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A Constructive Study of Landau's Summability Theorem

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Abstract: A summability theorem of Landau, which classically is a simple consequence of the uniform boundedness theorem, is examined within Bishop-style constructive mathematics. It is shown that the original theorem is nonconstructive, and that a natural weakening of the theorem is constructively equivalent to Ishihara's principle **BD-N**. The paper ends with a number of results that, while not as strong as Landau's theorem, nevertheless contain positive computational information related to its conclusion.
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Edmund Landau (1877–1938) is known for many contributions to mathematics. In this paper we examine his summability theorem,

> *If* p, q *are* ***conjugate exponents***—*numbers such that* $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$—*and if* $\mathbf{a} = (a_n)_{n\geqslant 1}$ *is a sequence in* $\mathbf{C}$ *such that* $\sum_{n=1}^{\infty} a_n x_n$ *converges for each* $\mathbf{x} = (x_n)_{n\geqslant 1}$ *in the Banach space* l_p, *then* $\mathbf{a} \in l_q$,

from the viewpoint of Bishop's constructive mathematics **(BISH)**—that is, mathematics developed with intuitionistic logic and a suitable set-theoretic foundation such as the Aczel-Rathjen-Myhill CST [Aczel and Rathjen 2001, Myhill 1975]. Here, l_p denotes the vector space of all complex sequences $\mathbf{x} \equiv (x_n)_{n\geq 1}$ that are ***p*-summable**, in the sense that the norm

$$\|\mathbf{x}\|_p \equiv \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}$$

exists; for more on (the generalisation of) these spaces, see Chapter 7 of [Bishop and Bridges 1985].

The standard functional-analytic proof goes as follows. For each $\mathbf{x} = (x_n)_{n\geqslant 1}$ in l_p and each k define

$$s_k(\mathbf{x}) = \sum_{n=1}^{k} a_n x_n.$$

Then

$$|s_k(\mathbf{x})| \leqslant \left(\sum_{n=1}^{k} |a_n|^q\right)^{1/q} \left(\sum_{n=1}^{k} |x_p|^p\right)^{1/p} \leqslant \left(\sum_{n=1}^{k} |a_n|^q\right)^{1/q} \|\mathbf{x}\|_p ,$$

from which it follows that s_k is a bounded linear functional on l_p with norm

$$\|s_k\| = \left(\sum_{n=1}^{k} |a_n|^q\right)^{1/q} .$$

Also, the sequence $(s_k(\mathbf{x}))_{k\geqslant 1}$ converges in $\mathbf{C}$ and so is bounded. Applying the uniform boundedness theorem to the sequence $(s_k)_{k\geqslant 1}$, we now obtain $M > 0$ such that $\|s_k\| \leqslant M$ for each k. The partial sums of the series $\sum_{n=1}^{\infty} |a_n|^q$ are therefore bounded, so the series converges in $\mathbf{R}$.

From a constructive viewpoint, there are two problems with this proof. First, the uniform boundedness theorem in the form applied there is not the constructive one. Secondly, boundedness of the partial sums of a series of positive terms is not enough to ensure its convergence (see pages 60–64 of [Bridges and Richman 1985]). In fact, a Brouwerian example shows that Landau's summability theorem in its classical form is not constructively valid: under its hypotheses we cannot even prove, in general, that $a_n \to 0$ as $n \to \infty$. To see this, take $\mathbf{a}$ as a binary sequence with at most one term equal to 1, and consider the case $p = q = 2$. The series $\sum_{n=1}^{\infty} a_n x_n$ certainly converges for each $\mathbf{x}$ in l_2. But if $a_n \to 0$ as $n \to \infty$, we can find N such that $a_n = 0$ for all $n > N$; by testing $a_1, \ldots, a_N$, we can decide whether $a_n = 0$ for all n or there exists n such that $a_n = 1$. Thus the statement

> For each sequence $\mathbf{a}$ of complex numbers, if $\sum_{n=1}^{\infty} a_n x_n$ converges for all $\mathbf{x} \in l_2$, then $\mathbf{a} \in l_2$

implies the essentially nonconstructive **limited principle of omniscience**,

> **LPO**: For each binary sequence $\mathbf{a}$, either $a_n = 0$ for all n or else there exists n such that $a_n = 1$.

