}$ are only sequentially continuous.

4.1 Complete subalgebras

When $\mathcal{A}$ is a classical complete premetric Σ-algebra we may ask whether for every subalgebra $\mathcal{B} \leq \mathcal{A}$ there exists a smallest *complete* subalgebra $\overline{\mathcal{B}} \leq \mathcal{A}$ which contains $\mathcal{B}$. The answer is positive: because intersections of complete subalgebras preserve completeness, $\overline{\mathcal{B}}$ is the intersection of all complete subalgebras that contain $\mathcal{B}$. This rather uninformative description of $\overline{\mathcal{B}}$ can be improved if we require operations of $\mathcal{B}$ to be locally uniformly continuous.

Proposition 9. *Let $\mathcal{A}$ be a classical complete premetric Σ-algebra. The closure $\overline{|\mathcal{B}|}$ of the carrier of a subalgebra $\mathcal{B} \leq \mathcal{A}$ is the least complete subalgebra of $\mathcal{A}$ containing $\mathcal{B}$, provided the operations on $\mathcal{B}$ are locally uniformly continuous.*

[7] In Bishop's constructive mathematics pointwise continuity is implied by sequential continuity and Ishihara's BD-N principle [Ishihara 1992]. However, our formulation of metric spaces in terms of premetric is more general than the standard constructive one, and the connection with BD-N does not apply.

Proof. By Proposition 5 each $\mathsf{f}^{\mathcal{B}} : |\mathcal{B}|^n \to |\mathcal{B}|$ extends to an operation $\mathsf{f}^{\overline{\mathcal{B}}} : \overline{|\mathcal{B}|}^n \to \overline{|\mathcal{B}|}$, hence $\overline{|\mathcal{B}|}$ is a Σ-algebra which we denote by $\overline{\mathcal{B}}$. We still have to show that $\overline{\mathcal{B}}$ is a subalgebra of $\mathcal{A}$, i.e., that $\mathsf{f}^{\mathcal{A}}$ restricted to $\overline{|\mathcal{B}|}^n$ is $\mathsf{f}^{\overline{\mathcal{B}}}$. Consider any $x \in \overline{|\mathcal{B}|}$. There is a sequence $(u_n)_n$ in $|\mathcal{B}|$ such that $\lim_n u_n = x$. Because $\mathsf{f}^{\mathcal{A}}$ and $\mathsf{f}^{\overline{\mathcal{B}}}$ are continuous it follows that

$$\mathsf{f}^{\mathcal{A}}(x) = \lim_n \mathsf{f}^{\mathcal{A}}(u_n) = \lim_n \mathsf{f}^{\mathcal{B}}(u_n) = \lim_n \mathsf{f}^{\overline{\mathcal{B}}}(u_n) = \mathsf{f}^{\overline{\mathcal{B}}}(x).$$

□

We state the effective version of the previous proposition in terms of sequential continuity so that it is applicable to our situation.

Proposition 10. *Let $\mathcal{A}$ be an effective Σ-algebra whose carrier is an effectively complete premetric space and the operations are effectively sequentially continuous. The effective closure $\overline{|\mathcal{B}|}$ of the carrier of an effective subalgebra $\mathcal{B} \leq \mathcal{A}$ is the least effectively complete subalgebra of $\mathcal{A}$ containing $\mathcal{B}$, provided the operations on $\mathcal{B}$ are effectively locally uniformly continuous.*

Proof. We may reuse the proof of Proposition 9 because Proposition 5 is constructively valid. The only change is that we refer to sequential rather than pointwise continuity of $\mathsf{f}^{\mathcal{A}}$ in the last sentence. □

We remark that the complete subalgebra $\overline{\mathcal{B}}$ generated by $\mathcal{B}$ is modest if $\mathcal{B}$ is modest, even if $\mathcal{A}$ is not. The explicit description of $\overline{\mathcal{B}}$ is as follows. Let $\mathbf{C} = (C, \Vdash_C) = \mathsf{Cauchy}(|\mathcal{B}|)$ be the assembly of fast Cauchy sequences in $|\mathcal{B}|$, which is modest because $|\mathcal{B}|$ is. Define the coincidence relation $\sim$ on C by

$$a \sim b \iff \forall\, n \in \mathbb{N} \,.\, d(a_n, b_n) \leq 2^{-n+1}.$$

Then $\overline{|\mathcal{B}|}$ is the modest assembly whose underlying set is $C/{\sim}$ and the realizability relation is

$$\mathtt{r} \Vdash_{\overline{|\mathcal{B}|}} [a]_{\sim} \iff \mathtt{r} \Vdash_C a.$$

Of course, in an actual implementation we are free to use any other representation, as long as it is effectively isomorphic to this one.

5 Main Theorems

Let $\mathcal{A}$ be a classical premetric Σ-algebra. In general there will be many effective subalgebras $\mathcal{B} \leq \nabla\mathcal{A}$, each carving out a different piece of $\mathcal{A}$ with its own effective structure. Our first theorem gives conditions which severely cut down the number of possibilities.

Theorem 11. *Suppose $\mathcal{A}$ is a classical premetric Σ-algebra in which the initial subalgebra $\langle\emptyset\rangle_{\mathcal{A}}$ is classically dense. Up to effective isomorphism, there is at most one effectively complete subalgebra $\mathcal{B} \leq \nabla\mathcal{A}$ on which the relation $\nabla d(x, y) < q$ is semidecidable in $x, y \in |\mathcal{B}|, q \in \mathbf{Q}$.*

Proof. We prove the theorem by showing that $\mathcal{B}$, if it exists, is the effective completion of the initial subalgebra $\mathcal{I} = \langle\emptyset\rangle_{\nabla\mathcal{A}}$. Because any two metric completions of $|\mathcal{I}|$ are unique up to isomorphism it follows that there is at most once such $\mathcal{B}$, up to isomorphism. As $\mathcal{B}$ is presumed to be effectively complete, we only need to show that $\mathcal{I}$ is effectively dense in $\mathcal{B}$.

