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A Heuristic Approach to Positive Root Isolation for Multiple Power Sums

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Abstract: Given a multiple power sum (extending polynomial's exponents to real numbers), the positive root isolation problem is to find a list of disjoint intervals, satisfying that they contain all positive roots and each of them contains exactly distinct one. In this paper, we develop the pseudo-derivative sequences for multiple power sums, then generalize Fourier's theorem and Descartes' sign rule for them to overestimate the number of their positive roots. Furthermore we bring up some formulas of linear and quadratic complexity to compute complex root bounds and positive root bounds based on Descartes' sign rule and Cauchy's theorem. Besides, we advance a factorization method for multiple power sums with rational coefficients utilizing $\mathbb{Q}$-linear independence, thus reduce the computational complexity in the isolation process. Finally we present an efficient algorithm to isolate all positive roots under any given minimum root separation.

Key Words: multiple power sums, root isolation, root bounds, Descartes' sign rule, Fourier's theorem
Category: F.2.1, G.1.5

1 Introduction

Real root computation of univariate functions is a fundamental problem in constructive and computational mathematics [Bridges 1994]. Applications of real root isolation are numerous from theoretical research to engineering practice. However it is quite difficult to get all real roots, since there is no finite root extraction for a univariate polynomial of degree 5 or higher. Even for some low-degree polynomials, their roots, calculated by the root extractions, are in complex irrational form that will involve more efforts in successive calculation. Thus we would rather isolate them in disjoint intervals, each containing distinct one and together containing all, than compute the exact values generally. The real roots isolation for polynomials with integer coefficients has a series of classic work utilizing some substantial properties of polynomials as follows:

1. Cauchy index and Sturm sequence (Sturm 1829, [Collins and Loos 1983, Knuth 1997, and references therein]): It utilizes a well-founded GCD sequence and alternates signs to generate Sturm sequence, which can determine the exact number of the distinct real roots in arbitrary interval, then isolates them through bisection.

2. Descartes' sign rule and Vincent's method: Descartes' sign rule overestimates the number of all positive roots, which is equal to the number of sign variations in polynomial coefficient sequence. The famous algorithm in [Collins and Akritas 1976] isolates each distinct real root by repeatedly bisecting this interval via the continued fraction transformation like $(cx+d)^m$ $f(\frac{ax+b}{cx+d})$ from Vincent's theorem (Vincent 1836, [Uspensky 1948, and references therein]). Moreover [Akritas 1978] introduced a new continued fraction method for largely shortening the scope of positive roots in each bisecting iteration, which is one of the most efficient isolation algorithms by increasingly refining the positive root upper and lower bounds [Akritas et al. 2006, Akritas et al. 2008] and integrated into most computer algebra systems including MATHEMATICA, MAPLE and SYNAPS [Tsigaridas and Emiris 2006]. For another progress, [Rouillier and Zimmermann 2004] proposed a semi-numerical algorithm for handling polynomials with interval coefficients.

3. The differentiation method: [Collins and Loos 1976], based on Rolle's theorem, advocated a sinking and lifting procedure to isolate all real roots of a polynomial by the monotonicity in its derivative's complement isolation list. Due to the relationship between algebraic numbers and their minimal defining polynomials, it is feasible to test whether a root is multiple, which makes this method sound and complete.

4. The complete discrimination system for polynomials: The complete discrimination system [Yang et al. 1996, Yang 1999] can determine the number and

multiplicities of all complex roots by computing the sub-resultant chains of Sylvester resultants. To some extent, it functionally extends Sturm sequence and largely improves efficiency in tackling with parameterized coefficients.