At this stage, it remains a possibility that, under the hypotheses of Landau's theorem, the series $\sum_{n=1}^{\infty} |a_n|^q$ has bounded partial sums. To explore this possibility, we need some background information from constructive functional analysis.

A linear functional ϕ on a normed space X is said to be **normed** (or **normable**) if its norm

$$\|\phi\| = \sup \{\|\phi(x) : x \in X, \ \|x\| \leqslant 1\|\}$$

exists. Every linear functional on a finite-dimensional Banach space is normed; but if every bounded linear functional on an infinite-dimensional Hilbert space is normed, then we can prove **LPO**. The following is the constructive version of the representation theorem for l_p spaces ([Bishop and Bridges 1985], Chapter 7, Theorem (3.25)).

Theorem 1. *If p, q are conjugate exponents, then a bounded linear functional ϕ on l_p is normed if and only if there exists a (perforce unique) vector $\mathbf{a} \in l_q$ such that $\phi(\mathbf{x}) = \sum_{n=1}^{\infty} a_n x_n$ for each $\mathbf{x} \in l_p$, in which case $\|\phi\| = \|\mathbf{a}\|_q$.*

We shall also need the constructive **uniform boundedness theorem:**

Theorem 2. *Let $(T_n)_{n\geqslant 1}$ be a sequence of bounded linear mappings from a Banach space X into a normed space Y, and $(x_n)_{n\geqslant 1}$ a sequence of unit vectors in X, such that $\|T_n x_n\| \to \infty$ as $n \to \infty$. Then there exists $x \in X$ such that the sequence $(\|T_n x\|)_{n\geqslant 1}$ is unbounded.*

Proof. See [Bridges and Vîţă 2006] (Corollary 6.2.12) or [Royden 1985].

The next result follows from Theorem 7 of [Ishihara 1997]; we include the proof here to clarify the role played by the uniform boundedness theorem in our work.

Theorem 3. *Let $(T_n)_{n\geqslant 1}$ be a sequence of bounded linear mappings of a separable Banach space X into a normed space Y, converging pointwise to a linear mapping $T : X \to Y$. Then T is sequentially continuous.*

Proof. Let $(x_n)_{n\geqslant 1}$ be a sequence converging to 0 in X, and let $\varepsilon > 0$. By Ishihara's tricks [Ishihara 1997] (Lemma 2), either $\|Tx_n\| < \varepsilon$ for all sufficiently large n or else $\|Tx_n\| > \varepsilon/2$ for infinitely many n. It suffices to rule out the latter alternative. To that end, we may suppose that $\|Tx_n\| > \varepsilon/2$ and $\|x_n\| < 1/n$ for each n. Then $y_n = \|x_n\|^{-1} x_n$ is a unit vector such that $\|Ty_n\| > n\varepsilon/2$. Since $T_n x \to Tx$ for each $x \in X$, we can construct inductively a strictly increasing sequence $(n_k)_{k\geqslant 1}$ of positive integers such that $\|T_{n_k} y_k\| > k\varepsilon/2$ for each k. Applying the uniform boundedness theorem, we obtain a unit vector $y \in X$ such that the sequence $(\|T_{n_k} y\|)_{k\geqslant 1}$ is unbounded. This is absurd, since $T_{n_k} y \to Ty$ as $k \to \infty$.

Corollary 4. *Let $p > 1$, and let $\mathbf{a}$ be a sequence of complex numbers such that*

$$f(\mathbf{x}) = \sum_{n=1}^{\infty} a_n x_n \tag{1}$$

converges for each $\mathbf{x} \in l_p$. Then f is a sequentially continuous linear functional on l_p.