The assumption that $\langle\emptyset\rangle_{\mathcal{A}}$ is dense in $\mathcal{A}$ is a classical statement, but we can still express it in Asm as a statement about $\nabla\langle\emptyset\rangle_{\mathcal{A}}$ and $\nabla\mathcal{A}$:

$$\models_{\mathsf{Asm}} \forall\, y \in \nabla|\mathcal{A}| \,.\, \forall\, q \in \mathbb{Q} \,.\, (q > 0 \implies \neg\neg\exists\, y \in \nabla\langle\emptyset\rangle_{\mathcal{A}} \,.\, \nabla d(x, y) \leq q) \,. \tag{8}$$

Notice how we translated the classical $\exists$ to $\neg\neg\exists$ in Asm. Because $\mathcal{I}$ is the initial subalgebra of $\nabla\mathcal{A}$, it is contained in both $\nabla\langle\emptyset\rangle_{\mathcal{A}}$ and in $\mathcal{B}$. In fact, the mono $|\mathcal{I}| \rightarrowtail |\nabla\langle\emptyset\rangle_{\mathcal{A}}|$ is $\neg\neg$-dense because the underlying set of $|\mathcal{I}|$ is precisely the carrier of $\langle\emptyset\rangle_{\mathcal{A}}$.

From now on we argue in the internal language of Asm. Because $\mathcal{I}$ is $\neg\neg$-dense in $\nabla\langle\emptyset\rangle_{\mathcal{A}}$, statement (8) is equivalent to

$$\forall\, y \in \nabla|\mathcal{A}| \,.\, \forall\, q \in \mathbb{Q} \,.\, (q > 0 \implies \neg\neg\exists\, y \in |\mathcal{I}| \,.\, \nabla d(x, y) < q) \,.$$

We restrict the outer quantifier to $|\mathcal{B}|$,

$$\forall\, y \in |\mathcal{B}| \,.\, \forall\, q \in \mathbb{Q} \,.\, (q > 0 \implies \neg\neg\exists\, y \in |\mathcal{I}| \,.\, \nabla d(x, y) < q) \,, \tag{9}$$

and thereby make the relation $\nabla d(x, y) < q$ semidecidable. Recall from Section 3.1 that the subalgebra $\mathcal{I}$ is effectively enumerated by a map $e : \mathbf{N} \to \{\star\} + |\mathcal{I}|$. The statement (9) is equivalent to

$$\forall\, y \in |\mathcal{B}| \,.\, \forall\, q \in \mathbb{Q} \,.\, (q > 0 \implies \neg\neg\exists\, n \in \mathbb{N} \,.\, e(n) \neq \star \wedge \nabla d(e(n), y) < q) \,,$$

Because the relation inside the existential quantifier is semidecidable, by Markov's principle we may erase the double negation in front of $\exists$. Then we pass back to quantification over $\mathcal{I}$ to obtain effective density of $\mathcal{I}$ in $\mathcal{B}$:

$$\forall\, y \in |\mathcal{B}| \,.\, \forall\, q \in \mathbb{Q} \,.\, (q > 0 \implies \exists\, x \in \mathcal{I} \,.\, \nabla d(x, y) < q) \,.$$

This completes the proof. □

When the initial subalgebra $\langle\emptyset\rangle_{\mathcal{A}}$ is not dense, Theorem 11 cannot be applied. Quite often this can be fixed with a judicious addition of new constants and operations. For example, the initial subring of the ring $\mathcal{C}[0, 1]$ of continuous real functions on the closed unit interval is the ring of integers (embedded as constant functions), which is not dense. If we adjoin the identity function and the constant function $\frac{1}{2}$ as primitive constants, the initial subalgebra will be the ring of polynomials whose coefficients are dyadic rationals,[8] which is dense by the (classical) Stone-Weierstraß theorem.

The next theorem complements Theorem 11 by giving conditions for existence of subalgebras.

Theorem 12. *Let $\mathcal{A}$ be a classical complete premetric Σ-algebra. Suppose the relation $\nabla d(x, y) < q$ is semidecidable on $\langle\emptyset\rangle_{\nabla\mathcal{A}}$ and the operations of $\langle\emptyset\rangle_{\nabla\mathcal{A}}$ are effectively locally uniformly continuous. Then $\nabla\mathcal{A}$ has an effective complete subalgebra on which the relation $\nabla d(x, y) < q$ is semidecidable.*

Proof. We know from the proof of Theorem 11 that the desired subalgebra must be the completion $\overline{\mathcal{I}}$ of $\mathcal{I} = \langle\emptyset\rangle_{\nabla\mathcal{A}}$. By Proposition 10, $\overline{\mathcal{I}}$ is an effective subalgebra of $\nabla\mathcal{A}$. It remains to be shown that $\nabla d(x, y) < q$ is semidecidable on $\overline{\mathcal{I}}$. We argue in the internal language of Asm. Consider any $x, y \in \overline{\mathcal{I}}$ and $q \in \mathbf{Q}$. There exist fast Cauchy sequences $(u_n)_n, (v_n)_n \in \mathsf{Cauchy}(\mathcal{I})$ such that $x = \lim_n u_n$ and $y = \lim_n v_n$. A little thinking reveals that

$$\nabla d(x, y) < q \iff \exists\, s \in \mathbb{Q} \,.\, \exists\, m \in \mathbb{N} \,.\, \nabla d(u_m, v_m) < q - 2^{-m+1},$$

which is semidecidable because it is a countable existential quantification of a semidecidable statement. $\square$