These researches of real roots for polynomials are rather mature and have been embedded into many important algorithms, such as *Cylindrical Algebraic Decomposition* [Collins 1975]. Nevertheless the real root isolation for multiple power sums of the form $\sum_{i=0}^{m} q_i x^{\lambda_i}$ with both real coefficients and real exponents, where $q_i \neq 0$ and $\lambda_0 < \lambda_1 < \cdots < \lambda_m$, receives less attentions in past. Owing to the restriction with the domain of power functions, we only consider the positive root isolation problem. Unfortunately those traditional methods do not work for multiple power sums, because the existing GCD algorithm, continued fraction transformations, resultants and algebraic roots are just defined for polynomials. Furthermore the roots of those special but important univariate functions are much harder to be rigorously determined. A notable result proved by [Qu and Wong 1999] is that $(v - a_k(\frac{v}{2})^{\frac{1}{3}}, v - a_k(\frac{v}{2})^{\frac{1}{3}} + \frac{3a_k^2}{20}(\frac{2}{v})^{\frac{1}{3}})$, where a_k is the k-th negative root of the Airy function, are the "best possible" bounds for the k-th positive root of the Bessel function of positive order v. Besides, [Jin and Wong 1999, Qiu and Wong 2004] yielded some useful asymptotic estimates for the roots of the Meixner polynomials and the Krawtchouk polynomials. These special functions (including multiple power sums) are possible templet solutions to the differential system of fractional order, who can withstand noises or perturbations in stochastic sampling [Li 2010, Li and Li 2010], and widely occur in many physical systems such as *Traffic Modeling* [Li and Zhao 2010].

In this paper, we develop the pseudo-derivative sequences for multiple power sums inspired by [Achatz et al. 2008, Strzeboński 2008]. Therefore we can generalize Fourier's theorem, which is a more powerful tool than Descartes' sign rule and applicable to polynomials in customary, to analyze the positive roots of multiple power sums. So Descartes' sign rule follows immediately. Furthermore some useful complex root bounds and positive root bounds of linear and quadratic complexity are also obtained by Descartes' sign rule and Cauchy's theorem. Besides, we advance a factorization method for multiple power sums with rational coefficients to produce potential multiple power sums without any multiple roots by $\mathbb{Q}$-linear independence. Finally we give an efficient algorithm for isolating all positive roots of multiple power sums based on Fourier's theorem and positive root bounds under the given positive root separation. Our algebraic algorithm widely differs with the numerical one. In principle, a numerical root-finder based on the theory of approximation can not always compute the exact roots because it works with finite and fixed precision. For instance, Russian constructivist school has proved that there is no algorithm to isolate multiple roots of polynomials with real coefficients, even when these real numbers are defined by explicit recursive functions, due to floating-point errors. To the opposite, our algebraic method using intervals can dynamically adjust the precision in execution as we need. So it is more reliable at the price of the moderately larger use of time and memory.

The remainder of this paper is organized as follows. In Section 2-3, the basic notations of multiple power sums and factorization will be introduced. In Section 4, we extend Fourier's theorem and Descartes' sign rule to multiple power sums. We raise some useful root bounds in Section 5 and present the main isolation algorithm in Section 6. Section 7 is the conclusion.

2 Multiple power sums

In this section, we give a formal definition of multiple power sum and its pseudo-derivative sequence. Then an important property of this pseudo-derivative sequence is described, which is similar to Fourier's sequence (i.e. ordinary-derivative sequence) in some sense.

Let $q(y_0, y_1, \cdots, y_m)$ be a multivariate polynomial. We can define a ring homomorphism that substitutes each variable y_i with power function x^{λ_i} ($\lambda_i \in \mathbb{R}$):

$$\mathrm{hom} : q(y_0, y_1, \cdots, y_m) \mapsto q(x^{\lambda_0}, x^{\lambda_1}, \cdots, x^{\lambda_m}).$$

Definition 1. Given a $q(y_0, y_1, \cdots, y_m) \in \mathbb{R}[y_0, y_1, \cdots, y_m]$, its image under the mapping hom

$$q^*(x) = \mathrm{hom}(q(y_0, y_1, \cdots, y_m)) = q(x^{\lambda_0}, x^{\lambda_1}, \cdots, x^{\lambda_m})$$

is a multiple power sum (MPS for short).

Let $q^*(x) = \sum_{i=0}^{m} q_i x^{\lambda_i}$ ($q_i \neq 0 \wedge \lambda_0 < \lambda_1 < \cdots < \lambda_m$) be a MPS. Then the number of nonzero terms in q^*, denoted by $\mathrm{num}(q^*)$, is $m+1$. The tail coefficient $\mathrm{tc}(q^*)$ is q_0 and the tail exponent $\mathrm{te}(q^*)$ is λ_0 while the head coefficient $\mathrm{hc}(q^*)$ is q_m and the head exponent $\mathrm{he}(q^*)$ is λ_m. A MPS $q^*(x)$ is canonical if $\mathrm{te}(q^*) = 0$.