Proof. Noting that

$$f_k(\mathbf{x}) = \sum_{n=1}^{k} a_n x_n$$

defines a normed, and *a fortiori* bounded, linear functional on X with

$$\|f_k\| = \left(\sum_{n=1}^{k} |a_k|^q\right)^{1/q},$$

we apply Theorem 3 with $X = l_p$.

Observe that the linear functional f in this corollary is continuous/bounded if and only if the partial sums of the series $\sum_{i=1}^{\infty} |a_i|^q$ are bounded. Indeed, if f has a bound $c > 0$ and k is any positive integer, then, assuming that $a_i \neq 0$ for $i \leqslant k$, take

$$\mathbf{x} = \left(a_1^* \, |a_1|^{q-2}, \ldots, a_k^* \, |a_k|^{q-2}, 0, 0, \ldots\right),$$

where * denotes complex conjugation. We obtain

$$\begin{aligned}
\sum_{n=1}^{k} |a_n|^q = \sum_{n=1}^{k} a_n x_n &= f(\mathbf{x}) \\
&\leqslant c \, \|\mathbf{x}\|_p = c \left(\sum_{n=1}^{k} \left|a_n^* \, |a_n|^{q-2}\right|^p\right)^{1/p} \\
&= c \left(\sum_{n=1}^{k} |a_n|^{p(q-1)}\right)^{1/p} = c \left(\sum_{n=1}^{k} |a_n|^q\right)^{1/p}
\end{aligned}$$

and therefore

$$\left(\sum_{n=1}^{k} |a_n|^q\right)^{1/q} = \left(\sum_{n=1}^{k} |a_n|^q\right)^{1-1/p} \leqslant c.$$

A simple approximation argument shows that this inequality holds even if we remove the assumption that $a_i \neq 0$ for $i \leqslant k$. Conversely, if c is a positive number such that c^q is a bound for the partial sums of $\sum_{n=1}^{\infty} |a_n|^q$, then for each $\mathbf{x} \in l_2$ and each k we have

$$\begin{aligned}
|f(x_1, x_2, \ldots, x_k, 0, 0, \ldots)| &= \left|\sum_{n=1}^{k} a_n x_n\right| \\
&\leqslant \left(\sum_{n=1}^{k} |a_n|^q\right)^{1/q} \left(\sum_{n=1}^{k} |x_n|^p\right)^{1/p} \leqslant c \, \|\mathbf{x}\|_p .
\end{aligned}$$

Since (by Corollary 4) f is sequentially continuous and

$$\mathbf{x} = \lim_{k\to\infty} (x_1, x_2, \ldots, x_k, 0, 0, \ldots)$$

in l_p, it follows that $|f(\mathbf{x})| \leqslant c \|\mathbf{x}\|_p$. Thus our suggestion that, under the hypotheses of Landau's theorem, the series $\sum_{n=1}^{\infty} |a_n|^q$ has bounded partial sums is equivalent to the corresponding linear functional, defined at (1), being continuous. This equivalence, taken with work of Ishihara [Ishihara 1992], suggests that we bring into play the following notions.

We say that a subset S of $\mathbf{N}$ is **pseudobounded** if $\lim_{n\to\infty} n^{-1}s_n = 0$ for each sequence $(s_n)_{n\geqslant 1}$ in S. Following Ishihara [Ishihara 1992], we consider the principle

BD-N *Every inhabited, countable, pseudobounded subset of the set* $\mathbf{N}^+$ *of positive integers is bounded,*

which holds in the intuitionistic and recursive models of BISH, but, being independent of Heyting arithmetic [Lietz 2004], is not provable within BISH. In [Ishihara 1992], Ishihara proved that the statement 'Every sequentially continuous linear mapping from a separable metric space into a metric space is pointwise continuous' is equivalent to **BD-N**.

