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A Note on Closed Subsets in Quasi-zero-dimensional Qcb-spaces

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Abstract: We introduce the notion of quasi-zero-dimensionality as a substitute for the notion of zero-dimensionality, motivated by the fact that the latter behaves badly in the realm of qcb-spaces. We prove that the category QZ of quasi-zero-dimensional qcb_0-spaces is cartesian closed. Prominent examples of spaces in QZ are the spaces of the Kleene-Kreisel continuous functionals equipped with the respective sequential topology. Moreover, we characterise some types of closed subsets of QZ-spaces in terms of their ability to allow extendability of continuous functions. These results are related to a problem in Computable Analysis.

Key Words: Computable Analysis, Qcb-spaces, Extendability
Category: F.1.1

1 Introduction

The category QCB of quotients of countably based spaces [Simpson 03] has excellent closure properties. For example, it is cartesian closed, in contrast to the category Top of all topological spaces, see [Escardó et al. 04, Schröder 03]. This means that QCB allows us to form products and function spaces with the usual transposing properties. Qcb-spaces are known to form exactly the class of topological spaces which can be handled appropriately by the representation based approach to Computable Analysis, the Type Two Model of Effectivity (TTE) [Weihrauch 00].

Unfortunately, exponentiation in QCB behaves badly in terms of preservation of classical topological notions like regularity, normality and zero-dimensionality. For example, the function space $\mathbb{N}^{(\mathbb{N}^{\mathbb{N}})}$ formed in QCB is not zero-dimensional and not even normal (see [Schröder 09a]) despite the fact that both the exponent $\mathbb{N}^{\mathbb{N}}$ and the basis $\mathbb{N}$ are zero-dimensional Polish spaces. In [Schröder 09b] the notion of quasi-normality is introduced as a substitute for normality in the realm of qcb-spaces [see Section 2.7]. This notion has the advantage of being preserved by exponentiation in QCB, while admitting an Extension Theorem for continuous real-valued functions similar to the classical Tietze-Urysohn Extension Theorem for normal topological spaces.

In an analogous way we introduce the notion of a quasi-zero-dimensional qcb-space [see Section 3]. The category QZ of quasi-zero-dimensional qcb-spaces turns out to be an exponential ideal of QCB. In [Section 4] we investigate extendability of continuous functions that have as target space either a quasi-zero-dimensional qcb-space or the real numbers. We prove that a subspace X of a QZ-space Y admits continuous extendability of all continuous functions from X to $\mathbb{N}$ if, and only if, X is closed in the zero-dimensional reflection of Y. Analogously, we characterise functionally closed subspaces of a quasi-normal qcb-space as those subspaces that admit continuous extendability of all continuous real-valued functions defined on them.

In [Section 5] we discuss the relationship of our results with a question in Computable Analysis. The question is whether two hierarchies of functionals over the reals coincide, see [Bauer et al. 02].

2 Preliminaries

We recall some notions and basic facts about sequential spaces, qcb-spaces, pseudobases, and quasi-normal spaces. Moreover, we remind the reader of the definition of the completely regular reflection and of the zero-dimensional reflection of a sequential space.

2.1 Notations

We use sans-serif letters like X, Y etc. to denote topological spaces. We write $\mathcal{O}(\mathsf{X})$ for the topology of a topological space X and $\mathcal{A}(\mathsf{X})$ for the family of closed sets of X. In abuse of notation, we will denote the carrier set of a space X by the same symbol X.

We use the following symbols for relevant topological spaces: $\mathbb{R}$ for the space of real numbers endowed with the Euclidean topology, $\mathbb{I}$ for the unit interval $[0, 1]$ endowed with the Euclidean subspace topology, $\mathbb{N}$ for the discrete topological space of natural numbers $\{0, 1, 2, \dots\}$, $\mathbb{J}$ for the one-point compactification of $\mathbb{N}$ with carrier set $\mathbb{N} \cup \{\infty\}$, and the symbol 2 for the two-point discrete space with points 0 and 1.

2.2 Sequential spaces, sequential coreflections

A subset A of a topological space X is called *sequentially closed*, if A contains any limit of any convergent sequence of points in A. Complements of sequentially closed sets are called *sequentially open*. For a given topology τ, we denote the topology of sequentially open sets by $seq(\tau)$. Spaces such that every sequentially open set is open are called *sequential*. The sequential coreflection (or sequentialisation) $seq(\mathsf{X})$ of X is the topological space that carries the topology $seq(\mathcal{O}(\mathsf{X}))$

consisting of all sequentially open sets of X. The operator *seq* is idempotent. Importantly, a function between two sequential spaces is topologically continuous if, and only if, it is sequentially continuous.

For more information about sequential spaces we refer to [Engelking 89, Willard 70].

2.3 Qcb-spaces

A qcb-space [Simpson 03] is a topological *q*uotient of a *c*ountably-*b*ased topological space. Qcb$_0$-spaces, i.e. qcb-spaces that satisfy the T_0-property, are well-established to be exactly the class of sequential spaces which can be handled by the Type Two Model of Effectivity [Weihrauch 00].

Qcb-spaces are hereditarily Lindelöf (i.e. any open cover of any subset has a countable subcover) and sequential. The category QCB of qcb-spaces as objects and of continuous functions as morphisms is cartesian closed. Moreover, QCB has all countable limits and all countable colimits. For two qcb-spaces A and B we denote by $\mathsf{A} \times \mathsf{B}$ their product, by $\mathsf{A} + \mathsf{B}$ their coproduct, and by B^{A} their function space formed in QCB. These spaces agree with their counterparts formed in the category of sequential spaces and, in the case that A, B are Hausdorff spaces, with their counterparts formed in the category of Hausdorff k-spaces [Escardó et al. 04]. QCB-subspaces carry the *subsequential* topology: this is the sequentialisation of the usual subspace topology. Note that the subsequential topology and the subspace topology agree, if the underlying set of the subspace is closed in the superspace. This is due to the fact that closed topological subspaces of sequential spaces are sequential [Engelking 89].

More information can be found in [Escardó et al. 04, Schröder 02, Schröder 03, Simpson 03, Simpson et al. 07].

2.4 Pseudobases and pseudo-open decompositions

Given a topological space X, we say that a family $\mathcal{A}$ of subsets of X is a *pseudo-open decomposition* of a subset M, if $M = \bigcup \mathcal{A}$ holds and for every sequence $(x_n)_n$ that converges to some element $x_\infty \in M$ there is some set $B \in \mathcal{A}$ and some $n_0 \in \mathbb{N}$ such that $\{x_n, x_\infty \,|\, n \geq n_0\} \subseteq B \subseteq M$ holds. Clearly, a set has a pseudo-open decomposition if, and only if, it is sequentially open.

A *(sequential) pseudobase* for X is a family $\mathcal{B}$ of subsets such that every open set has a pseudo-open decomposition into members of $\mathcal{B}$. Any base of topological space is a pseudobase, but not vice versa. Pseudobases characterise qcb-spaces: a sequential space is a qcb-space if, and only if, it has a countable pseudobase. Any countably pseudobased space is hereditarily Lindelöf and its sequential coreflection is a qcb-space. In this paper we will only deal with spaces having a countable pseudobase. More information can be found in [Escardó et al. 04, Schröder 03, Simpson 03].

2.5 Completely regular reflections, functionally open sets

Let X be a sequential space. The *completely regular reflection* of X is defined to carry the topology that is induced by the base

$$\mathcal{B} := \{h^{-1}(0,1] \,|\, h\colon \mathsf{X} \to \mathbb{I} \text{ is continuous}\}.$$

We denote this topological space by $\mathcal{R}(\mathsf{X})$. It has the property that every real-valued function f on X is continuous w.r.t. the original topology $\mathcal{O}(\mathsf{X})$ if, and only if, f is continuous w.r.t. the topology $\mathcal{O}(\mathcal{R}(\mathsf{X}))$. If $\mathcal{R}(\mathsf{X})$ is a T_0-space, then $\mathcal{R}(\mathsf{X})$ is a Tychonoff space.

A subset A of X is called *functionally closed*, if there is a continuous function h from X to the unit interval $\mathbb{I} = [0,1]$ such that $h^{-1}\{0\} = A$. Complements of functionally closed sets are called *functionally open*. A common term for "functionally closed set" is *zero-set*, and for "functionally open set" is *cozero-set*. We denote the family of functionally closed sets of X by $\mathcal{FA}(\mathsf{X})$ and the family of functionally open sets by $\mathcal{FO}(\mathsf{X})$. If X is a hereditarily Lindelöf space, then $\mathcal{FO}(\mathsf{X})$ forms the topology of the completely regular reflection $\mathcal{R}(\mathsf{X})$ of X. Otherwise $\mathcal{FO}(\mathsf{X})$ need not be a topology.

Regularity, normality and perfect normality[1] are equivalent for hereditarily Lindelöf spaces, thus for countably pseudobased spaces and for qcb-spaces.

2.6 Zero-dimensional spaces, zero-dimensional reflections

A *zero-dimensional space* is a topological space that has a base consisting of *clopen* (= closed-and-open) sets. Any zero-dimensional T_0-space is regular. Zero-dimensional hereditarily Lindelöf spaces X are even *strongly zero-dimensional*, meaning that any pair of disjoint closed sets A, B can be separated by a clopen set C (i.e. $A \subseteq C \subseteq \mathsf{X} \setminus B$). Strongly zero-dimensional T_1-spaces are zero-dimensional and normal (see [Engelking 89]).