6 Partial algebras

A *partial Σ-algebra* in a category C is given by a carrier object $|\mathcal{A}|$, and for each function symbol f with arity n, a *partial* morphism $\mathsf{f}^{\mathcal{A}} : |\mathcal{A}|^n \rightharpoonup |\mathcal{A}|$, which is a morphism $\mathsf{f}^{\mathcal{A}} : \mathrm{dom}(\mathsf{f}^{\mathcal{A}}) \to |\mathcal{A}|$ whose domain is a subobject $\mathrm{dom}(\mathsf{f}^{\mathcal{A}}) \rightarrowtail |\mathcal{A}|^n$, called the *support* of the operation. The operation $\mathsf{f}^{\mathcal{A}}$ is *total* if $\mathrm{dom}(\mathsf{f}^{\mathcal{A}}) = |\mathcal{A}|^n$. We require that function symbols with arity 0, namely the constants, be total. This is necessary if we want to avoid the bizarre phenomenon that the initial subalgebra of a partial algebra is always the empty one. We adapt the results established so far to partial algebras.

The first change is that the interpretation of a term $t \in \mathsf{Term}(\Sigma)$ with variables $x_1, \ldots, x_n$ is a partial morphism $t^{\mathcal{A}} : |\mathcal{A}|^n \rightharpoonup |\mathcal{A}|$. We may still use the same Gödel numbering for terms, as long as we keep in mind that some terms may signify undefined values. In order to be able to make useful statements about the partial algebra, we augment extended formulas over Σ with primitive formulas $t\!\downarrow$ whose informal reading is "the term t is defined". If t contains free variables

[8] A dyadic rational is one of the form $n/2^k$.

$x_1, \dots, x_n$, then $t\downarrow$ denotes the mono $\mathrm{dom}(t^{\mathcal{A}}) \rightarrowtail |\mathcal{A}|^n$. In particular cases $t\downarrow$ might be expressible with other logical primitives, for example in the theory of real numbers $x^{-1}\downarrow$ is equivalent to $x \neq 0$.

A *subalgebra* $\mathcal{B}$ of a partial Σ-algebra $\mathcal{A}$ is a partial Σ-algebra with a mono $|\mathcal{B}| \rightarrowtail |\mathcal{A}|$ such that the operations $\mathsf{f}^{\mathcal{B}}$ are restrictions of $\mathsf{f}^{\mathcal{A}}$. In particular this means that $\mathrm{dom}(\mathsf{f}^{\mathcal{B}})$ is the restriction of $\mathrm{dom}(\mathsf{f}^{\mathcal{A}})$ so that the following diagram is a pullback:

$$\begin{tikzcd} \mathrm{dom}(\mathsf{f}^{\mathcal{B}}) \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{dom}(\mathsf{f}^{\mathcal{A}}) \arrow[d, tail] \\ |\mathcal{B}|^n \arrow[r, tail] & |\mathcal{A}|^n \end{tikzcd}$$

Functors Γ and ∇ preserve partial Σ-algebras because they preserve finite limits. If $\mathcal{A}$ is a classical partial Σ-algebra then $\nabla\mathcal{A}$ is an effective partial Σ-algebra whose operations have $\neg\neg$-stable supports because ∇ maps subsets to $\neg\neg$-stable monos. Thus we simplify the treatment by considering only those partial algebras in Asm whose operations have $\neg\neg$-stable domains.

Proposition 13. *A partial Σ-algebra $\mathcal{A}$ in* Asm *satisfies the same extended negative formulas as $\Gamma\mathcal{A}$ does in* Set.

Proof. Analogous to the proof of Proposition 1. The only change is the extra base case $t\downarrow$, which goes through because we limit attention to partial algebras with $\neg\neg$-stable supports. □

Corollary 14. *An effective partial Σ-algebra and a $\neg\neg$-dense subalgebra satisfy the same extended negative formulas over Σ.*

Proof. Analogous to the proof of Corollary 2. □

Proposition 15. *A partial Σ-algebra $\mathcal{A}$ in* Set *satisfies the same extended negative formulas as $\nabla\mathcal{A}$ does in* Asm.

Proof. By Proposition 13 $\nabla\mathcal{A}$ satisfies the same extended negative formulas as $\Gamma(\nabla\mathcal{A})$, but $\Gamma(\nabla\mathcal{A}) = \mathcal{A}$. □

Every subset $C \subseteq |\mathcal{A}|$ of a classical partial Σ-algebra $\mathcal{A}$ is contained in the smallest subalgebra $\langle C\rangle_{\mathcal{A}} \leq \mathcal{A}$, namely the intersection of all subalgebras that contain it. Similarly, a subassembly $\mathbf{C} \rightarrowtail \mathcal{A}$ of an effective partial Σ-algebra $\mathcal{A}$ is contained in the smallest subalgebra $\langle \mathbf{C}\rangle_{\mathcal{A}} \leq \mathcal{A}$. We describe explicitly the initial subalgebra $\mathcal{I} = \langle\emptyset\rangle_{\mathcal{A}}$. The carrier of $\mathcal{I}$ is the assembly $\mathbf{I}$ whose underlying set is

$$I = \{x \in |\mathcal{A}| \mid \text{for a closed } \Sigma\text{-term } t,\ t^{\mathcal{A}} \text{ is defined and } x = t^{\mathcal{A}}\ \}$$

with the realizability relation (recall that ν is a standard enumeration of Σ-terms)

$$\overline{n} \Vdash_I x \iff \text{for a closed } \Sigma\text{-term } t,\ t^{\mathcal{A}} \text{ is defined, } \nu(n) = t \text{ and } t^{\mathcal{A}} = x.$$

An important difference with respect to total algebras is that $\mathbf{I}$ need not be effectively enumerated because the predicate "$t^{\mathcal{A}}$ is defined" may be arbitrarily complicated. A sufficient condition for $\mathbf{I}$ to be effectively enumerable is that the predicate

"n is Gödel code of a closed Σ-term t such that $t^{\mathcal{A}}$ is defined"

denotes a semidecidable mono into $\mathbf{N}$.