Our next result (whose proof has, unsurprisingly, some similarities to that of Lemma 20 in [Ishihara 2001]) belongs to constructive reverse mathematics, a relatively new field in which theorems are classified according to their equivalence, over some formal or (in this case) informal system for constructive mathematics, to certain principles such as **BD-N**. For more on this topic, see [Ishihara 2006].

Theorem 5. *The following statements are equivalent.*

(i) *There exist conjugate exponents* p, q *such that if* $\mathbf{a}$ *is any sequence of complex numbers such that* $\sum_{n=1}^{\infty} a_n x_n$ *converges for each* $\mathbf{x} \in l_p$*, then* $\sum_{n=1}^{\infty} |a_n|^q$ *has bounded partial sums.*

(ii) **BD-N**.

(iii) *For all conjugate exponents* p *and* q*, if* $\mathbf{a}$ *is any sequence of complex numbers such that* $\sum_{n=1}^{\infty} a_n x_n$ *converges for each* $\mathbf{x} \in l_p$*, then* $\sum_{n=1}^{\infty} |a_n|^q$ *has bounded partial sums.*

Proof. The implication from **BD-N** to (iii) is a consequence of Corollary 4 and the result of Ishihara mentioned before the statement of this proposition. The implication from (iii) to (i) is trivial. So it remains to prove that (i) implies **BD-N**. Assuming, then, that the conjugate exponents p, q have the property described in (i). Let

$$S \equiv \{s_1, s_2, \ldots\}$$

be an inhabited, countable, pseudobounded subset of $\mathbf{N}$. In order to prove that S is bounded, we may, if necessary, replace s_n by $\max\{s_1, \ldots, s_n\}$. Thus we may assume that $s_1 \leqslant s_2 \leqslant \cdots$. Setting

$$b_1 \equiv \sqrt[q]{s_1},\ b_{n+1} \equiv \sqrt[q]{s_{n+1} - s_n},$$

we need only prove that $\sum_{n=1}^{\infty} b_n x_n$ converges for each $\mathbf{x} \in l_p$: for then the partial sums of the series $\sum_{n=1}^{\infty} |b_n|^q$ are bounded, which implies the boundedness of the set S. Accordingly, fix $\mathbf{x} \in l_p$ and let $(n_k)_{k \geqslant 1}$ be a strictly increasing sequence of positive integers such that

$$\sum_{n=n_k}^{\infty} |x_n|^p < \left(\frac{1}{2^{k+1}k}\right)^p \tag{2}$$

for each k. Define

$$I_k \equiv \{n_k, n_k + 1, \ldots, n_{k+1} - 1\}\ .$$

Since S is pseudobounded, there exists κ such that $s_{n_{k+1}} < k$ for all $k \geqslant \kappa$. For $k' > k \geqslant \kappa$ we have

$$\begin{aligned}\left|\sum_{n=n_k}^{n_{k'}-1} b_n x_n\right| &\leqslant \sum_{j=k}^{k'} \left(\sum_{i \in I_j} |b_i x_i|\right) \leqslant \sum_{j=k}^{k'} \left(\sqrt[q]{\sum_{i \in I_j} |b_i|^q}\ \sqrt[p]{\sum_{i \in I_j} |x_i|^p}\right) \\ &\leqslant \sum_{j=k}^{k'} \frac{s_{n_{j+1}} - s_{n_j}}{2^{j+1}j} \leqslant \sum_{j=k}^{k'} \frac{s_{n_{j+1}}}{j} 2^{-j-1} \leqslant \sum_{j=k}^{k'} 2^{-j-1} < 2^{-k}.\end{aligned}$$

It readily follows that the partial sums of $\sum_{n=1}^{\infty} b_n x_n$ form a Cauchy sequence, and hence that the series converges in $\mathbf{C}$. This completes the proof that (i) implies (ii).