A topological space is called *totally disconnected*, if every singleton is an intersection of clopen sets, and *hereditarily disconnected*, if each of its components contains at most one point. Zero-dimensionality implies total disconnectedness which in turn implies hereditary disconnectedness (see [Engelking 89]).

The *zero-dimensional reflection* of a topological space X is defined to be the space that carries the topology induced by the base

$$\mathcal{B} := \{h^{-1}\{1\} \,|\, h\colon \mathsf{X} \to \mathsf{2} \text{ is continuous}\}.$$

We denote this space by $\mathcal{Z}(\mathsf{X})$. Clearly, $\mathcal{Z}(\mathsf{X})$ is zero-dimensional. If X is hereditarily Lindelöf, then the zero-dimensional reflection $\mathcal{Z}(\mathsf{X})$ is hereditarily Lindelöf as well and thus strongly zero-dimensional.

[1] A *normal* space is a T_1-space such that for a pair of disjoint closed sets (A, B) there exists a pair of disjoint open sets (U, V) such that $A \subseteq U$ and $B \subseteq V$. A perfectly normal space is a T_1-space in which every closed set is functionally closed. Note that some authors omit the T_1-condition.

2.7 Quasi-normal spaces and the category QN

A *quasi-normal* space is defined to be the sequential coreflection of some normal space [Schröder 09b]. The category of quasi-normal qcb-spaces, which we denote by QN, is cartesian closed and inherits finite products and exponentials from its supercategory QCB. By contrast, the category of normal qcb-spaces is not cartesian closed. Any continuous function $f\colon \mathsf{X} \to \mathbb{R}$ from a functionally closed subspace X of a space $\mathsf{Y} \in \mathsf{QN}$ can be extended to a continuous function $F\colon \mathsf{Y} \to \mathbb{R}$. Details can be found in [Schröder 09b].

3 Quasi-zero-dimensional Qcb-Spaces

In this section we introduce and investigate the notion of a quasi-zero-dimensional qcb-space.

The category QCB of qcb-spaces is known to be cartesian closed. However, forming function spaces in QCB does not preserve classical topological notions like regularity, normality and zero-dimensionality. For example, the function space $\mathbb{N}^{(\mathbb{N}^{\mathbb{N}})}$ formed in QCB is neither zero-dimensional nor normal (see [Schröder 09a]), although both $\mathbb{N}$ and $\mathbb{N}^{\mathbb{N}}$ are zero-dimensional and normal. Hence the final topology of the natural TTE-representation for $\mathbb{N}^{(\mathbb{N}^{\mathbb{N}})}$, which is equal to the topology of $\mathbb{N}^{(\mathbb{N}^{\mathbb{N}})}$, is not zero-dimensional. By contrast, the compact-open topology on $\mathbb{N}^{(\mathbb{N}^{\mathbb{N}})}$ is even strongly zero-dimensional.

This fact motivates the introduction of an appropriate substitute for the property of zero-dimensionality in the realm of qcb-spaces. We use the same idea as in [Schröder 09b], where the notion of quasi-normality is defined as a replacement for normality. The idea behind the following definition is the fact that finite products and function spaces in the category QCB are constructed as the sequential coreflection of their counterparts in classical topology, which enjoy the property of preserving zero-dimensionality.

Definition 1. A qcb-space X is called *quasi-zero-dimensional*, if X is the sequential coreflection of a zero-dimensional T_0-space.

So a qcb-space is quasi-zero-dimensional if, and only if, its convergence relation is induced by some zero-dimensional T_0-topology. Simple examples of quasi-zero-dimensional spaces are zero-dimensional separable metrisable spaces, because they are equal to their own sequentialisation. By QZ we denote the full subcategory of QCB consisting of all quasi-zero-dimensional qcb-spaces.

Since zero-dimensional T_0-spaces are totally disconnected Hausdorff spaces [see Section 2.6], we have:

Lemma 2. *Any quasi-zero-dimensional qcb-space is a totally disconnected and hereditarily disconnected Hausdorff space.*

Recall that a quasi-normal space is defined to be the sequential coreflection of a normal space [Schröder 09b]. From the fact that zero-dimensional hereditarily Lindelöf T_0-spaces are normal, we obtain:

Lemma 3. *Any* QZ*-space is a* QN*-space.*

3.1 Characterisation of quasi-zero-dimensionality

We will give now several characterisations of QZ-spaces. They are analogous to characterisations of quasi-normality given in [Schröder 09b]. We begin with the following observation.

Lemma 4. *A qcb$_0$-space* X *is quasi-zero-dimensional if, and only if, it is the sequential coreflection of its zero-dimensional reflection* $\mathcal{Z}(\mathsf{X})$*.*

Proof. If X' is a zero-dimensional T_0-space with $seq(\mathsf{X}') = \mathsf{X}$, then we have $\mathcal{O}(\mathsf{X}') \subseteq \mathcal{O}(\mathcal{Z}(\mathsf{X})) \subseteq \mathcal{O}(\mathsf{X})$. Hence X has the same sequentially open sets as $\mathcal{Z}(\mathsf{X})$ implying that X is the sequential coreflection of $\mathcal{Z}(\mathsf{X})$.
Conversely, if a qcb$_0$-space X is the sequential coreflection of $\mathcal{Z}(\mathsf{X})$, then $\mathcal{Z}(\mathsf{X})$ has the T_0-property as well. Clearly, $\mathcal{Z}(\mathsf{X})$ is zero-dimensional. □

For the second characterisation, we define two families of (respectively, closed and open) subsets of a topological space X by

$$\mathcal{ZA}(\mathsf{X}) := \big\{h^{-1}\{\infty\} \,\big|\, h\colon \mathsf{X} \to \mathbb{J} \text{ is continuous}\big\},$$
$$\mathcal{ZO}(\mathsf{X}) := \big\{h^{-1}[\mathbb{N}] \,\big|\, h\colon \mathsf{X} \to \mathbb{J} \text{ is continuous}\big\}.$$

Here $\mathbb{J}$ denotes the one-point compactification of $\mathbb{N}$. Obviously, every set in $\mathcal{ZA}(\mathsf{X})$ is closed in the zero-dimensional reflection of X. We will sometimes use the term *$\mathcal{Z}$-closed* for the members of $\mathcal{ZA}(\mathsf{X})$ and *$\mathcal{Z}$-open* for the members of $\mathcal{ZO}(\mathsf{X})$. If X is hereditarily Lindelöf, then $\mathcal{ZO}(\mathsf{X})$ is a topology.

Lemma 5. *Let* X *be a hereditarily Lindelöf space. Then* $\mathcal{ZO}(\mathsf{X})$ *is the family of all open sets of* $\mathcal{Z}(\mathsf{X})$*. Dually,* $\mathcal{ZA}(\mathsf{X})$ *is the family of all closed sets of* $\mathcal{Z}(\mathsf{X})$*.*

Proof. Let U be an open set in $\mathcal{Z}(\mathsf{X})$. As $\mathcal{Z}(\mathsf{X})$ is zero-dimensional and hereditarily Lindelöf, there is a sequence of clopen sets $(C_i)_i$ in X with $U = \bigcup_{i\in\mathbb{N}} C_i$. Clearly, the function $g\colon \mathsf{X} \to \mathbb{J}$ defined by $g(x) := \min\big\{\infty, i \in \mathbb{N} \,\big|\, x \in C_i\big\}$ is continuous satifying $g^{-1}[\mathbb{N}] = U$. Hence $U \in \mathcal{ZO}(\mathsf{X})$.

Conversely, if $h\colon \mathsf{X} \to \mathbb{J}$ is a continuous function, then $h^{-1}\{a\}$ is clopen for every $a \in \mathbb{N}$. Thus $h^{-1}[\mathbb{N}]$ is open in $\mathcal{Z}(\mathsf{X})$ by being equal to $\bigcup_{a\in\mathbb{N}} h^{-1}\{a\}$. □

Lemma 5 implies the following reformulation of Lemma 4.

Corollary 6. *A qcb$_0$-space* X *is quasi-zero-dimensional if, and only if, its convergence relation is induced by the topology* $\mathcal{ZO}(\mathsf{X})$*.*

Every $\mathcal{Z}$-closed subset of a quasi-zero-dimensional space is functionally closed, because $\mathbb{J}$ is homeomorphic to the closed subspace $\{0, 2^{-n} \mid n \in \mathbb{N}\}$ of $\mathbb{I}$. It is unknown for which QZ-spaces the converse is true as well.

We now characterise quasi-zero-dimensionality in terms of pseudobases. Recall that qcb-spaces are known to be those sequential spaces that have a countable pseudobase [see Section 2.4].

Proposition 7. *A qcb$_0$-space* X *is quasi-zero-dimensional if, and only if, it has a countable pseudobase consisting of sets in* $\mathcal{ZA}(\mathsf{X})$*.*

Note that quasi-normal qcb-spaces are characterised in a similar way, namely via the existence of a countable pseudobase consisting of functionally closed sets (see Proposition 4 in [Schröder 09b]). The proof of Proposition 7 is based on a series of lemmas. We start with the following observation.