Example 1. Consider the trivariate polynomial $q(y_0, y_1, y_2) = 3y_0 - 10y_0^2 y_1 - 2y_0 y_1 + 5y_1 - y_0^2 y_1^2 + 2y_2^2$ and the mapping $\mathrm{hom} : q(y_0, y_1, y_2)) \mapsto q(x^{-1}, x^2, x^{\pi})$. So $q^*(x) = 3x^{-1} - 10 - 2x + 4x^2 + 2x^{2\pi}$ is a MPS consisting of five nonzero terms. Then $\mathrm{num}(q^*) = 5$, $\mathrm{te}(q^*) = -1$, $\mathrm{tc}(q^*) = 3$, $\mathrm{he}(q^*) = 2\pi$ and $\mathrm{hc}(q^*) = 2$. Furthermore $xq^*(x) = 3 - 10x - 2x^2 + 4x^3 + 2x^{2\pi+1}$ is a canonical MPS.

Next we can construct a pseudo-derivative sequence of q^* as follows:

$$\begin{cases} G_0 = \dfrac{q^*}{x^{\mathrm{te}(q^*)}} = q_0 + \sum\limits_{i=1}^{m} q_i x^{\lambda_i - \lambda_0}, \\ G_1 = \dfrac{G_0'}{x^{\mathrm{te}(G_0')}} = q_1(\lambda_1 - \lambda_0) + \sum\limits_{i=2}^{m} q_i(\lambda_i - \lambda_0) x^{\lambda_i - \lambda_1}, \\ \cdots, \\ G_m = \dfrac{G_{m-1}'}{x^{\mathrm{te}(G_{m-1}')}} = q_m \prod\limits_{j=0}^{m-1} (\lambda_m - \lambda_j). \end{cases} \tag{1}$$

From the construction, it can easily verified that $\mathrm{sign}(G_{i+1}(x)) = \mathrm{sign}(G_i'(x))$ $(\mathrm{sign}(G_0(x)) = \mathrm{sign}(q^*(x)))$ for arbitrary $x > 0$ (which can be a weak Fourier's sequence defined in [Strzeboński 2008]) and each G_i is canonical. Let $\mathrm{PDS}(q^*) = [G_0, G_1, \cdots, G_m]$ denote the whole sequence and $\mathrm{PDS}_M(q^*) = [G_0, G_1, \cdots, G_M]$ denote the partial sequence restricted to $[0, 1, \cdots, M]$ $(M \leq m)$.

Lemma 2. *For $0 \leq i \leq i + j \leq m$, there exist $\lambda_{i,j}, a_{i,j,k} \in \mathbb{R}$ such that*

$$G_{i+j} = x^{\lambda_{i,j}} G_i^{(j)} + \sum_{1 \leq k < j} a_{i,j,k} x^{\lambda_{i,j} - j + k} G_i^{(k)}. \tag{2}$$

Proof. If $j = 0$, it plainly holds. Otherwise assume that $G_{i+j} = x^{\lambda_{i,j}} G_i^{(j)} + \sum_{1 \leq k < j} a_{i,j,k} x^{\lambda_{i,j} - j + k} G_i^{(k)}$, with $G_{i+j+1} = G'_{i+j} / x^{\mathrm{te}(G'_{i+j})}$, then define

$$\begin{cases} \lambda_{i,j+1} = \lambda_{i,j} - \mathrm{te}(G'_{i+j}), \\ a_{i,j+1,1} = a_{i,j,1}(\lambda_{i,j} - j + 1), \\ a_{i,j+1,k} = a_{i,j,k}(\lambda_{i,j} - j + k) + a_{i,j,k-1}, \quad (\text{for } 1 < k < j) \\ a_{i,j+1,j} = \lambda_{i,j} + a_{i,j,j-1}. \end{cases} \tag{3}$$

Hence $G_{i+j+1} = x^{\lambda_{i,j+1}} G_i^{(j+1)} + \sum_{1 \leq k < j+1} a_{i,j+1,k} x^{\lambda_{i,j+1} - j - 1 + k} G_i^{(k)}$. So it holds for all j by induction.