Perhaps the most significant aspect of Theorem 5 is this: Ishihara's original result relating **BD-N** and the passage from sequential to pointwise continuity used a relatively strange space as the domain of the sequentially continuous mapping; in contrast, and in view of Corollary 4 and the remarks following it, Theorem 5 carries through this relation using one of the standard spaces in functional analysis.

Our next result confirms that the use of the classical uniform boundedness theorem in proving Landau's theorem is not just a matter of convenience.

Proposition 6. *Statement* (iii) *of Theorem 5 is equivalent to the classical uniform boundedness theorem in the form*

UBT$_c$ *If* $(T_n)_{n \geqslant 1}$ *is a sequence of bounded linear mappings of a Banach space* X *into a Banach space* Y *such that*

$$\{T_n x : n \geqslant 1\}$$

is bounded for each $x \in X$*, then there exists* $c > 0$ *that is a bound for each of the operators* T_n*.*

Proof. Ishihara [Ishihara 2007] has shown that **UBT**$_c$ is equivalent to **BD-N**. The result now follows from Theorem 5.

The question now arises: *what can we say about Landau's theorem without assuming* **BD-N**? The next three lemmas take some distance in the direction of an answer.

Lemma 7. *Let* p, q *be conjugate exponents, let* $\mathbf{a}$ *be a sequence of complex numbers such that* $\sum_{n=1}^{\infty} a_n x_n$ *converges for each* $\mathbf{x}$ *in* l_p*, and let* $\phi : \mathbf{N}^+ \to \mathbf{N}^+$ *be a strictly increasing mapping. Let* $(\lambda_k)_{k \geqslant 1}$ *be an increasing binary sequence such that if* $\lambda_k = 1 - \lambda_{k-1}$*, then there exists* $\nu \geqslant k$ *such that* $\sum_{n=1}^{\nu} |a_n|^q > \phi(k)$*. Then either* $\lambda_k = 0$ *for all* k *or else there exists* K *such that* $\lambda_K = 1$*.*

Proof. Let $\mathbf{u}$ be a unit vector in l_q, set $\lambda_0 = 0$, and define a sequence $(f_k)_{k \geqslant 1}$ of normed linear functionals on l_p as follows. For each positive integer k if $\lambda_k = \lambda_{k-1}$, define

$$f_k(\mathbf{x}) = k \sum_{n=1}^{\infty} u_n x_n \quad (\mathbf{x} \in l_p)$$

and note that $\|f_k\| = k$. If $\lambda_k = 1 - \lambda_{k-1}$, then, choosing $\nu \geqslant k$ such that $\sum_{n=1}^{\nu} |a_n|^q > \phi(k)$, define

$$f_k(\mathbf{x}) = \sum_{n=1}^{\nu} a_n x_n \quad (\mathbf{x} \in l_p)$$

and note that $\|f_k\| > (\phi(k))^{1/q}$. Clearly, $\|f_k\| \to \infty$ as $k \to \infty$; so, by Theorem 2, there exists a unit vector $\mathbf{x} \in l_p$ such that $|f_k(\mathbf{x})| \to \infty$ as $k \to \infty$. Since $\sum_{n=1}^{\infty} a_n x_n$ converges, there exists K such that

$$|f_k(\mathbf{x})| > 1 + \left| \sum_{n=1}^{k} a_n x_n \right| \qquad (k \geqslant K). \tag{3}$$

Suppose that $\lambda_k = 1 - \lambda_{k-1}$ for some $k > K$. Then $f_k(\mathbf{x}) = \sum_{n=1}^{\nu} a_n x_n$ for some $\nu \geqslant k$, which is absurd in view of (3). Hence $\lambda_k = \lambda_K$ for all $k \geqslant K$, from which the desired conclusion follows.