Lemma 8. *Let* X *be a* T_0*-space equipped with a countable pseudobase consisting of sets in* $\mathcal{ZA}(\mathsf{X})$*. Then the singleton* $\{x\}$ *is a* $\mathcal{Z}$*-closed set for every* $x \in \mathsf{X}$*. Moreover,* X *is a totally disconnected Hausdorff space.*

Proof. For a given point $x \in \mathsf{X}$, we define the set A to be the countable intersection of all pseudobase sets containing x and show that $A = \{x\}$. Since X is a T_0-space, for any $y \neq x$ there is an open set V such that either $x \in V \not\ni y$ or $x \notin V \ni y$. In the first case there exists a pseudobase set B with $x \in B \subseteq V$, hence $y \notin A$. In the second case, there are, as the complements of the pseudobase sets are open, pseudobase sets B, D with $y \in B \subseteq V$ and $x \in D \subseteq \mathsf{X} \setminus B \subseteq \mathsf{X} \setminus \{y\}$. We conclude $A = \{x\}$. Hence $\{x\}$ is $\mathcal{Z}$-closed. Since for any $y \neq x$ there is a clopen set C with $x \in C \not\ni y$, X is a totally disconnected Hausdorff space.

Disjoint $\mathcal{Z}$-closed subsets satisfy the following separation lemma.

Lemma 9. *Let* X *be a hereditarily Lindelöf space, and let* A, B *be disjoint closed subsets of* $\mathcal{Z}(\mathsf{X})$*. Then there is a continuous function* $h\colon \mathsf{X} \to \mathbb{J}$ *with* $h^{-1}\{\infty\} = A$ *and* $B \subseteq h^{-1}\{0\}$*.*

Proof. By Lemma 5 there are continuous functions $f, g\colon \mathsf{X} \to \mathbb{J}$ with $f^{-1}\{\infty\} = A$ and $g^{-1}\{\infty\} = B$. One easily verifies that the function $h\colon \mathsf{X} \to \mathbb{J}$ defined by

$$h(x) := \begin{cases} f(x) & \text{if } f(x) \geq g(x) \\ 0 & \text{otherwise} \end{cases}$$

has the required properties. $\square$

This lemma is instrumental in proving the following lemma about sequentially open sets that are $\mathcal{G}_\delta$-sets in the zero-dimensional reflection of a QZ-space.

Lemma 10. *Let X be a qcb-space equipped with a countable pseudobase consisting of sets in $\mathcal{ZA}(\mathsf{X})$. Then every open set $V \in \mathcal{O}(\mathsf{X})$ that is a $\mathcal{G}_\delta$-set in $\mathcal{Z}(\mathsf{X})$ is open in $\mathcal{Z}(\mathsf{X})$. Dually, every closed set $A \in \mathcal{A}(\mathsf{X})$ that is an $\mathcal{F}_\sigma$-set in $\mathcal{Z}(\mathsf{X})$ is closed in $\mathcal{Z}(\mathsf{X})$.*

Proof. Let $G_0, G_1, \dots$ be a sequence of sets in $\mathcal{ZO}(\mathsf{X})$ such that $V := \bigcap_{j=0}^{\infty} G_j$ is open in X. Let $(\beta_i)_i$ be a pseudo-open decomposition of V [see Section 2.4] into pseudobase sets. Since $\bigcup_{i=0}^{n} \beta_i$ is closed in $\mathcal{Z}(\mathsf{X})$ and contained in G_n, there is a continuous function $h_n \colon \mathsf{X} \to \mathbb{J}$ with $h_n^{-1}\{\infty\} = \mathsf{X} \setminus G_n$ and $\bigcup_{i=0}^{n} \beta_i \subseteq h_n^{-1}\{0\}$ by Lemma 9. We define a function $f \colon \mathsf{X} \to \mathbb{J}$ by $f(x) := \sup_{n\in\mathbb{N}} h_n(x)$ and show that f is sequentially continuous with $f^{-1}\{\infty\} = \mathsf{X} \setminus V$.
Let $(x_n)_n$ be a sequence converging in X to some x_∞.

(1) Let $x_\infty \in V$. Then there is some $i_0, n_0 \in \mathbb{N}$ such that $\{x_n \,|\, n \geq n_0\} \subseteq \beta_{i_0}$. Thus for all $j \geq i_0$ and $n \geq n_0$ (including $n = \infty$) we have $h_j(x_n) = 0$ and $f(x_n) = \max\{h_0(x_n), \dots, h_{i_0}(x_n)\}$. This implies that $(f(x_n))_n$ converges to $f(x_\infty)$. Moreover, since $h_j(x_\infty) \neq \infty$ for all $j \leq i_0$, $f(x_\infty) \neq \infty$.

(2) Let $x_\infty \notin V$. Then there is some $j \in \mathbb{N}$ with $x_\infty \notin G_j$, hence $f(x_\infty) = h_j(x_\infty) = \infty$. Since $(h_j(x_n))_n$ converges to $h_j(x_\infty) = \infty$, $(f(x_n))_n$ converges to ∞ as well.

Hence f is sequentially continuous and therefore topologically continuous, because X is sequential. So f is a witness for $V \in \mathcal{ZO}(\mathsf{X})$. $\square$

Now we are ready to prove our characterisation of quasi-zero-dimensionality via pseudobases.

Proof of Proposition 7. First, assume $\mathsf{X} \in \mathsf{QZ}$. By Corollary 6, X is equal to the sequential coreflection of $\mathcal{Z}(\mathsf{X})$. Moreover, X has some countable pseudobase $\mathcal{B}$ by being a qcb-space. We define $\mathcal{B}'$ to be the family of the closures of all sets in $\mathcal{B}$ formed in the strongly zero-dimensional space $\mathcal{Z}(\mathsf{X})$ and show that $\mathcal{B}'$ is a pseudobase of $\mathcal{Z}(\mathsf{X})$.

Let $U \in \mathcal{ZO}(\mathsf{X}) = \mathcal{O}(\mathcal{Z}(\mathsf{X}))$ and let $(x_n)_n$ be a sequence converging to some element $x_\infty \in U$. By zero-dimensionality of $\mathcal{Z}(\mathsf{X})$, there is a clopen set C and a pseudobase element $B \in \mathcal{B}$ with $x_\infty \in B \subseteq C \subseteq U$ and $x_n \in B$ for almost all $n \in \mathbb{N}$. The closure of B formed in $\mathcal{Z}(\mathsf{X})$ is a subset of C and hence of U. Thus $\mathcal{B}'$ is a pseudobase for $\mathcal{Z}(\mathsf{X})$. By Lemma 10 in [Schröder 02], the family $\mathcal{A}$ of all finite intersections of sets in $\mathcal{B}'$ yields a countable pseudobase for the sequential coreflection of the space $\mathcal{Z}(\mathsf{X})$, which is equal to X by Lemma 4. Since the sets in $\mathcal{B}'$ are $\mathcal{Z}$-closed, $\mathcal{A}$ consists of sets in $\mathcal{ZA}(\mathsf{X})$.

Conversely, let X be qcb$_0$-space with a countable pseudobase consisting of sets in $\mathcal{ZA}(\mathsf{X})$. In order to show that $\mathcal{ZO}(\mathsf{X})$ induces the sequence convergence of X, let $x \in \mathsf{X}$ and $(a_n)_n$ be a sequence that does not converge in X to x. Then $(a_n)_n$ has a subsequence $(b_n)_n$ such that no subsequence of $(b_n)_n$ converges in X to x and x does not occur in $(b_n)_n$. We consider two cases:

(1) Assume that $(b_n)_n$ has a subsequence $(c_n)_n$ that converges in X to some point y. By Lemma 8, there is clopen set C with $x \in C \not\ni y$. As $(c_n)_n$ is eventually in $\mathsf{X} \setminus C$, there are infinitely many n with $b_n \in \mathsf{X} \setminus C$. Therefore neither $(b_n)_n$ nor $(a_n)_n$ converge to x in $\mathcal{Z}(\mathsf{X})$.

(2) Now assume that $(b_n)_n$ has no subsequence that converges in X. Since X is a Hausdorff space by Lemma 8, this implies that the set $A := \{b_n \mid n \in \mathbb{N}\}$ is sequentially closed. By Lemmas 8 and 10, A is closed in $\mathcal{Z}(\mathsf{X})$. Since $\mathsf{X} \setminus A$ contains x, but no element of $(b_n)_n$, neither $(b_n)_n$ nor $(a_n)_n$ converge to x in $\mathcal{Z}(\mathsf{X})$.

This shows that convergence in $\mathcal{Z}(\mathsf{X})$ implies convergence in X. The reverse implication follows from $\mathcal{ZO}(\mathsf{X}) \subseteq \mathcal{O}(\mathsf{X})$. So X and $\mathcal{Z}(\mathsf{X})$ induce the same convergence relation for sequences, implying $seq(\mathcal{Z}(\mathsf{X})) = \mathsf{X}$. We conclude that X is quasi-zero-dimensional by Corollary 6. $\square$

We give a further characterisation of quasi-zero-dimensional qcb-spaces in terms of embeddability into function spaces of the form $\mathsf{2}^{\mathsf{Z}}$. A continuous function $e\colon \mathsf{X} \to \mathsf{Y}$ between sequential spaces X, Y is said to *reflect convergent sequences*, if, for any sequence $(x_n)_n$ in X and any point $x_\infty \in \mathsf{X}$, $(x_n)_n$ converges to x_∞ in X whenever $(e(x_n))_n$ converges to $e(x_\infty)$ in Y.