Next we turn attention to premetric notions for partial algebras. We argue constructively so that all results apply to Set and Asm. In particular, we only rely on sequential continuity of operations. Let $\mathcal{A}$ be a complete premetric partial Σ-algebra, either in Set or Asm, by which we mean that $|\mathcal{A}|$ is a premetric space and the operations $\mathsf{f}^{\mathcal{A}} : |\mathcal{A}|^k \rightharpoonup |\mathcal{A}|$ are partial maps that are sequentially continuous on their supports. Every subalgebra $\mathcal{B} \leq \mathcal{A}$ is contained in the smallest complete subalgebra $\overline{\mathcal{B}} \leq \mathcal{A}$, namely the intersection of all complete subalgebras containing it. Just as in the total case, we seek conditions which allow us to conclude that $\overline{\mathcal{B}}$ is the completion of $\mathcal{B}$. The conditions should imply that each partial operation $\mathsf{f}^{\mathcal{B}}$ has a continuous extension to $\mathrm{dom}(\mathsf{f}^{\mathcal{A}}) \cap |\overline{\mathcal{B}}|^n$.

Proposition 16. *Let $X \subseteq Y$ be a dense subset of a complete premetric space Y and $U = \bigcup_{i \in I} B_Y(x_i, r_i)$ a union of open balls in Y with centers $x_i \in X$. Suppose $f : U \cap X \to Z$ is a map into a complete premetric space Z which is uniformly continuous on every $B_X(x_i, r_i)$. Then there is a unique extension $\overline{f} : U \to Z$ which is uniformly continuous on every $B_Y(x_i, r_i)$.*

Proof. Let us first define $\overline{f}$ at $y \in U$. There is $i \in I$ such that $y \in B_Y(x_i, r_i)$ and a Cauchy sequence $(u_n)_n$ in X such that $y = \lim u_n$. For large enough n we have $u_n \in B_X(x_i, r_i)$, so we may assume without loss of generality that the whole sequence $(u_n)_n$ is in $B_X(x_i, r_i)$. Because $(f(u_n))_n$ is the image of a Cauchy sequence by a uniformly continuous map, it is Cauchy and we may define $\overline{f}(y) = \lim_n f(u_n)$. The value $\overline{f}(y)$ does not depend on the choice of $(u_n)_n$. Indeed, if $(v_n)_n$ is another Cauchy sequence in X that converges to y then $\lim_n f(v_n) = \lim_n f(u_n)$ because $(v_n)_n$ is eventually in $B_X(x_i, r_i)$, and f is uniformly continuous on $B_X(x_i, r_i)$. It is not hard to verify that $\overline{f}$ so defined is uniformly continuous on each $B_Y(x_i, r_i)$. The map $\overline{f}$ is the only continuous extension of f because X is dense in Y. $\square$

This brings us to the main theorems for partial algebras. The proof of Theorem 11 uses the fact that $\langle \emptyset \rangle_{\mathcal{A}}$ is effectively countable, so we need to add an additional assumption in the partial case:

Theorem 17. *Suppose $\mathcal{A}$ is a classical partial premetric Σ-algebra in which the initial subalgebra $\langle\emptyset\rangle_{\mathcal{A}}$ is classically dense and the initial effective subalgebra $\langle\emptyset\rangle_{\nabla\mathcal{A}}$ is effectively enumerable. Up to isomorphism, there is at most one effectively complete subalgebra $\mathcal{B} \leq \nabla\mathcal{A}$ on which the relation $d(x, y) < q$ is semidecidable.*

Proof. The proof proceeds just like the proof of Theorem 11 where we use the effective enumeration of $\langle\emptyset\rangle_{\nabla\mathcal{A}}$ in place of the enumeration e. □

In the second main theorem we must add assumptions that secure applicability of Proposition 16. We give sufficient conditions which require that supports of operations are effectively open and that the operations are effectively locally uniformly continuous. With this we cover the usual examples of partial algebras and stay close to the way partial operations are actually implemented, see Section 8.2 for further discussion.

Let $\mathcal{A}$ be a classical complete premetric partial Σ-algebra. We say that a partial operation $\mathsf{f}^{\mathcal{A}} : |\mathcal{A}|^n \rightharpoonup |\mathcal{A}|$ is *acceptable for* $\mathcal{B} \leq \nabla\mathcal{A}$ when there is an effective sequence $(x, r) : \mathbf{N} \to |\mathcal{B}|^n \times \mathbf{Q}$ with $r_i > 0$ such that

1. $\mathrm{dom}(\mathsf{f}^{\mathcal{A}}) = \bigcup_{n\in\mathbb{N}} B_{|\mathcal{A}|^n}(x_n, r_n)$, and
2. the statement

$$\forall\, i \in \mathbf{N}\,.\ \text{“}\mathsf{f}^{\mathcal{B}} \text{ is uniformly continuous on } B_{|\mathcal{B}|^n}(x_i, r_i)\text{”} \tag{10}$$

   is realized.