Lemma 8. *Let p, q be conjugate exponents, let* $\mathbf{a}$ *be a sequence of complex numbers such that* $\sum_{n=1}^{\infty} a_n x_n$ *converges for each* $\mathbf{x}$ *in* l_p*, and let* $\phi : \mathbf{N}^+ \to \mathbf{N}^+$ *be a strictly increasing mapping. Let* $(\lambda_k)_{k\geqslant 1}$ *be an increasing binary sequence, and* $(n_k)_{k\geqslant 1}$ *an increasing sequence of positive integers, such that if* $\lambda_k = 0$*, then* $\sum_{n=1}^{n_k} |a_n|^q > \phi(k) - 1$*. Then there exists* K *such that* $\lambda_K = 1$*.*

Proof. Again let $\mathbf{u}$ be a unit vector in l_p and set $\lambda_0 = 0$. This time, for each $\mathbf{x}$ in l_p we define $f_k(\mathbf{x}) = \sum_{n=1}^{n_k} a_n x_n$ if $\lambda_k = 0$, and $f_k(\mathbf{x}) = k \sum_{n=1}^{\infty} u_n x_n$ if $\lambda_k = 1$. This produces a sequence $(f_k)_{k\geqslant 1}$ of normed linear functionals on l_p such that $\|f_k\| \to \infty$ as $k \to \infty$. Using Theorem 2, we produce a unit vector $\mathbf{x}$ in l_p such that $|f_k(\mathbf{x})| \to \infty$ as $k \to \infty$. Since $\sum_{n=1}^{\infty} a_n x_n$ converges, there exists K such that

$$|f_k(\mathbf{x})| > 1 + \left| \sum_{n=1}^{n_k} a_n x_n \right| \qquad (k \geqslant K).$$

If $\lambda_K = 0$, then $f_K(\mathbf{x}) = \sum_{n=1}^{n_K} a_n x_n$, which is absurd in view of our choice of K. Hence $\lambda_K = 1$.

Lemma 9. *Let p, q be conjugate exponents, let* $\mathbf{a}$ *be a sequence of complex numbers such that* $\sum_{n=1}^{\infty} a_n x_n$ *converges for each* $\mathbf{x}$ *in* l_p*, and let* $\phi : \mathbf{N}^+ \to \mathbf{N}^+$ *be a strictly increasing mapping. Then either* $\sum_{n=1}^{k} |a_n|^q < \phi(k)$ *for all* k *or else there exists* k *such that* $\sum_{n=1}^{k} |a_n|^q > \phi(k) - 1$*.*

Proof. Construct an increasing binary sequence $(\lambda_k)_{k\geqslant 1}$ such that

$$\lambda_k = 0 \Rightarrow \forall_{j\leqslant k} \left(\sum_{n=1}^{j} |a_n|^q < \phi(j) \right),$$

$$\lambda_k = 1 - \lambda_{k-1} \Rightarrow \sum_{n=1}^{k} |a_n|^q > \phi(k) - 1.$$

Now apply Lemma 7.

Proposition 10. *Let p, q be conjugate exponents, let* $\mathbf{a}$ *be a sequence of complex numbers such that* $\sum_{n=1}^{\infty} a_n x_n$ *converges for each* $\mathbf{x}$ *in* l_p*, and let* $\phi : \mathbf{N}^+ \to \mathbf{N}^+$ *be a strictly increasing mapping. Then there exists* K *such that* $\sum_{n=K+1}^{m} |a_n|^q < \phi(m)$ *for all* $m \geqslant K$*.*