Proposition 11. *A qcb-space* X *is quasi-zero-dimensional if, and only if, there exist a qcb-space* Z *and a continuous injection* $e\colon \mathsf{X} \to \mathsf{2}^{\mathsf{Z}}$ *that reflects convergent sequences.*

Proof. Let Z be a qcb-space and $e\colon \mathsf{X} \to \mathsf{2}^{\mathsf{Z}}$ be a continuous function reflecting convergent sequences. By Lemma 4.2.2 in [Schröder 03], the compact-open topology τ_{co} on the set $\mathsf{2}^{\mathsf{Z}}$ induces the convergence relation of continuous convergence, which is known to be the convergence relation of QCB-exponentials. Since $\mathsf{2}$ is zero-dimensional, τ_{co} is zero-dimensional. Hence $\mathsf{2}^{\mathsf{Z}}$ is a QZ-space.

Now let $(x_n)_n$ be a sequence that does not converge to x_∞ in X. Then $(e(x_n))_n$ does not converge to $e(x_\infty)$ in $\mathsf{2}^{\mathsf{Z}}$, because e reflects convergent sequences. So there is a set D which is clopen w.r.t. τ_{co} and satisfies $e(x_\infty) \in D$ and $e(x_n) \notin D$ for infinitely many n. Since e is continuous, $e^{-1}(D)$ is a clopen set in the sequential space X. Hence $(x_n)_n$ does not converge to x_∞ in $\mathcal{ZO}(\mathsf{X})$, because the clopen sets of X form a base for $\mathcal{ZO}(\mathsf{X})$. We conclude that $\mathcal{ZO}(\mathsf{X})$ induces the sequence convergence relation of X. By embedding into a Hausdorff space, X has the T_0-property. Corollary 6 implies that X is quasi-zero-dimensional.

For the only-if-part, let $\mathsf{X} \in \mathsf{QZ}$. We set $\mathsf{Z} := 2^{\mathsf{X}}$. By the above proof, 2^{Z} is a QZ-space. Cartesian closedness of QCB yields continuity of the function $e\colon \mathsf{X} \to 2^{\mathsf{Z}}$ defined by $e(x)(f) := f(x)$. Let $(x_n)_n$ be a sequence in X and x_∞ be a point in X such that $(e(x_n))_n$ converges to $e(x_\infty)$. Moreover, let C be a clopen set in X containing x_∞. Then the function $h\colon \mathsf{X} \to 2$ uniquely defined by $h^{-1}\{1\} = C$ is an element of 2^{X}. Since $e(x_\infty)(h) = 1$, we have $e(x_n)(h) = 1$ and thus $x_n \in C$ for almost all n. As the clopen sets form a base of the topology $\mathcal{ZO}(\mathsf{X})$ and, by Corollary 6, $\mathcal{ZO}(\mathsf{X})$ induces the sequence convergence relation of X, $(x_n)_n$ converges to x_∞ in X. Hence e reflects converging sequences. $\square$

3.2 Constructing quasi-zero-dimensional spaces

The category QZ of quasi-zero-dimensional qcb-spaces enjoys excellent closure properties. Like quasi-normality, QZ is an exponential ideal of QCB and hence inherits the cartesian closed structure of QCB.

Theorem 12. *The category* QZ *of quasi-zero-dimensional qcb-spaces is cartesian closed and has all countable limits and all countable colimits. Moreover,* QZ *is an exponential ideal of* QCB *and its countable limits are inherited from* QCB*.*

Proof. Similar to the proof of Theorem 6 in [Schröder 09b]. Alternatively, one can apply Proposition 11 and techniques from category theory.

So quasi-zero-dimensionality is preserved by forming in the category QCB of qcb-spaces (i) countable products, (ii) subspaces, (iii) countable coproducts, and (iv) function spaces.

Obviously, all zero-dimensional metrisable spaces are in QZ. Theorem 12 implies that all Kleene-Kreisel spaces [Escardó 09] of the form $\mathbb{N}^{\mathsf{Z}}$ belong to QZ. Furthermore, for all $k \in \mathbb{N}$ the space $\mathsf{N}\langle k\rangle$ of Kleene-Kreisel continuous functional of order k (see [Kleene 59, Kreisel 59, Normann 05]) is a quasi-zero-dimensional space. The hierarchy $(\mathsf{N}\langle k\rangle)_k$ is recursively defined by the formulae $\mathsf{N}\langle 0\rangle := \mathbb{N}$ and $\mathsf{N}\langle k+1\rangle := \mathbb{N}^{\mathsf{N}\langle k\rangle}$. On the other hand, the Euclidean space $\mathbb{R}$ is not quasi-zero-dimensional by being connected.

Remark. One can show that there is a cartesian closed embedding of QZ into the cartesian closed category $\mathsf{k_2 0dim}$ considered by G. Lukács in [Lukács 08]. This category is itself equivalent to a full reflective sub-ccc of the category of Hausdorff k-spaces. The embedding functor maps a space $\mathsf{X} \in \mathsf{QZ}$ to its zero-dimensional reflection $\mathcal{Z}(\mathsf{X})$.

4 Extendability of continuous functions

In this section we investigate extendability of continuous functions defined on subspaces of quasi-zero-dimensional spaces. Moreover, we classify subspaces in terms of their ability to admit extendability of continuous functions.

4.1 A transitivity property for $\mathcal{Z}$-closed sets

It is well-known that the subspace operator on topological spaces has the following transitivity property: Any closed subset of a closed subspace is closed in the original space. However, the analogous statement for functionally closed sets is false in general (see Ex. 2.1.B in [Engelking 89]).

In [Schröder 09b] it is shown that functionally closed sets in quasi-normal qcb-spaces do have the transitivity property. Recall that functionally closed sets of a QN-space Y are exactly the closed sets of the completely regular reflection of Y. In Proposition 7 and Lemma 5 we have seen that $\mathcal{Z}$-closed sets play a similar role for QZ-spaces as functionally closed sets do for QN-spaces.

Validity of the transitivity property for $\mathcal{Z}$-closed sets is related to extendability of continuous functions with zero-dimensional codomains as follows: Let X be a $\mathcal{Z}$-closed subspace of a sequential space Y. If any continuous function from X to $\mathbb{J}$ (the one-point compactification of $\mathbb{N}$) is extendable onto Y, then any subset $A \subseteq \mathsf{X}$ that is closed in $\mathcal{Z}(\mathsf{X})$ is also closed in $\mathcal{Z}(\mathsf{Y})$: Choose continuous functions $f \colon \mathsf{X} \to \mathbb{J}$ and $g \colon \mathsf{Y} \to \mathbb{J}$ with $f^{-1}\{\infty\} = A$, $g^{-1}\{\infty\} = \mathsf{X}$ and extend f to a continuous function $F \colon \mathsf{Y} \to \mathbb{J}$. Then the function $h \colon \mathsf{Y} \to \mathbb{J}$ defined by $h(y) := \min\{F(y), g(y)\}$ is a continuous function witnessing that A is closed in $\mathcal{Z}(\mathsf{Y})$.

Fortunately, the transitivity property for $\mathcal{Z}$-closed sets is valid in the realm of QZ-spaces. So the zero-dimensional reflection of any $\mathcal{Z}$-closed subspace is a subspace of the zero-dimensional reflection of its QZ-superspace.

Proposition 13. *Let $\mathsf{Y} \in \mathsf{QZ}$. Let X be a subspace of Y such that X is closed in $\mathcal{Z}(\mathsf{Y})$. Then every set that is closed in $\mathcal{Z}(\mathsf{X})$ is closed in $\mathcal{Z}(\mathsf{Y})$. Moreover, $\mathcal{Z}(\mathsf{X})$ is a topological subspace of $\mathcal{Z}(\mathsf{Y})$.*

Proof. Since the preimage of a clopen set under a continuous function is clopen, the subspace topology τ_{sub} induced by $\mathcal{Z}(\mathsf{Y})$ on X is contained in $\mathcal{ZO}(\mathsf{X})$.

Let $A \in \mathcal{ZA}(\mathsf{X})$. Then the set $U := \mathsf{Y} \setminus A = (\mathsf{X} \setminus A) \cup (\mathsf{Y} \setminus \mathsf{X})$ is sequentially open. Let $B \in \mathcal{ZA}(\mathsf{Y})$ be a subset of U. Since $B \cap \mathsf{X}$ and A are disjoint closed sets in the strongly zero-dimensional space $\mathcal{Z}(\mathsf{X})$, there is a clopen set D in X such that $B \cap \mathsf{X} \subseteq D \subseteq \mathsf{X} \setminus A \subseteq U$.
Let $V := D \cup (\mathsf{Y} \setminus \mathsf{X})$. Then V is sequentially open and satisfies $B \subseteq V \subseteq U$. As Y has a pseudobase consisting of sets closed in $\mathcal{Z}(\mathsf{Y})$ and D is the complement

of an sequentially open set, D and thus V are $\mathcal{G}_\delta$-set in $\mathcal{Z}(\mathsf{Y})$. Lemma 10 yields $V \in \mathcal{ZO}(\mathsf{Y})$. So U is a union of sets in $\mathcal{ZO}(\mathsf{Y})$. As $\mathcal{ZO}(\mathsf{Y})$ is a topology, this implies $U \in \mathcal{ZO}(\mathsf{Y})$ and $A \in \mathcal{ZA}(\mathsf{Y})$.

The second statement follows from the first statement. $\square$

4.2 Extendability of continuous functions into QZ-spaces

Now we study extendability of continuous functions defined on subspaces of QZ-spaces. It turns out that extendability is guaranteed, if the subspace is a $\mathcal{Z}$-closed subspace (i.e. an intersection of clopen sets) and the target space is a zero-dimensional Polish space. We start by showing that clopens of $\mathcal{Z}$-closed subspaces extend to clopens of the whole space, provided that the latter is in QZ.