Concretely, the second condition amounts to the existence of a realizer which accepts (realizers of) $i, k \in \mathbb{N}$ and computes a (realizer of) $m \in \mathbb{N}$ such that, for all $u, v \in |\mathcal{B}|^n$, if $\nabla d_{|\mathcal{A}|^n}(x_i, u) < r_i$, $\nabla d_{|\mathcal{A}|^n}(x_i, v) < r_i$, and $\nabla d_{|\mathcal{A}|^n}(u, v) \leq 2^{-m}$ then $\nabla d_{|\mathcal{A}|^n}(f^{\mathcal{B}}(u), f^{\mathcal{B}}(v)) \leq 2^{-k}$.

Proposition 18. *Suppose $\mathcal{A}$ is a classical complete premetric partial Σ-algebra and $\mathcal{B} \leq \nabla\mathcal{A}$ such that the operations of $\mathcal{A}$ are acceptable for $\mathcal{B}$ and $\nabla d(x, y) < q$ is semidecidable in $x, y \in |\mathcal{B}|, q \in \mathbf{Q}$. Then the effective completion of $|\mathcal{B}|$ is the smallest effectively complete partial Σ-algebra containing $\mathcal{B}$.*

Proof. Let $\mathbf{C} \rightarrowtail \nabla\mathcal{A}$ be the effective completion of $|\mathcal{B}|$. We first show that each $\mathsf{f}^{\nabla\mathcal{A}} : \mathrm{dom}(\mathsf{f}^{\nabla\mathcal{A}}) \to |\nabla\mathcal{A}|$ restricts to $\mathrm{dom}(\mathsf{f}^{\nabla\mathcal{A}}) \cap \mathbf{C}^n \to \mathbf{C}$. Let $(x, r) : \mathbf{N} \to |\mathcal{B}|^n \times \mathbf{Q}$ be the effective sequence witnessing the fact that $\mathsf{f}^{\mathcal{A}}$ is acceptable for $\mathcal{B}$. We claim that

$$\models_{\mathsf{Asm}} \mathrm{dom}(\mathsf{f}^{\nabla\mathcal{A}}) \cap \mathbf{C}^n = \bigcup_{i\in\mathbb{N}} B_{\mathbf{C}^n}(x_i, r_i).$$

Because $\mathrm{dom}(\mathsf{f}^{\nabla\mathcal{A}}) = \nabla\mathrm{dom}(\mathsf{f}^{\mathcal{A}}) = \nabla\bigcup_{i\in\mathbb{N}} B_{|\mathcal{A}|^n}(x_i, r_i)$, this is equivalent to having a realizer for

$$\forall\, x \in \mathbf{C}^n \,.\, (\neg\neg\exists\, i \in \mathbb{N} \,.\, \nabla d(x, x_i) < r_i) \iff (\exists\, i \in \mathbb{N} \,.\, \nabla d(x, x_i) < r_i).$$

Because $\nabla d(x, y) < q$ is semidecidable on $|\mathcal{B}|$, it is also semidecidable on its completion $\mathbf{C}$, by the same argument as the one in the proof of Theorem 12. Hence the above equivalence holds by Markov's principle.

It is our intention to apply the interpretation of Proposition 16 in Asm with $X = |\mathcal{B}|^n$, $Y = \mathbf{C}^n$, $U = \mathrm{dom}(\mathsf{f}^{\nabla\mathcal{A}}) \cap \mathbf{C}^n = \bigcup_{i\in\mathbb{N}} B_{\mathbf{C}^n}(t_i, r_i)$, $Z = \mathbf{C}$, and $f = \mathsf{f}^{\mathcal{B}}$. For this we need to check several conditions. First, $|\mathcal{B}|^n$ is effectively dense in $\mathbf{C}^n$ because $\mathbf{C}$ is the effective completion of $|\mathcal{B}|$. Second, above we verified that $\mathrm{dom}(\mathsf{f}^{\nabla\mathcal{A}}) \cap \mathbf{C}^n = \bigcup_{i\in\mathbb{N}} B_{\mathbf{C}^n}(t_i, r_i)$ holds effectively. The last condition is that (10) be realized, which it is because $f^{\mathcal{B}}$ is acceptable. We may indeed apply Proposition 16 to obtain an extension of $\mathsf{f}^{\mathcal{B}}$ to $\mathrm{dom}(\mathsf{f}^{\nabla\mathcal{A}}) \cap \mathbf{C}^n \to \mathbf{C}$ that is effectively uniformly continuous on each $B_{\mathbf{C}^n}(x_i, r_i)$. We now know that $\mathbf{C}$ is the carrier of an effective complete Σ-algebra $\mathcal{C}$, but we still have to show that $\mathcal{C}$ is a subalgebra of $\nabla\mathcal{A}$. This is done like in the proof of Proposition 10. □

The second main theorem for partial algebras reads as follows.

Theorem 19. *Let $\mathcal{A}$ be a classical complete premetric partial Σ-algebra with acceptable operations. If $d(x, y) < q$ is semidecidable in $x, y \in \langle\emptyset\rangle_{\nabla\mathcal{A}}$ and $q \in \mathbf{Q}$ then $\nabla\mathcal{A}$ has an effectively complete subalgebra on which the relation $d(x, y) < q$ is semidecidable in x, y and $q \in \mathbf{Q}$.*

Proof. We know from the proof of Theorem 17 that the desired subalgebra must be the completion of the carrier of $\mathcal{I} = \langle\emptyset\rangle_{\nabla\mathcal{A}}$. By Proposition 18 the operations extend from $|\mathcal{I}|$ to its completion $\overline{|\mathcal{I}|}$. It remains to be shown that $d(x, y) < q$ is semidecidable in $x, y \in \overline{|\mathcal{I}|}$ and $q \in \mathbb{Q}$. The proof now proceeds as the proof of Theorem 12. □

We have considered only those partial operations whose supports are open sets. This excludes relevant examples on real numbers, such as $\sqrt{\ }$ and arcsin which are defined on *closed* subsets $[0, \infty)$ and $[-1, 1]$, respectively. However, notice that their domains are (effective) retracts of open sets, which means that we can easily extend them to open sets so that the second main theorem becomes applicable.