Proof. In view of the previous lemma, we may suppose that there exists n_1 such that $\sum_{n=1}^{n_1} |a_n|^q > \phi(n_1) - 1$. Setting $\lambda_1 = 0$ and applying Lemma 9 to the sequence $(0, 0, \ldots, 0, a_{n_1+1}, a_{n_1+2}, \ldots)$, we see that either $\sum_{n=n_1+1}^{m} |a_n|^q < \phi(m)$ for all $m > n_1$ or else there exists $n_2 > n_1$ such that $\sum_{n=n_1+1}^{n_2} |a_n|^q > \phi(n_2) - 1$. In the first case we set $\lambda_k = 1$ and $n_k = n_1$ for all $k \geqslant 2$; in the second we set $\lambda_2 = 0$. Carrying on in this way, we construct an increasing binary sequence $(\lambda_k)_{k\geqslant 1}$ and an increasing sequence $(n_k)_{k\geqslant 1}$ of positive integers such that

- if $\lambda_{k+1} = 0$, then $n_{k+1} > n_k$ and $\sum_{n=n_k+1}^{n_{k+1}} |a_n|^q > \phi(n_{k+1}) - 1$;
- if $\lambda_{k+1} = 1 - \lambda_k$, then $\sum_{n=n_k+1}^{m} |a_n|^q < \phi(m)$ for all $m > n_k$, and $n_j = n_k$ for all $j \geqslant k$.

Applying Lemma 8, we obtain the desired conclusion.

It follows, for example, that under the hypotheses of Landau's theorem, for each positive integer m there exists N such that

$$\sum_{i=N}^{n} |a_i|^q < \underbrace{\log(\log(\cdots(\log n)\cdots))}_{m \text{ instances of "}\log\text{"}}$$

for all $n > N$. This is a long way from showing that the partial sums of $\sum_{i=1}^{\infty} |a_i|^q$ are bounded, but it is a step towards that aim (one that is constructively unattainable, in view of Theorem 5).

We now have a constructive substitute for the convergence of a_n to 0 in Landau's theorem.

Proposition 11. *Let p, q be conjugate exponents, and let $\mathbf{a}$ be a sequence of complex numbers such that the series $\sum_{n=1}^{\infty} a_n x_n$ converges for each $\mathbf{x}$ in l_p. Then for each $\varepsilon > 0$ and each positive integer ν there exists k such that $\sum_{n=(k-1)\nu}^{k\nu} |a_n|^q < \varepsilon$.*

Proof. Fix a unit vector $\mathbf{u}$ in l_q. For each positive integer k, construct an increasing binary sequence $(\lambda_k)_{k \geqslant 1}$ such that

$$\lambda_k = 0 \Rightarrow \forall_{j \leqslant k} \left(\sum_{n=(j-1)\nu}^{j\nu} |a_n|^q > \frac{\varepsilon}{2} \right),$$

$$\lambda_k = 1 - \lambda_{k-1} \Rightarrow \sum_{n=(j-1)\nu}^{j\nu} |a_n|^q < \varepsilon.$$

Applying Lemma 8 with $\phi(k) = 1 + \frac{k\varepsilon}{2}$, we see that there exists N such that $\lambda_N = 1$; whence $\sum_{n=(k-1)\nu}^{k\nu} |a_n|^q < \varepsilon$ for some $k \leqslant N$.

Corollary 12. *Let p, q be conjugate exponents, and let $\mathbf{a}$ be a sequence of complex numbers such that the series $\sum_{n=1}^{\infty} a_n x_n$ converges for each $\mathbf{x}$ in l_p. Then there exists a sequence $(n_k)_{k \geqslant 1}$ of positive integers such that for each k, $n_k + k < n_{k+1}$ and*

$$\sum_{n=n_k+1}^{n_k+k} |a_n|^q < 2^{-k}.$$

Proof. By Proposition 11, there exists n_1 such that $|a_{n_1}|^q < 2^{-1}$. Having computed n_k with the desired properties, apply Proposition 11 to the sequence $(a_n)_{n>n_k+k}$, to produce $n_{k+1} > n_k + k$ such that $\sum_{n=n_{k+1}+1}^{n_{k+1}+k+1} |a_n|^q < 2^{-k-1}$. This completes the inductive construction of the sequence $(n_k)_{k\geqslant 1}$.