Lemma 14. *Let $\mathsf{Y} \in \mathsf{QZ}$, and let X be a subspace of Y with $\mathsf{X} \in \mathcal{ZA}(\mathsf{Y})$. Then for every set D that is clopen in X there is a clopen C in Y with $D = C \cap \mathsf{X}$.*

Proof. By Proposition 13, both D and $\mathsf{X} \setminus D$ are closed sets in $\mathcal{Z}(\mathsf{Y})$. By strong zero-dimensionality of $\mathcal{Z}(\mathsf{Y})$, there is a clopen set C in Y with $D \subseteq C \subseteq \mathsf{X} \setminus D$. Clearly, $C \cap \mathsf{X} = D$. $\square$

Lemma 14 can be reformulated by stating that any continuous function from a $\mathcal{Z}$-closed subset into the two-point discrete space 2 has a continuous extension.

We now investigate the full subcategory ZEXT of QCB consisting of those spaces $\mathsf{Z} \in \mathsf{QZ}$ that have the following property: For all spaces $\mathsf{Y} \in \mathsf{QZ}$, for all $\mathcal{Z}$-closed subspaces X of Y and for all continuous functions $f\colon \mathsf{X} \to \mathsf{Z}$ there exists a continuous function $F\colon \mathsf{Y} \to \mathsf{Z}$ extending f. By Lemma 14 the two point discrete space 2 is an object of ZEXT.

Given two qcb-spaces Y and B, we say that a subspace X of Y *admits a continuous extension operator for* B, if there exists a continuous function $E\colon \mathsf{B}^{\mathsf{X}} \to \mathsf{B}^{\mathsf{Y}}$ satisfying $E(f)(x) = f(x)$ for all $x \in \mathsf{X}$ and all continuous functions $f\colon \mathsf{X} \to \mathsf{B}$. Cartesian closedness of QZ (see Theorem 12) yields the following characterisation of the objects in ZEXT.

Proposition 15. *A space $\mathsf{Z} \in \mathsf{QZ}$ is an object of ZEXT if, and only if, any $\mathcal{Z}$-closed subspace X of any space $\mathsf{Y} \in \mathsf{QZ}$ admits a continuous extension operator $E\colon \mathsf{Z}^{\mathsf{X}} \to \mathsf{Z}^{\mathsf{Y}}$ for Z.*

Proof. The if-part is obvious. For the only-if-part, let $\mathsf{Z} \in \mathsf{ZEXT}$ and let X be a $\mathcal{Z}$-closed subspace of a QZ-space Y. By Theorem 12, the space $\mathsf{Z}^{\mathsf{X}} \times \mathsf{Y}$ formed in QCB is in QZ. Moreover, the set $\mathsf{Z}^{\mathsf{X}} \times \mathsf{X}$ is a $\mathcal{Z}$-closed subset of $\mathsf{Z}^{\mathsf{X}} \times \mathsf{Y}$, because for a given continuous function $g\colon \mathsf{Y} \to \mathbb{J}$ with $g^{-1}\{\infty\} = \mathsf{X}$ the continuous function $G\colon \mathsf{Z}^{\mathsf{X}} \times \mathsf{Y} \to \mathbb{J}$ defined by $G(h, y) := g(y)$ satisfies $G^{-1}\{\infty\} = \mathsf{Z}^{\mathsf{X}} \times \mathsf{X}$. Since

$\mathsf{Z} \in \mathsf{ZEXT}$, the continuous evaluation function $ev \colon \mathsf{Z}^{\mathsf{X}} \times \mathsf{X} \to \mathsf{Z}$ can be extended to a continuous function $EV \colon \mathsf{Z}^{\mathsf{X}} \times \mathsf{Y} \to \mathsf{Z}$. Using cartesian closedness of QCB, we define the function $E_{\mathsf{Z}} \colon \mathsf{Z}^{\mathsf{X}} \to \mathsf{Z}^{\mathsf{Y}}$ as the continuous transpose of EV. Then all continuous functions $h \colon \mathsf{X} \to \mathsf{Z}$ and all $x \in \mathsf{X}$ satisfy $E_{\mathsf{Z}}(h)(x) = EV(h,x) = ev(h,x) = h(x)$. So $E_{\mathsf{Z}}(h)$ extends h. $\square$

The category ZEXT enjoys excellent closure properties. It is an exponential ideal of QCB closed under forming sums, retracts and open subspaces.

Proposition 16.

1. *If* $\mathsf{A}, \mathsf{B} \in \mathsf{ZEXT}$*, then* $\mathsf{A} \times \mathsf{B} \in \mathsf{ZEXT}$*.*
2. *If* $\mathsf{B} \in \mathsf{ZEXT}$ *and* $\mathsf{A} \in \mathsf{QCB}$*, then* $\mathsf{B}^{\mathsf{A}} \in \mathsf{ZEXT}$*.*
3. *If* $\mathsf{A}, \mathsf{B} \in \mathsf{ZEXT}$*, then* $\mathsf{A} + \mathsf{B} \in \mathsf{ZEXT}$*.*
4. *If* $\mathsf{B} \in \mathsf{ZEXT}$ *and* A *is a* QCB*-retract of* B*, then* $\mathsf{A} \in \mathsf{ZEXT}$*.*
5. *If* $\mathsf{B} \in \mathsf{ZEXT}$ *and* A *is a* $\mathcal{Z}$*-open subspace of* B*, then* $\mathsf{A} \in \mathsf{ZEXT}$*.*

Proof. Let X be a $\mathcal{Z}$-closed subspace of a space $\mathsf{Y} \in \mathsf{QZ}$. By Proposition 15 there is a continuous extension operator $E_{\mathsf{B}} \colon \mathsf{B}^{\mathsf{X}} \to \mathsf{B}^{\mathsf{Y}}$ for any space $\mathsf{B} \in \mathsf{ZEXT}$.

1. Let $f \colon X \to A \times B$ be continuous. Let $pr_1 \colon \mathsf{A} \times \mathsf{B} \to \mathsf{A}$ and $pr_2 \colon \mathsf{A} \times \mathsf{B} \to \mathsf{B}$ denote the respective projection functions. Then the continuous functions $pr_1 \circ f$ and $pr_2 \circ f$ can be extended to continuous functions $G \colon \mathsf{Y} \to \mathsf{A}$ and $H \colon \mathsf{Y} \to \mathsf{B}$, respectively. The function $F \colon \mathsf{Y} \to \mathsf{A} \times \mathsf{B}$ defined by $F(y) := \big(G(y), H(y)\big)$ is continuous and extends f. Therefore $\mathsf{A} \times \mathsf{B} \in \mathsf{ZEXT}$.

2. By cartesian closedness of QCB, the function $T \colon (\mathsf{B}^{\mathsf{A}})^{\mathsf{X}} \times \mathsf{A} \to \mathsf{B}^{\mathsf{X}}$ defined by $T(f,a)(x) := f(x)(a)$ is continuous. We define $E_{\mathsf{B}^{\mathsf{A}}} \colon (\mathsf{B}^{\mathsf{A}})^{\mathsf{X}} \to (\mathsf{B}^{\mathsf{A}})^{\mathsf{Y}}$ by

$$E_{\mathsf{B}^{\mathsf{A}}}(f)(y)(a) := E_{\mathsf{B}}\big(T(f,a)\big)(y)$$

   for all $f \in (\mathsf{B}^{\mathsf{A}})^{\mathsf{X}}$, $y \in \mathsf{Y}$ and $a \in \mathsf{A}$. Then $E_{\mathsf{B}^{\mathsf{A}}}$ is sequentially continuous. Moreover, $E_{\mathsf{B}^{\mathsf{A}}}(f)(x) = f(x)$ holds for all continuous functions $f \colon \mathsf{X} \to \mathsf{B}^{\mathsf{A}}$ and all $x \in \mathsf{X}$. Hence $\mathsf{B}^{\mathsf{A}} \in \mathsf{ZEXT}$.

3. Let $f \colon \mathsf{X} \to \mathsf{A} + \mathsf{B}$ be continuous. The carrier set of the coproduct $\mathsf{A} + \mathsf{B}$ of A and B can be assumed to be $(\{1\} \times \mathsf{A}) \cup (\{2\} \times \mathsf{B})$. We choose two points $a_0 \in \mathsf{A}$ and $b_0 \in \mathsf{B}$ and define functions $g_2 \colon \mathsf{X} \to 2$, $g_{\mathsf{A}} \colon \mathsf{X} \to \mathsf{A}$ and $g_{\mathsf{B}} \colon \mathsf{X} \to \mathsf{B}$ by

$$g_2(x) := \begin{cases} 1 & \text{if } f(x) \in \{1\} \times \mathsf{A} \\ 0 & \text{otherwise} \end{cases}, \quad g_{\mathsf{A}}(x) := \begin{cases} a_0 & \text{if } g_2(x) = 0 \\ a & \text{if } f(x) = (1,a) \end{cases}$$

and

$$g_{\mathsf{B}}(x) := \begin{cases} b_0 & \text{if } g_2(x) = 1 \\ b & \text{if } f(x) = (2, b). \end{cases}$$

Since $\{1\} \times \mathsf{A}$ and $\{2\} \times \mathsf{B}$ are clopen subsets of $\mathsf{A} + \mathsf{B}$, these functions are continuous. As $2, \mathsf{A}, \mathsf{B} \in \mathsf{ZEXT}$, the functions $g_2, g_{\mathsf{A}}, g_{\mathsf{B}}$ extend to continuous functions $G_2 \colon \mathsf{Y} \to 2$, $G_{\mathsf{A}} \colon \mathsf{Y} \to \mathsf{A}$ and $G_{\mathsf{B}} \colon \mathsf{Y} \to \mathsf{B}$, respectively. It is not difficult to verify that the function $F \colon \mathsf{Y} \to \mathsf{A} + \mathsf{B}$ defined by

$$F(y) := \begin{cases} \big(1, G_{\mathsf{A}}(y)\big) & \text{if } G_2(y) = 1 \\ \big(2, G_{\mathsf{B}}(y)\big) & \text{otherwise} \end{cases}$$

is a continuous function extending f.