7 Multi-sorted algebras

Our theorems are formulated for single-sorted algebras. They can be extended to multi-sorted algebras with a considerable overhead in notation but no conceptual insights. We briefly outline how the multi-sorted case works. A multi-sorted signature Σ consists of:

– a list of sort symbols $\mathsf{S}_1, \ldots, \mathsf{S}_m$,

– a list of function symbols $\mathsf{f}_1, \ldots, \mathsf{f}_k$,

– each function symbol has a specified *type* $\mathsf{S}_{i_1} \times \cdots \times \mathsf{S}_{i_n} \to \mathsf{S}_j$.

Each variable is assigned a sort symbol, which we call the *type* of the variable. Terms are built inductively as in the single-sorted case: variables are terms; if f has type $\mathsf{S}_{i_1} \times \cdots \times \mathsf{S}_{i_n} \to \mathsf{S}_j$ and $t_1, \ldots, t_n$ are terms of types $\mathsf{S}_{i_1}, \cdots, \mathsf{S}_{i_n}$, respectively, then $\mathsf{f}(t_1, \ldots, t_n)$ is a term of type S_j.

A Σ-algebra $\mathcal{A}$ in a category C is given by carrier objects $\mathsf{S}^{\mathcal{A}}$, one for each sort symbol S, and morphisms $\mathsf{f}^{\mathcal{A}}$ with appropriate domains and codomains, one for each function symbol f. A subalgebra $\mathcal{B} \leq \mathcal{A}$ is a Σ-algebra $\mathcal{B}$ with monos $\mathsf{S}^{\mathcal{B}} \rightarrowtail \mathsf{S}^{\mathcal{A}}$, one for each sort symbol S, such that the operations $\mathsf{f}^{\mathcal{A}}$ restrict to operations $\mathsf{f}^{\mathcal{B}}$.

A premetric Σ-algebra is a Σ-algebra in which the carriers are premetric spaces and the operations are (sequentially) continuous maps. A premetric Σ-algebra is complete when its carriers are complete premetric spaces.

The multi-sorted variant of Theorem 11 requires that for each sort symbol S the carrier $\mathsf{S}^{\langle\emptyset\rangle_{\nabla\mathcal{A}}}$ of the initial subalgebra $\langle\emptyset\rangle_{\mathcal{A}} \leq \mathcal{A}$ is classically dense and concludes that there is at most one effectively complete subalgebra $\mathcal{B} \leq \nabla\mathcal{A}$ such that, for each sort symbol $\mathcal{S}$, the relation $\nabla d_{\mathsf{S}}(x, y) < q$ is semidecidable in $x, y \in \mathsf{S}^{\mathcal{B}}$ and $q \in \mathbb{Q}$.

The multi-sorted variant of Theorem 12 requires that for each sort symbol S the relation $\nabla d_{\mathsf{S}}(x, y) < q$ is semdecidable on the carrier $\mathsf{S}^{\langle\emptyset\rangle_{\nabla\mathcal{A}}}$, and that all operations of $\langle\emptyset\rangle_{\nabla\mathcal{A}}$ are effectively locally uniformly continuous. The conclusion is that $\nabla\mathcal{A}$ has an effective complete subalgebra $\mathcal{B}$ such that for each sort symbol S the relation $d_{\mathsf{S}}(x, y) < q$ is semidecidable in $x, y \in \mathsf{S}^{\mathcal{B}}$ and $q \in \mathbb{Q}$.

We leave it to the reader to formulate the main theorems for the multi-sorted partial case.

8 Applications

In this section we apply the results to several examples.

8.1 Discrete premetric spaces

The simplest kind of complete premetric algebras are the discrete ones. Let $\mathcal{A}$ be a classical Σ-algebra and define the *discrete premetric* on $|\mathcal{A}|$ by

$$d(x, y) \leq q \iff (q < 1 \implies x = y),$$

which of course corresponds to the metric that takes on only values 0 and 1. The discrete premetric is complete and every map with discrete codomain is uniformly continuous. Therefore, half of the conditions in Theorems 11 and 12 are trivially satisfied. Furthermore, a discrete premetric is semidecidable on $\mathcal{B} \leq \nabla\mathcal{A}$ if, and only if, equality is semidecidable on $\mathcal{B}$, because $x = y \iff d(x, y) < 1$ and $d(x, y) < q \iff (q > 1 \vee x = y)$. Thus we obtain the following result.

Proposition 20. *Suppose $\mathcal{A}$ is a* finitely generated *classical Σ-algebra. Up to isomorphism, there is at most one effective structure on $\mathcal{A}$ for which the operations and the generators are effective, and equality is semidecidable. Furthermore, if there is such an effective structure, it is isomorphic to the effective subalgebra $\langle\{a_1, \ldots, a_n\}\rangle_{\nabla\mathcal{A}}$ of $\nabla\mathcal{A}$ generated by the generators $a_1, \ldots, a_n$ for $\mathcal{A}$.*

Proof. More precisely, the first part of the theorem states that there is at most one realizability relation $\Vdash_{|\mathcal{A}|}$ on $|\mathcal{A}|$ such that the assembly $\mathbf{A} = (|\mathcal{A}|, \Vdash_{|\mathcal{A}|})$ has semidecidable equality, the operations of $\mathcal{A}$ are realized as maps $\mathsf{f}^{\mathcal{A}} : \mathbf{A}^n \to \mathbf{A}$, and the generators have realizers in A_{eff}.