The conclusion of Corollary 12 holds for any binary sequence with at most one term equal to 1, and so is not enough to yield constructively the result that, under the hypotheses of that corollary and with $p = q = 2$, $a_n \to 0$ as $n \to \infty$.

We conclude the paper by proving a constructive version of Landau's summability theorem that is classically equivalent to the classical version but has stronger hypotheses and conclusion than Corollary 4. For this we recall the constructive **least-upper-bound principle**:

> In order that an inhabited set S of real numbers that is bounded above have a supremum, it is necessary and sufficient that S be **order located**, in the sense that for all α, β with $\alpha < \beta$, either β is an upper bound for S or else there exists $x \in S$ such that $x > \alpha$ ([Bishop and Bridges 1985], page 37, Proposition (4.3)).

Theorem 13. *Let p, q be conjugate exponents, and let $\mathbf{a}$ be a sequence of complex numbers such that $\sum_{n=1}^{\infty} a_n x_n$ converges for each $\mathbf{x}$ in l_p. Then the following are equivalent.*

(i) *The series $\sum_{n=1}^{\infty} |a_n|^q$ is convergent.*

(ii) *For all α, β with $0 < \alpha < \beta$, either $\sum_{n=1}^{k} |a_n|^q < \beta$ for all k or else there exists k such that $\sum_{n=1}^{k} |a_n|^q > \alpha$.*

Proof. It is clear that if $\sum_{n=1}^{\infty} |a_n|^q$ converges, then (ii) holds. Conversely, assuming (ii), construct an increasing binary sequence $(\lambda_k)_{k\geqslant 1}$ and an increasing sequence $(n_k)_{k\geqslant 0}$ of positive integers with $n_0 = 0$, such that

▷ if $\lambda_k = 0$, then $n_k > n_{k-1}$ and $\sum_{i=1}^{n_k} |a_i|^q > k$, and

▷ if $\lambda_k = 1$, then $n_k = n_{k-1}$ and $\sum_{i=1}^{j} |a_i|^q < k + 1$ for all j.

To do so, first observe that either $\sum_{i=1}^{j} |a_i|^q < 2$ for all j or else there exists $n_1 \geqslant 1$ such that $\sum_{i=1}^{n_1} |a_i|^q > 1$. In the first case set $\lambda_1 = n_1 = 1$; in the second, set $\lambda_1 = 0$. Now suppose we have found λ_{k-1} and n_{k-1} with the applicable properties. If $\lambda_{k-1} = 1$, set $\lambda_k = 1$ and $n_k = n_{k-1}$. If $\lambda_{k-1} = 0$, then by (ii), either $\sum_{i=1}^{j} |a_i|^q < k + 1$ for all j, in which case we set $\lambda_k = 1$ and $n_k = n_{k-1}$; or else there exists n_k such that $\sum_{i=1}^{n_k} |a_i|^q > k$. In the latter case, replacing

n_k by a sufficiently large positive integer, we may assume that $n_k > n_{k-1}$; we then set $\lambda_k = 0$ to complete the inductive construction. Taking $\phi(k) = k+1$ in Lemma 8, we obtain K such that $\lambda_K = 1$. The partial sums of $\sum_{i=1}^{\infty} |a_i|^q$ are therefore bounded above by $K+1$. It follows from (ii) and the constructive least-upper-bound principle that $\sum_{i=1}^{\infty} |a_i|^q$ converges in $\mathbf{R}$.

In view of the constructive least-upper-bound principle, it is curious that condition (ii) is used to prove that the partial sums of $\sum_{n=1}^{\infty} |a_n|^2$ are bounded before it is again invoked to prove that their supremum exists.

For related work within the framework of Weihrauch's theory of Type Two Effectivity [Weihrauch 2000], see [Brattka 2005]. For connections between that theory and Bishop-style constructive mathematics, see [Bauer 2005].

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