Theorem 3. *If α is an M-multiple positive root of $G_i(x)$, then $G_{i+j}(x)$ and $G_i^{(j)}(x)$ share the same sign in an ϵ-neighborhood of α for each $0 \leq j \leq M$.*

Proof. Let ϵ be a positive number such that G_{i+j} $(0 \leq j \leq M)$ has no root in $\delta(\alpha; \epsilon) \triangleq (\alpha - \epsilon, \alpha) \cup (\alpha, \alpha + \epsilon)$. On one hand, by Lemma 2, we have

$$\begin{pmatrix} G_{i+1}(x) \\ G_{i+2}(x) \\ \vdots \\ G_{i+M}(x) \end{pmatrix} = (t_{jk})_{M \times M} \begin{pmatrix} G_i'(x) \\ G_i^{(2)}(x) \\ \vdots \\ G_i^{(M)}(x) \end{pmatrix}, \tag{4}$$

where

$$t_{jk} = \begin{cases} 0, & \text{for } j < k, \\ x^{\lambda_{i,j}}, & \text{for } j = k, \\ a_{i,j,k} x^{\lambda_{i,j} - j + k}, & \text{for } j > k. \end{cases}$$

Since the transition matrix $(t_{jk})_{M \times M}$ is lower triangular and nonsingular for $x > 0$, $G_i(\alpha) = G_i'(\alpha) = \cdots = G_i^{(M-1)}(\alpha) = 0 \neq G_i^{(M)}(\alpha)$ if and only if $G_i(\alpha) = G_{i+1}(\alpha) = \cdots = G_{i+M-1}(\alpha) = 0 \neq G_{i+M}(\alpha)$.

On the other hand, we will prove inductively on all $0 \leq j \leq M$ that $G_i^{(j)}(x) > 0$ if and only if $G_{i+j}(x) > 0$ for each x in $\delta(\alpha; \epsilon)$. If $j = 0$, it plainly holds; otherwise the following statements are equivalent to each other:

1. $G_i^{(j+1)}(x) > 0$.
2. If $x < \alpha$, then $G_i^{(j)}(x) < 0$; else $G_i^{(j)}(x) > 0$. (by $G_i^{(j)}(\alpha) = 0$)
3. If $x < \alpha$, then $G_{i+j}(x) < 0$; else $G_{i+j}(x) > 0$. (by Inductive Hypothesis)
4. $G'_{i+j}(x) > 0$. (by $G_{i+j}(\alpha) = 0$)
5. $G_{i+j+1}(x) > 0$. (by Construction)

Therefore $\mathrm{sign}(G_{i+j}(x)) = \mathrm{sign}(G_i^{(j)}(x))$ for arbitrary $x \in (\alpha - \epsilon, \alpha + \epsilon)$.

3 Factorization

Next we introduce a factorization method for multiple power sums with rational coefficients by $\mathbb{Q}$-linear independence [Hardy and Wright 1979].

Definition 4. A set of numbers $\{\mu_0, \mu_1, \cdots, \mu_k\}$ is $\mathbb{Q}$-linearly independent if no linear relation $a_0\mu_0 + a_1\mu_1 + \cdots + a_k\mu_k = 0$ with rational coefficients a_is, not all zero, holds between them.

For a MPS $q^*(x) = \sum_{i=0}^{m} q_i x^{\lambda_i}$ $(q_i \in \mathbb{Q})$, we can factor it as follows:

1. To start with, we partition $\Lambda = \{\lambda_0, \lambda_1, \cdots, \lambda_m\}$ into $\sum_{j=0}^{k} \Lambda_j$, which are pairwise $\mathbb{Q}$-linearly independent:

   Basic Step: $\Lambda_0 = \{\lambda_i : \ \lambda_i = a_{i0} \in \mathbb{Q}\}$;

   Inductive Step: If there exists a μ_{l+1} in $\Lambda \setminus \sum_{j=0}^{l} \Lambda_j$, then

$$\Lambda_{l+1} = \{\lambda_i : \ \lambda_i = a_{i0} + \sum_{j=1}^{l+1} a_{ij}\mu_j \wedge a_{i0}, a_{i1}, \cdots, a_{il+1} \in \mathbb{Q} \wedge a_{il+1} \neq 0\}.$$