We first consider the case when $\mathcal{A}$ is generated by the empty set, $\langle\emptyset\rangle_{\mathcal{A}} = \mathcal{A}$. This means that $\langle\emptyset\rangle_{\nabla\mathcal{A}}$ is a $\neg\neg$-dense subalgebra of $\nabla\mathcal{A}$. We equip $\mathcal{A}$ with the discrete premetric, which turns $\nabla\mathcal{A}$ and all of its subalgebras into effectively complete premetric spaces. Because the smallest subalgebra $\langle\emptyset\rangle_{\nabla\mathcal{A}}$ is $\neg\neg$-dense, all of them are. Therefore, by Theorem 11, there is at most one $\neg\neg$-dense subalgebra $\mathcal{B} \leq \nabla\mathcal{A}$ with semidecidable equality. This proves the first part of the theorem. To prove the second part, observe that as soon as there is a subalgebra $\mathcal{B} \leq \nabla\mathcal{A}$ with semidecidable equality, then $\langle\emptyset\rangle_{\nabla\mathcal{A}}$ has semidecidable equality as well, because it is contained in $\mathcal{B}$.

If $\mathcal{A}$ is generated by the elements $a_1, \ldots, a_n \in |\mathcal{A}|$, we change the signature Σ to $\Sigma' = \Sigma + \{\mathsf{a}_1, \ldots, \mathsf{a}_n\}$ where the arities of the newly added function symbols are 0. If we interpret a_i as the element a_i, then $\mathcal{A}$ is a Σ'-algebra generated by the empty set, which is the case we already considered. □

In the context of type 1 effectivity Proposition 20 was first proved by Mal'cev, see [Mal'cev 1961] and [Mal'cev 1971, Theorem 4.1.2]. He actually considered two versions, one with general recursive functions and another with partial recursive functions. Our result corresponds to the partial recursive case because all partial recursive functions are representable in a PCA.

8.2 The real numbers

The real numbers $(\mathbb{R}, 0, 1, +, -, \times, ^{-1})$ form a classical ordered field, and a classical complete premetric space with the usual premetric $d(x, y) \leq q \iff |x-y| \leq q$. The relation $|x - y| < q$ is semidecidable, even decidable when x, y and q are rationals. The initial algebra is the ring of rational numbers, which is dense in $\mathbb{R}$ and is clearly effectively enumerable.

In order to apply Theorem 19 we need to verify that the opeations are acceptable. Only acceptability of the inverse requires some thought. It is decidable whether an open ball with rational radius and centred on a rational point contains 0. So the sequence verifying that inverse is an acceptable operation can, for example, be computed by effectively listing all pairs of rationals (t_n, r_n) such that $0 < r_n < |t_n|$. Second, the inverse is uniformly continuous on any open ball not containing 0, and the modulus of uniform continuity on (t_n, r_n) can easily be computed from n.

We may replace semidecidability of $d(x, y) < q$ with semidecidability of the strict order relation $x < y$ because $d(x, y) < q \iff -q < x - y < q$ and

$$x < y \iff \\ \exists\, q, r \in \mathbb{Q} \,.\, \exists\, k \in \mathbb{N} \,.\, \big(d(x, q) < 2^{-k} \wedge d(y, r) < 2^{-k} \wedge q + 2^{-k+2} < r\big)\,.$$

From the above observation we get the following result.

Proposition 21. *Up to isomorphism, there is exactly one effectively complete effective subfield of the real numbers for which the strict linear order is is semidecidable.*

Proof. By Theorems 17 and 19. □

When the proposition is specialized to type 2 effectivity it gives Hertling's result [Hertling 1999] about type 2 representations of reals, while the interpretation in type 1 effectivity corresponds to a result of Moschovakis [Moschovakis 1965].

We note that the sequence witnessing the acceptability of a partial operation often is related to practical computations of the operations. For example, the computation of inverse on the reals starts by bounding the input x away from zero. Once a lower bound for the distance from zero is found, an approximate value of x^{-1} is computed and local uniform continuity (expressed as a local Lipschitz coefficient) is used to compute an error bound for the approximation.

There are other interesting topological fields to consider, of which we mention the *p-adic numbers.* For a prime p define the p-adic absolute value of a non-zero $x \in \mathbb{Q}$ by $|x|_p = p^{m-n}$ if $x = ap^n/bp^m$ and p does not divide ab. For zero, let $|0|_p = 0$. A p-adic premetric can now be defined by $d_p(x, y) \le q \iff |x-y|_p \le q$. The field of p-adic numbers is the metric completion of $\mathbb{Q}$ with respect to d_p. Similarly to the real case we have that, up to isomorphism, there is exactly one effectively complete effective subfield of the p-adic numbers for which the premetric is semidecidable.

8.3 Banach spaces

Let us apply our results to Banach spaces. First we look at the particular Banach space $X = L^p(\mathbb{R})$, where $p > 1$ is rational. In this case the signature Σ is multi-sorted since we have the sort X of vectors and the sort $\mathbb{R}$ of scalars. The operations are those of a normed vector space over a field of scalars:

- on the vector space we have 0, $+$ and $-$,
- on the scalars we have 0, 1, $+$, $-$, $\times$, and ${}^{-1}$,
- there are two operations which map between the sorts, namely the norm $\|-\|_p : X \to \mathbb{R}$ and scalar multiplication $\cdot : \mathbb{R} \times X \to X$.