4. There are continuous functions $s \colon \mathsf{A} \to \mathsf{B}$ and $r \colon \mathsf{B} \to \mathsf{A}$ with $r \circ s = id_{\mathsf{A}}$. Let $f \colon \mathsf{X} \to \mathsf{A}$ be continuous. Then $sf \colon \mathsf{X} \to \mathsf{B}$ can be extended to a continuous function $G \colon \mathsf{Y} \to \mathsf{B}$. It is easy to verify that $F := r \circ G$ is a continuous function extending f. Thus $\mathsf{A} \in \mathsf{ZEXT}$.

5. There exist continuous functions $g \colon \mathsf{Y} \to \mathbb{J}$ and $h \colon \mathsf{B} \to \mathbb{J}$ with $g^{-1}\{\infty\} = \mathsf{X}$ and $h^{-1}\{\infty\} = \mathsf{B} \backslash \mathsf{A}$. Let $f \colon \mathsf{X} \to \mathsf{A}$ be continuous. Then there is a continuous function $F_{\mathsf{B}} \colon \mathsf{Y} \to \mathsf{B}$ with $F_{\mathsf{B}}(x) = f(x)$ for all $x \in \mathsf{X}$. We choose some point $a_0 \in A$ and define $F_{\mathsf{A}} \colon \mathsf{Y} \to \mathsf{A}$ by

$$F_{\mathsf{A}}(y) := \begin{cases} F_{\mathsf{B}}(y) & \text{if } g(y) \geq h\big(F_{\mathsf{B}}(y)\big) \\ a_0 & \text{otherwise.} \end{cases}$$

Since at least one of the values $g(y)$ and $h\big(F_{\mathsf{B}}(y)\big)$ is different from ∞, F_{A} is sequentially continuous and hence continuous. Obviously, F_{A} extends f. Therefore $\mathsf{A} \in \mathsf{ZEXT}$. $\square$

In the category of sequential spaces and hence in QCB the discrete space $\mathbb{N}$ is homeomorphic to the function space $2^{2^{\mathbb{N}}}$ by Proposition 3 in [Bauer et al. 02]. Moreover, by Theorem 7.8 in [Kechris 95] every zero-dimensional Polish space is homeomorphic to a closed subset of the Baire space $\mathbb{N}^{\mathbb{N}}$. In turn, this closed subspace is a retract of $\mathbb{N}^{\mathbb{N}}$ by Proposition 2 in [Bauer et al. 02]. We obtain by Proposition 16 and Lemma 14:

Example 1. The following spaces are objects of ZEXT:

(a) the discrete space $\mathbb{N}$,

(b) the Baire space $\mathbb{N}^{\mathbb{N}}$,

(c) any zero-dimensional Polish space,

(d) hence the one-point compactification $\mathbb{J}$ of $\mathbb{N}$,

(e) for any $k \in \mathbb{N}$ the space $\mathsf{N}\langle k \rangle$ of all Kleene-Kreisel continuous functionals of order k equipped with the sequential topology [see Section 3.2].

4.3 Subspaces that admit continuous extendability

Now we study under which conditions a subspace admits continuous extendability of continuous functions. We start with the following simple observation.

Lemma 17. *Let* $\mathsf{Y} \in \mathsf{QZ}$ *and let* X *be a* QCB*-subspace of* Y *such that for every subset* $D \subseteq \mathsf{X}$ *that is clopen in* X *there exists a clopen* C *in* Y *with* $D = C \cap \mathsf{X}$*. Then* X *is sequentially closed.*

Proof. Suppose that $(x_n)_n$ were a sequence in X converging to an element $y \in \mathsf{Y} \setminus \mathsf{X}$. Since Y is a Hausdorff space, $(x_n)_n$ does not contain any subsequence that converges to an element inside X. Hence $(x_n)_n$ contains an injective subsequence $(a_n)_n$. Furthermore, $\mathsf{A} := \{a_n \,|\, n \in \mathbb{N}\}$ forms a sequentially closed subspace of X. As X is a QZ-space by Theorem 12, Lemma 10 implies $\mathsf{A} \in \mathcal{ZA}(\mathsf{X})$. Moreover, the set $E := \{a_{2n} \,|\, n \in \mathbb{N}\}$ is clopen in A. An application of Lemma 14 to the QZ-space X, its $\mathcal{Z}$-closed subspace A and the clopen set E yields a clopen set D in X with $D \cap \mathsf{A} = E$. But D cannot be extended to a clopen set C in Y, because y is in the closure of both D and $\mathsf{Y} \setminus D$, as $(a_{2n})_n$ and $(a_{2n+1})_n$ converge to y. This yields the contradiction. □

We have already seen that the property of X being closed in $\mathcal{ZA}(\mathsf{Y})$ is sufficient to guarantee extendability of all continuous $\mathbb{N}$-valued functions defined on X. We show that this condition is also necessary.

Lemma 18. *Let* $\mathsf{Y} \in \mathsf{QZ}$*, and let* X *be a* QCB*-subspace of* Y*. If every continuous function* $h\colon \mathsf{X} \to \mathbb{N}$ *can be extended to a continuous function* $H\colon \mathsf{Y} \to \mathbb{N}$*, then* $\mathsf{X} \in \mathcal{ZA}(\mathsf{Y})$*.*

Proof. Let $y \in \mathsf{Y} \setminus \mathsf{X}$. By Lemma 8 there is a continuous function $G\colon \mathsf{Y} \to \mathbb{J}$ with $G^{-1}\{\infty\} = \{y\}$. The formula $h(x) := G(x)$ defines a total continuous function h from X to $\mathbb{N}$. By assumption, h has a continuous extension $H\colon \mathsf{Y} \to \mathbb{N}$. It is easy to verify that

$$C := \big\{ z \in \mathsf{Y} \,\big|\, H(z) = H(y) \text{ and } G(z) > H(y) \big\}$$

is a clopen set in Y with $y \in C \subseteq \mathsf{Y} \setminus \mathsf{X}$. Hence the set $\mathsf{Y} \setminus \mathsf{X}$ is a union of clopen sets of Y. Lemma 5 yields $\mathsf{X} \in \mathcal{ZA}(\mathsf{Y})$. □

We obtain as an easy consequence:

Corollary 19. *Let* A *be a retract of a space* $\mathsf{Y} \in \mathsf{QZ}$*. Then* A *is homeomorphic to a* $\mathcal{Z}$*-closed subspace of* Y*.*

Lemma 18 generalises to all non-compact QZ-spaces replacing $\mathbb{N}$ as target space.

Proposition 20. *Let* $\mathsf{Z} \in \mathsf{QZ}$ *such that* Z *is not compact. Let* X *be a* QCB*-subspace of a space* $\mathsf{Y} \in \mathsf{QZ}$ *such that every continuous function* $f\colon \mathsf{X} \to \mathsf{Z}$ *can be extended to a continuous function* $F\colon \mathsf{Y} \to \mathsf{Z}$*. Then* $\mathsf{X} \in \mathcal{ZA}(\mathsf{Y})$*.*

Proposition 20 is a consequence of Lemma 18 and the following equivalence.

Lemma 21. *A space* $\mathsf{X} \in \mathsf{QZ}$ *is not compact if, and only if,* $\mathbb{N}$ *is a retract of* X*.*

Proof. First, let $\mathbb{N}$ be a retract of X, and let $r\colon \mathsf{X} \to \mathbb{N}$ be the retraction map. Since r is surjective, $\mathbb{N}$ is a continuous image of X. Because $\mathbb{N}$ is not compact, neither is X.

Now assume that X is not compact. Since X is hereditarily Lindelöf, X is neither countably compact nor sequentially compact. So there exists a sequence $(x_n)_n$ that does not have any convergent subsequence. Since each member of the sequence only occurs finitely often, $(x_n)_n$ has an injective subsequence $(a_n)_n$. We define A to be the subspace of X with carrier set $\{a_n \,|\, n \in \mathbb{N}\}$. Since X is a Hausdorff space and $(a_n)_n$ does not have convergent subsequences, A is a countable, discrete, sequentially closed subspace of X. Hence the bijection $s\colon \mathbb{N} \to \mathsf{A}$ defined by $s(n) := a_n$ and its inverse are continuous. As the set A the countable union of the singletons $\{a_n\}$, A is closed in $\mathcal{Z}(\mathsf{X})$ by Lemmas 8 and 10. Since $\mathbb{N} \in \mathsf{ZEXT}$, there exists a continuous extension $r\colon \mathsf{X} \to \mathbb{N}$ of the continuous inverse of s. Clearly, $r(s(n)) = n$ for all $n \in \mathbb{N}$. Hence $\mathbb{N}$ is a retract of X. □

We do not know whether Lemma 18 is valid for the two-point discrete space 2 replacing $\mathbb{N}$. However, the (possibly) stronger condition on a subspace X to admit a continuous extension operator for the continuous functions with codomain 2 is enough to ensure that X is $\mathcal{Z}$-closed.