2. Now we have a $\mathbb{Q}$-linearly independent set $\{\mu_0, \mu_1, \cdots, \mu_k\}$: (with $\mu_0 = 1$)

$$\begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_m \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} & \cdots & a_{0k} \\ a_{10} & a_{11} & a_{12} & \cdots & a_{1k} \\ a_{20} & a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m0} & a_{m1} & a_{m2} & \cdots & a_{mk} \end{pmatrix} \begin{pmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \\ \vdots \\ \mu_k \end{pmatrix}. \tag{5}$$

   Further we choose another $\mathbb{Q}$-linearly independent set $\{\nu_0, \nu_1, \cdots, \nu_k\}$, satisfying $(\lambda_0, \lambda_1, \cdots, \lambda_m)^{\mathrm{T}} = (w_{ij})_{(m+1)\times(k+1)}(\nu_0, \nu_1, \cdots, \nu_k)^{\mathrm{T}}$ where $w_{ij} \in \mathbb{Z}$ and $\gcd_i\{w_{ij}\} = \pm 1$ for all j.

3. Define the "reverse" mapping $q^*(x) \rightarrow q^{**}(z_0, z_1, \cdots, z_k)$, i.e. the back-substitution of each x^{λ_i} with $\prod_{j=0}^{k} z_j^{w_{ij}}$ subject to $\lambda_i = \sum_{j=0}^{k} w_{ij}\nu_j$.
4. Let $w_j = \min_i\{w_{ij}\}$, the factorization of $q^*(x)/\prod_j x^{w_j\nu_j}$ corresponds to that of $q^{**}(z_0, z_1, \cdots, z_k)/\prod_j z_j^{w_j}$, a multivariate polynomial with rational coefficients.

Example 2. Consider the MPS

$$\begin{aligned} q^*(x) = {} & 49 - 98x - 28x^{\sqrt{2}} + 49x^2 + 28x^{\sqrt{2}+1} + 4x^{2\sqrt{2}} - 14x^3 + 14x^4 \\ & +4x^{\sqrt{2}+3} + x^6 + 14x^{2\pi} - 14x^{2\pi+1} - 4x^{2\pi+\sqrt{2}} - 2x^{2\pi+3} + x^{4\pi}. \end{aligned} \tag{6}$$

Its Λ is $\{0, 1, \sqrt{2}, 2, \sqrt{2}+1, 2\sqrt{2}, 3, 4, \sqrt{2}+3, 6, 2\pi, 2\pi+1, 2\pi+\sqrt{2}, 2\pi+3, 4\pi\}$, where $\{1, \sqrt{2}, 2\pi\}$ is a $\mathbb{Q}$-linearly independent set. After variable transformation, (with $z_0 = x, z_1 = x^{\sqrt{2}}, z_2 = x^{2\pi}$)

$$\begin{aligned} q^{**}(z_0, z_1, z_2) = {} & 49 - 98z_0 - 28z_1 + 49z_0^2 + 28z_0z_1 + 4z_1^2 - 14z_0^3 + 14z_0^4 \\ & +4z_0^3z_1 + z_0^6 + 14z_2 - 14z_0z_2 - 4z_1z_2 - 2z_0^3z_2 + z_2^2. \end{aligned} \tag{7}$$

Factor $q^{**}(z_0, z_1, z_2)$, we have $q^{**}(z_0, z_1, z_2) = (7 - 7z_0 - 2z_1 - z_0^3 + z_2)^2$, thus $q^*(x) = (7 - 7x - 2x^{\sqrt{2}} - x^3 + x^{2\pi})^2$.

Conjecture 5. *The only possible multiple positive root of a square-free MPS with rational coefficients is* 1.

In other words, it requires: (1) the only possible multiple positive root of an irreducible MPS with rational coefficients is 1; (2) the only possible common positive root of two relatively prime MPSs with rational coefficients is 1.

4 Generalized Fourier's theorem

Now Fourier's theorem is extended from polynomials to multiple power sums by the pseudo-derivative sequence described in Section 2. Thus the number of positive roots in any nonnegative interval can be overestimated. As a result, generalized Descartes' sign rule also holds.