With this signature the initial subalgebra $I_\mathbb{R} \leq \mathbb{R}$ is dense because it contains the rational numbers, whereas the initial subalgebra $I_X \leq X$ is the trivial vector space $\{0\}$, which is not dense in X. According to the advice from Section 5 we need to add something to the signature. It is well known that the rational step functions are dense in $L^p(\mathbb{R})$. To have them appear in I_X, we adjoin to the signature the constant $c \in X$, which is the function defined by

$$c(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{otherwise,} \end{cases}$$

and binary operations $t : \mathbb{R} \times X \to X$ and $s : \mathbb{R} \times X \to X$ for translation and scaling, defined by

$$t(\lambda, f)(x) = f(x + \lambda) \qquad \text{and} \qquad s(\lambda, f)(x) = f(\lambda x).$$

Translations and scalings of c by rational amounts form a spanning set. The initial subalgebra I_X for the extended signature will contain the rational combinations of the spanning set, and hence, I_X will be dense in X.

In fact, $I_\mathbb{R}$ and I_X contain more than just the rational numbers and the rational step functions. For example, $I_\mathbb{R}$ contains norms $\|r\|_p$ of rational step functions r, which are algebraic numbers of the form

$$\|r\|_p = (b_1 a_1^p + \cdots + b_k a_k^p)^{1/p}$$

where $a_1, \ldots, a_k, b_1, \ldots, b_k$ are positive rationals that can be computed from (a realizer of) r. Once such numbers appear in $I_\mathbb{R}$ they can be used to scale and translate step functions by non-rational amounts, which further enlarges I_X, and so on. Nevertheless, all the numbers so generated are algebraic. It is decidable whether the inverse of an algebraic number is defined, hence $I_\mathbb{R}$ is effectively enumerable. The conditions of (the multi-sorted partial version of) Theorem 11 are therefore satisfied.

In order to apply (the multi-sorted partial version of) Theorem 12 we need to verify that the operations are acceptable and that the relations $\|f - g\|_p < q$ and $|u - v| < q$ are semidecidable in $q \in \mathbb{Q}$, $f, g \in I_X$, and $u, v \in I_\mathbb{R}$, respectively. Acceptability of operations is not problematic, while for the second part it suffices to check that the relations $\|f\|_p < 1$ and $|u| < 1$ are semidecidable in $f \in I_X$, $u \in I_{\mathbb{R}}$. Once again, because $\|f\|_p$ and $|u|$ are algebraic numbers computable from (realizers of) the terms denoting f and u, these relations are even decidable.

If we are willing to adjoin a third sort to the signature, namely the natural numbers $\mathbb{N}$, there is another way of making X satisfy the conditions of the main theorems: rather than trying to generate the desired dense subset by scalings and translations of a basic step function, we may directly adjoin an operation $e : \mathbb{N} \to X$ which enumerates a spanning set. The initial subalgebra I_X is dense because it contains the rational linear combinations of the spanning set enumerated by e. Once again we have to think about effective enumerability of $I_{\mathbb{R}}$ and semidecidability of predicates $\|f\|_p < 1$ and $|u| < 1$, but as long as basic information about $e(n)$ is computable from n we should not expect any trouble.

The situation with $L^p(\mathbb{R})$ is typical for Banach spaces that occur in practice. In general, we will have to ensure density of the initial subalgebra by adding something to the signature, and as long as the norms of the elements in the initial algebra are computable reals, the initial subalgebra of $\mathbb{R}$ will be effectively enumerable, and relations $\|f\| < 1$ and $|u| < 1$ will be semidecidable. It would be interesting to work out the details for other separable spaces, such as Hilbert, Sobolev, Frechet and Riesz spaces.

9 Conclusion

The relation d on a premetric space (X, d) induces a uniform structure on X whose (basic) entourages are $E_q = \{(x, y) \in X \times X \mid d(x, y) \leq q\}$, for rational $q > 0$. This suggests that one should look for a generalization to uniform spaces based on a suitable constructive treatment of uniform spaces and their completions.

Another direction which might be worth investigating follows the work of Blanck et al. [Blanck, Stoltenberg-Hansen, and Tucker 2008] who formulated general results about stability of effective algebras in type 1 effectivity. Their theorems do not translate into our settings easily, because they assume a structure which is not metric, but rather like that of sequential or limit spaces. Again, to incorporate such results we would require a constructive theory of limit spaces and their completions.

References

[Bauer 2000] Bauer, A.: *The Realizability Approach to Computable Analysis and Topology* PhD thesis, Carnegie Mellon University, 2000 Technical report CMU-CS-00-164.

[Blanck, Stoltenberg-Hansen, and Tucker 2008] Blanck, J., Stoltenberg-Hansen, V., and Tucker, J. V.: Stability for effective algebras; *Electronic Notes in Theoretical Computer Science*, 221:3 – 15, 2008 Proceedings of the Fifth International Conference on Computability and Complexity in Analysis (CCA 2008).

[Hertling 1999] Hertling, P.: A real number structure that is effectively categorical; *Mathematical Logic Quarterly*, 45(2):147–182, 1999.

[Ishihara 1992] Ishihara, H.: Continuity properties in constructive mathematics; *The Journal of Symbolic Logic*, 57(2):557–565, 1992.

[Mal'cev 1961] Mal'cev, A. I.: Constructive algebras. I; *Uspehi Mat. Nauk*, 16(3 (99)):3–60, 1961.

[Mal'cev 1971] Mal'cev, A. I.: *The Metamathematics of Algebraic Systems, Collected Papers: 1936–1967* North-Holland, 1971 Translated by Wells, B. F.

[Moschovakis 1965] Moschovakis, Y. N.: Notation systems and recursive ordered fields; *Compositio Mathematica*, 17:40–71, 1965.

[Richman 2008] Richman, F.: Real numbers and other completions; *Mathematical Logic Quarterly*, 54:98–108, 2008.

[van Oosten 2008] van Oosten, J.: *Realizability: An Introduction to its Categorical Side*, volume 152 of *Studies in Logic and the Foundations of Mathematics* Elsevier, 2008.

[Weihrauch 2000] Weihrauch, K.: *Computable Analysis* Springer, Berlin, 2000.