Proposition 22. *Let* $\mathsf{Y} \in \mathsf{QZ}$*. Let* X *be a* QCB*-subspace of* Y *that admits a continuous extension operator* $E\colon \mathsf{2}^{\mathsf{X}} \to \mathsf{2}^{\mathsf{Y}}$*. Then* $\mathsf{X} \in \mathcal{ZA}(\mathsf{Y})$*.*

Proof. We show that for any element $y \in \mathsf{Y} \setminus \mathsf{X}$ there is a clopen set C with $y \in C \subseteq \mathsf{Y} \setminus \mathsf{X}$. By Proposition 7, Y has a countable pseudobase $\mathcal{A}$ consisting of $\mathcal{Z}$-closed sets. Let $(\alpha_i)_i$ be a sequence with $\{\alpha_i \,|\, i \in \mathbb{N}\} = \{A \in \mathcal{A} \,|\, y \notin A\}$. By zero-dimensionality of $\mathcal{Z}(\mathsf{Y})$, there is a continuous function $H_n\colon \mathsf{Y} \to 2$ satisfying $H_n(y) = 1$ and $\bigcup_{i \leq n} \alpha_i \subseteq H_n^{-1}\{0\}$. Let $h_n\colon \mathsf{X} \to 2$ be the continuous restriction of H_n to X, and let $\mathbf{0}\colon \mathsf{X} \to 2$ be the constant zero-function.
To prove that $(h_n)_n$ converges continuously to $\mathbf{0}$, let $(x_n)_n$ be a sequence that converges in X to some x_∞. Since $\mathcal{A}$ is a pseudobase of Y and $\mathsf{Y} \setminus \{y\}$ is sequentially open, there are some $i, n_0 \in \mathbb{N}$ such that $x_n \in \alpha_i$ for all $n \geq n_0$. Hence $h_n(x_n) = 0$ for all $n \geq \max\{i, n_0\}$ implying that $(h_n(x_n))_n$ converges to $\mathbf{0}(x_\infty)$. We conclude that $(h_n)_n$ converges in $\mathsf{2}^{\mathsf{X}}$ to $\mathbf{0}$. By continuity of E, $(E(h_n))_n$ converges to $E(\mathbf{0})$ in $\mathsf{2}^{\mathsf{Y}}$. We consider two cases:

(1) There is some $i \in \mathbb{N}$ with $E(h_i)(y) \neq 1$. Then $C := \{z \in \mathsf{Y} \mid E(h_i)(z) \neq H_i(z)\}$ is a clopen set with $y \in C \subseteq \mathsf{Y} \setminus \mathsf{X}$.

(2) For all $i \in \mathbb{N}$, $E(h_i)(y) = 1$. Then $E(\mathbf{0})(y) = 1$. For all $x \in X$, $E(h_n)(x) = h_n(x) = H_n(x)$ is equal to 0 for almost all n, because x is contained in some α_i; hence $E(\mathbf{0})(x) = 0$. Therefore the set $C := (E(\mathbf{0}))^{-1}\{1\}$ is a clopen set with $y \in C \subseteq \mathsf{Y} \setminus \mathsf{X}$.

This shows that $\mathsf{Y} \setminus \mathsf{X}$ is a union of clopens. Lemma 5 implies $\mathsf{X} \in \mathcal{ZA}(\mathsf{Y})$. □

We obtain a characterisation of ZEXT which parallels Proposition 11.

Proposition 23. *A qcb-space* X *is an object of* ZEXT *if, and only if, there is a qcb-space* Z *such that* X *is a retract of* 2^{Z}.

Proof. The if-part follows from Proposition 16(2) and (4).
For the only-if-part, let $\mathsf{X} \in \mathsf{ZEXT}$. We set $\mathsf{Z} := 2^{\mathsf{X}}$ and $\mathsf{Y} := 2^{\mathsf{Z}}$. From the proof of Proposition 11 we know that the transpose $e\colon \mathsf{X} \to \mathsf{Y}$ of the evaluation function $ev\colon 2^{\mathsf{X}} \times \mathsf{X} \to 2$ defined by $e(x)(g) := g(x)$ is a continuous injection reflecting convergent sequences. Hence its inverse $f\colon e(\mathsf{X}) \to \mathsf{X}$ is sequentially continuous. Let X' be the QCB-subspace of Y with underlying set $e(\mathsf{X})$. We define $E\colon 2^{(\mathsf{X}')} \to 2^{\mathsf{Y}}$ by $E(h)(y) := y(h \circ e)$. Then E is sequentially continuous and satisfies $E(h)(a) = h(a)$ for all $a \in e(\mathsf{X})$ and all continuous functions $h\colon \mathsf{X}' \to 2$. By Proposition 22, X' is a $\mathcal{Z}$-closed subspace of Y. Since $\mathsf{X} \in \mathsf{ZEXT}$, f can be extended to a continuous function $r\colon \mathsf{Y} \to \mathsf{X}$. As $r(e(x)) = f(e(x)) = x$ holds for all $x \in \mathsf{X}$, the space X is a retract of $2^{2^{\mathsf{X}}}$. □

We summerise some of the above results in a characterisation theorem for sets that are closed in the zero-dimensional reflection.

Theorem 24 (Characterisation of $\mathcal{Z}$-closed subsets). *Let* Y *be a quasi-zero-dimensional qcb-space, and let* X *be a* QCB*-subspace of* Y*. Then the following statements are equivalent:*

(a) The set X *is closed in the zero-dimensional reflection* $\mathcal{Z}(\mathsf{Y})$ *of* Y*.*

(b) The set X *is* $\mathcal{Z}$*-closed (i.e.* $\mathsf{X} \in \mathcal{ZA}(\mathsf{Y})$*).*

(c) The subspace X *admits a continuous extension operator* $E\colon 2^{\mathsf{X}} \to 2^{\mathsf{Y}}$*.*

(d) The subspace X *admits a continuous extension operator* $E\colon \mathbb{N}^{\mathsf{X}} \to \mathbb{N}^{\mathsf{Y}}$*.*

(e) Any continuous function $f\colon \mathsf{X} \to \mathbb{N}$ *can be extended to a continuous function* $F\colon \mathsf{Y} \to \mathbb{N}$*.*

(f) There is a non-compact quasi-zero-dimensional qcb-space Z *such that any continuous function* $f\colon \mathsf{X} \to \mathsf{Z}$ *can be extended to a continuous function* $F\colon \mathsf{Y} \to \mathsf{Z}$*.*

4.4 Characterisation of functionally closed subsets

In this section we present a characterisation of all functionally closed subsets of quasi-normal spaces that is similar to Theorem 24.

In [Schröder 09b] it is shown that real-valued functions defined on a functionally closed subspace can be extended to the whole space, provided the latter is a quasi-normal qcb-space. The cartesian closedness of QN implies the following uniform versions of this extendability result.

Proposition 25. *Let* X *be a functionally closed subspace of a space* $\mathsf{Y} \in \mathsf{QN}$*. Then* X *admits continuous extension operators* $E_{\mathbb{I}} \colon \mathbb{I}^{\mathsf{X}} \to \mathbb{I}^{\mathsf{Y}}$ *and* $E_{\mathbb{R}} \colon \mathbb{R}^{\mathsf{X}} \to \mathbb{R}^{\mathsf{Y}}$*.*

Proof. By Theorem 6 in [Schröder 09b] the space $\mathsf{Y}' := \mathbb{R}^{\mathsf{X}} \times \mathsf{Y}$ formed in QCB is quasi-normal. As the subset $\mathbb{R}^{\mathsf{X}} \times \mathsf{X}$ is functionally closed in Y', the sequentially continuous evaluation function $ev \colon \mathbb{R}^{\mathsf{X}} \times \mathsf{X} \to \mathbb{R}$ can be extended to a sequentially continuous function $EV \colon \mathbb{R}^{\mathsf{X}} \times \mathsf{Y} \to \mathbb{R}$. Using the cartesian closedness of QCB, we define the function $E_{\mathbb{R}} \colon \mathbb{R}^{\mathsf{X}} \to \mathbb{R}^{\mathsf{Y}}$ as the continuous transpose of EV. One easily verifies that $E_{\mathbb{R}}$ is indeed an extension operator. A continuous extension operator $E_{\mathbb{I}} \colon \mathbb{I}^{\mathsf{X}} \to \mathbb{I}^{\mathsf{Y}}$ is constructed analogously. $\square$

Now we investigate under which condition a subspace admits continuous extendability of continuous real-valued functions. We begin with the following simple observation which is analogous to Lemma 17.

Lemma 26. *Let* $\mathsf{Y} \in \mathsf{QN}$*. Let* X *be a* QCB*-subspace of* Y *such that every continuous function* $f \colon \mathsf{X} \to \mathbb{I}$ *can be extended to a continuous function* $F \colon \mathsf{Y} \to \mathbb{I}$*. Then* X *is sequentially closed.*

Proof. Suppose for contradiction that $(x_n)_n$ is a sequence in X converging to an element $y \in \mathsf{Y} \setminus \mathsf{X}$. Since Y is a Hausdorff space, $(x_n)_n$ does not contain any subsequence that converges to an element inside X. Hence $(x_n)_n$ contains an injective subsequence $(a_n)_n$. We define $\mathsf{A} := \{a_n \,|\, n \in \mathbb{N}\}$. Since A is a sequentially closed subspace of the quasi-normal space X and is a countable union of functionally closed sets, A is functionally closed by Lemma 5 in [Schröder 09b]. Moreover, the set $E := \{a_{2n} \,\big|\, n \in \mathbb{N}\}$ is clopen in A. So the function $g \colon \mathsf{A} \to \mathbb{I}$ defined by

$$g(a_n) := \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

is continuous and can be extended to a continuous function $f \colon \mathsf{X} \to \mathbb{I}$ by Theorem 13 in [Schröder 09b]. But f does not extend to a continuous function $F \colon \mathsf{Y} \to \mathbb{I}$, because y is the limit of a sequence in $f^{-1}\{0\}$ and of a sequence in $f^{-1}\{1\}$, a contradiction. $\square$

The following observation follows basically from the fact that the completely regular reflection of a quasi-normal qcb-space is realcompact by being a regular Lindelöf space.