Definition 6. Given a finite numerical sequence $S = [s_0, s_1, \cdots, s_m]$, the number of the sign variations $\mathcal{V}(S)$ is the number of pairs (i, j) with $(0 \le i < j \le m)$ satisfying:

$$(s_i s_j < 0) \wedge (\forall j > \tilde{j} > i :\ s_{\tilde{j}} = 0).$$

Theorem 7. *Given a MPS* $q^*(x) = \sum_{i=0}^{m} q_i x^{\lambda_i}$ *and a nonnegative interval* (a, b)*, the number of positive roots (counting multiple roots as their multiplicities) in* (a, b) *is* $\mathcal{V}([\mathrm{PDS}(q^*)]_{x=a}) - \mathcal{V}([\mathrm{PDS}(q^*)]_{x=b})$ *or* $2N$ *less than the difference.*

Proof. Only G_i's roots in (a, b) can cause these sign variations. Without loss of generality, we assume that $G_i \neq 0$ at endpoints a and b for all $0 \leq i \leq m$, otherwise we choose $a+\epsilon$ or $b-\epsilon$ as an endpoint. Let $\alpha \in (a, b)$ be an M-multiple root of $G_i(x)$, by Taylor's theorem, we have that $G_i(x) = \sum_{k=0}^{+\infty} \frac{(x-\alpha)^k}{k!} G_i^{(k)}(\alpha) = \sum_{k=M}^{+\infty} \frac{(x-\alpha)^k}{k!} G_i^{(k)}(\alpha)$ and further $\operatorname{sign}(G_i^{(j)}(x)) = \operatorname{sign}((x-\alpha)^{M-j} G_i^{(M)}(\alpha))$ for $x \in \delta(\alpha; \epsilon)$. Then we will discuss two distinct cases.

1. First Case: $i = 0$. Then $\operatorname{sign}(G_j(x)) = \operatorname{sign}(G_0^{(j)}(x))$ for $x \in \delta(\alpha; \epsilon)$ by Theorem 3. We have $\mathcal{V}([\mathrm{PDS}_M(G_0)]_{x=\alpha-\epsilon}) - \mathcal{V}([\mathrm{PDS}_M(G_0)]_{x=\alpha+\epsilon}) = M$, since

$$\begin{cases} \mathcal{V}([\mathrm{PDS}_M(G_0)]_{x=\alpha-\epsilon}) \\ \quad = \mathcal{V}([(-1)^M G_0^{(M)}(\alpha), (-1)^{M-1} G_0^{(M)}(\alpha), \cdots, G_0^{(M)}(\alpha)]) \\ \quad = M, \\ \mathcal{V}([\mathrm{PDS}_M(G_0)]_{x=\alpha+\epsilon}) \\ \quad = \mathcal{V}([G_0^{(M)}(\alpha), G_0^{(M)}(\alpha), \cdots, G_0^{(M)}(\alpha)]) \\ \quad = 0. \end{cases} \tag{8}$$

2. Second Case: $i > 0 \wedge G_{i-1}(\alpha) \neq 0$. Similarly we have $\mathcal{V}([\mathrm{PDS}_M(G_i)]_{x=\alpha-\epsilon}) - \mathcal{V}([\mathrm{PDS}_M(G_i)]_{x=\alpha+\epsilon}) = M$. Further

   (a) If M is even,

$$\begin{aligned} \mathcal{V}([\mathrm{PDS}_1(G_{i-1})]_{x=\alpha-\epsilon}) &= \mathcal{V}([G_{i-1}(\alpha), G_i(\alpha-\epsilon)]) \\ &= \mathcal{V}([G_{i-1}(\alpha), G_i(\alpha+\epsilon)]) \\ &= \mathcal{V}([\mathrm{PDS}_1(G_{i-1})]_{x=\alpha+\epsilon}). \end{aligned} \tag{9}$$

   (b) Otherwise

$$\begin{aligned} \mathcal{V}([\mathrm{PDS}_1(G_{i-1})]_{x=\alpha-\epsilon}) &= \mathcal{V}([G_{i-1}(\alpha), G_i(\alpha-\epsilon)]) \\ &= \mathcal{V}([G_{i-1}(\alpha), -G_i(\alpha+\epsilon)]) \\ &= \mathcal{V}([\mathrm{PDS}_1(G_{i-1})]_{x=\alpha+\epsilon}) \pm 1. \end{aligned} \tag{10}$$

   Hence $\mathcal{V}([\mathrm{PDS}_{M+1}(G_{i-1})]_{x=\alpha-\epsilon}) - \mathcal{V}([\mathrm{PDS}_{M+1}(G_{i-1})]_{x=\alpha+\epsilon})$ is a nonnegative even number.