Lemma 27. *Let* $\mathsf{Y} \in \mathsf{QN}$. *Let* X *be a* QCB*-subspace of* Y *such that every continuous function* $f \colon \mathsf{X} \to \mathbb{R}$ *can be extended to a continuous function* $F \colon \mathsf{Y} \to \mathbb{R}$. *Then the set* X *is functionally closed in* Y.

Proof. At first we show that for any element $y \in \mathsf{Y} \setminus \mathsf{X}$ there is a continuous function $h_y \colon \mathsf{Y} \to \mathbb{I}$ with $h_y(y) \neq 0$ and $\mathsf{X} \subseteq h^{-1}\{0\}$. By Proposition 4 in [Schröder 09b], Y has a countable pseudobase $\mathcal{A}$ consisting of functionally closed sets. The singleton $\{y\}$ is functionally closed by being the intersection of all pseudobase sets containing y. So there is a continuous function $g \colon \mathsf{Y} \to \mathbb{I}$ with $g^{-1}\{0\} = \{y\}$. We define the total continuous function $f \colon \mathsf{X} \to \mathbb{R}$ by $f(x) := 1/g(x)$. By assumption, f can be extended to a continuous function $F \colon \mathsf{Y} \to \mathbb{R}$ with $F(x) = f(x)$ for all $x \in \mathsf{X}$. We define $h_y \colon \mathsf{Y} \to \mathbb{I}$ by

$$h_y(z) := \Big|\frac{1}{\max\{1, F(z)\}} - g(z)\Big| \quad \text{for all } z \in \mathsf{Y}.$$

Then h_y is a continuous function with $h_y(y) \neq 0$ and $h_y(x) = 0$ for all $x \in \mathsf{X}$.

Since Y is a hereditarily Lindelöf, there is a sequence $(y_i)_i$ of elements in $\mathsf{Y} \setminus \mathsf{X}$ with $\bigcup_{i \in \mathbb{N}} h_{y_i}^{-1}(0,1] = \mathsf{Y} \setminus \mathsf{X}$. Clearly, the function $h \colon \mathsf{Y} \to \mathbb{I}$ defined by $h(z) := \sup\big\{h_{y_i}(z)/2^i \,\big|\, i \in \mathbb{N}\big\}$ is continuous and satisfies $\mathsf{X} = h^{-1}\{0\}$. $\square$

We obtain the following consequence which parallels Corollary 19.

Corollary 28. *Let* A *be a retract of a space* $\mathsf{Y} \in \mathsf{QN}$. *Then* A *is homeomorphic to a functionally closed subspace of* Y.

We do not know whether non-uniform extendability of all continuous functions on X into the unit interval $\mathbb{I} = [0,1]$ implies that X is functionally closed. However, if X admits a continuous extension operator for $\mathbb{I}$ as target space, then X must be functionally closed. This result resembles Proposition 22.

Proposition 29. *Let* Y *be a quasi-normal qcb-space. Let* X *be a* QCB*-subspace of* Y *that admits a continuous extension operator* $E \colon \mathbb{I}^{\mathsf{X}} \to \mathbb{I}^{\mathsf{Y}}$. *Then* X *is functionally closed in* Y.

Proof. With the same argument as in the proof of Lemma 27, it suffices to show that for any element $y \in \mathsf{Y} \setminus \mathsf{X}$ there is a continuous function $h_y \colon \mathsf{Y} \to \mathbb{I}$ with $h_y(y) \neq 0$ and $\mathsf{X} \subseteq h_y^{-1}\{0\}$. By Proposition 4 in [Schröder 09b], Y has a countable pseudobase $\mathcal{A}$ consisting of functionally closed sets. Let $(\alpha_i)_i$ be a sequence with

$\{\alpha_i \mid i \in \mathbb{N}\} = \{A \in \mathcal{A} \mid y \notin A\}$. Since $\bigcup_{i \leq n} \alpha_i$ is functionally closed, there is a continuous function $G_n \colon \mathsf{Y} \to \mathbb{I}$ satisfying

$$G_n(y) = 1 \text{ and } \bigcup_{i \leq n} \alpha_i = G_n^{-1}\{0\}\,.$$

Let $g_n \colon \mathsf{X} \to \mathbb{I}$ be the continuous restriction of G_n to X. Moreover, let $\mathbf{0} \colon \mathsf{X} \to \mathbb{I}$ the constant zero-function.

To prove that $(g_n)_n$ converges continuously to $\mathbf{0}$, let $(x_n)_n$ be a sequence that converges in X to some x_∞. Since $\mathcal{A}$ is a pseudobase of Y and $\mathsf{Y} \setminus \{y\}$ is an open neighbourhood of x_∞, there are some $i, n_0 \in \mathbb{N}$ such that $x_n \in \alpha_i$ for all $n \geq n_0$. Hence $g_n(x_n) = 0$ for all $n \geq \max\{i, n_0\}$ implying that $(g_n(x_n))_n$ converges to $\mathbf{0}(x_\infty)$. Therefore $(g_n)_n$ converges in $\mathbb{I}^{\mathsf{X}}$ to $\mathbf{0}$. By continuity of E, $(E(g_n))_n$ converges to $E(\mathbf{0})$ in $\mathbb{I}^{\mathsf{Y}}$. We consider two cases:

(1) There is some $i \in \mathbb{N}$ with $E(g_i)(y) \neq 1$. Then $h_y \colon \mathsf{Y} \to \mathbb{I}$ defined by $h_y(z) := \big|G_i(z) - E(g_i)(z)\big|$ is a continuous function with $h_y(y) \neq 0$ and $\mathsf{X} \subseteq h_y^{-1}\{0\}$.

(2) For all $i \in \mathbb{N}$, $E(g_i)(y) = 1$. Then $E(\mathbf{0})(y) = 1$. For all $x \in \mathsf{X}$, $E(g_n)(x)$ is equal to 0 for almost all n, because x is contained in some α_i; hence $E(\mathbf{0})(x) = 0$. Therefore $h_y := E(\mathbf{0})$ is a continuous function with $\mathsf{X} \subseteq h_y^{-1}\{0\}$ and $h_y(y) = 1$.

So in both cases there is a continuous function $h_y \colon \mathsf{Y} \to \mathbb{I}$ separating y from the set X. $\square$

We summerise the above results in a characterisation theorem for functionally closed subsets of quasi-normal qcb-spaces.

Theorem 30 (Characterisation of functionally closed subsets).
Let Y be a quasi-normal qcb-space, and let X be a QCB-subspace of Y. Then the following statements are equivalent:

(a) The set X is functionally closed in Y (i.e. $\mathsf{X} \in \mathcal{FA}(\mathsf{Y})$).

(b) The set X is closed the completely regular reflection $\mathcal{R}(\mathsf{Y})$ of Y.

(c) The subspace X admits a continuous extension operator $E \colon \mathbb{I}^{\mathsf{X}} \to \mathbb{I}^{\mathsf{Y}}$.

(d) The subspace X admits a continuous extension operator $E \colon \mathbb{R}^{\mathsf{X}} \to \mathbb{R}^{\mathsf{Y}}$.

(e) Any continuous function $f \colon \mathsf{X} \to \mathbb{R}$ can be extended to a continuous function $F \colon \mathsf{Y} \to \mathbb{R}$.

5 Discussion

We have seen that the category QZ of quasi-zero-dimensional qcb-spaces and the category QN of quasi-normal qcb-spaces enjoy several similarities, for example they are exponential ideals of QCB. Moreover, both classes of topological spaces possess a distinguished family of closed subsets ($\mathcal{Z}$-closed subsets in the case of QZ and functionally closed subsets in the case of QN) with the following property: each of both classes is characterised by the existence of a countable pseudobase consisting of sets in the respective family of closed subsets. Functionally closed subspaces of QN-spaces are characterised as those subspaces that admit continuous extendability of real-valued functions, while $\mathcal{Z}$-closed subsets are exactly the class of sets which allow the extension of continuous functions defined on them with a Kleene-Kreisel space of the form $\mathbb{N}^{\mathsf{Z}}$ as codomain.

The question which QZ-spaces have the property that all their functionally closed sets are $\mathcal{Z}$-closed is related to a problem in Computable Analysis, namely whether or not two natural hierarchies of continuous functionals over the reals (called the intensional hierarchy and the extensional hierarchy, see [Bauer et al. 02]) coincide. D. Normann proved in [Normann 05] that the two hierarchies agree if, and only if, for all $k \geq 2$ the space $\mathsf{N}\langle k \rangle$ of Kleene-Kreisel continuous functionals of type k (which is a QZ-space [see Section 3.2]) has the property that every functionally closed subsets is $\mathcal{Z}$-closed, i.e., an intersection of clopen sets.

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