Therefore $\mathcal{V}([\mathrm{PDS}(q^*)]_{x=a}) - \mathcal{V}([\mathrm{PDS}(q^*)]_{x=b})$ is the number of q^*'s real roots in (a, b), or $2N$ more than the number.

By Theorem 7, we obtain an overestimate of the number of positive roots for q^* in (a, b), which can not determine the exact number unless the difference of sign variations number at endpoints is 0 or 1.

Theorem 8. *Given a MPS $q^*(x) = \sum_{i=0}^{m} q_i x^{\lambda_i}$, the number of positive roots (counting multiple roots as their multiplicities) is $\mathcal{V}(q^*) \triangleq \mathcal{V}([q_0, q_1, \cdots, q_m])$ or $2N$ less than the difference.*

Proof. Reviewing (1), we have

$$\begin{cases} \mathcal{V}([\mathrm{PDS}(q^*)]_{x=0}) & = \mathcal{V}([\mathrm{tc}(G_0), \mathrm{tc}(G_1), \cdots, \mathrm{tc}(G_m)]) \\ & = \mathcal{V}([q_0, q_1(\lambda_2 - \lambda_1), \cdots, q_m \prod_{j=0}^{m-1}(\lambda_m - \lambda_j)]) \\ & = \mathcal{V}(q^*), \\ \mathcal{V}([\mathrm{PDS}(q^*)]_{x=+\infty}) & = \mathcal{V}([\mathrm{hc}(G_0), \mathrm{hc}(G_1), \cdots, \mathrm{hc}(G_m)]) \\ & = \mathcal{V}([q_m, q_m(\lambda_m - \lambda_1), \cdots, q_m \prod_{j=0}^{m-1}(\lambda_m - \lambda_j)]) \\ & = 0. \end{cases} \tag{11}$$

So it follows immediately by Theorem 7.

5 Root bounds

By extending the inequality tips from polynomials in [Mignotte 1983], some useful root bounds of linear and quadratic complexity are obtained. A criteria for the relationship between upper and lower bounds for positive roots is also given.

Definition 9. Given a function f, a positive number U is the complex root bound if no magnitude of f's complex roots is greater than U, i.e.

$$\forall\ \alpha \in \mathbb{C}:\ f(\alpha) = 0 \Rightarrow U \geq |\alpha|.$$

A positive number u is the positive root upper bound if no f's positive root is greater than u, while a positive number l is the positive root lower bound if no f's positive root is less than l, i.e.

$$\forall\ \alpha \in \mathbb{R}^+:\ f(\alpha) = 0 \Rightarrow u \geq \alpha \geq l.$$

Lemma 10. *For a MPS $q^*(x) = \sum_{i=0}^{m} q_i x^{\lambda_i}$, a positive number U is a complex root bound if*

$$U^{\lambda_m} \geq \sum_{i=0}^{m-1} \left| \frac{q_i}{q_m} \right| U^{\lambda_i}. \tag{12}$$

Proof. Consider the auxiliary MPS $A(x) = x^{\lambda_m} - \sum_{i=0}^{m-1} |\frac{q_i}{q_m}| x^{\lambda_i}$, $0 \leq A(U) < A(+\infty)$. Let α be an arbitrary complex root of $q^*(x)$. However $A(|\alpha|) \leq 0$, since $|\alpha|^{\lambda_m} = |\alpha^{\lambda_m}| = |\sum_{i=0}^{m-1} \frac{q_i}{q_m} \alpha^{\lambda_i}| \leq \sum_{i=0}^{m-1} |\frac{q_i}{q_m}||\alpha|^{\lambda_i}$. Since $\mathcal{V}(A(x)) = 1$, by Theorem 8, $A(x)$ has and only has one positive root, thus $U \in [|\alpha|, +\infty)$. Hence U is a complex root bound by the arbitrariness of α.

Given a MPS $q^*(x) = \sum_{i=0}^{m} q_i x^{\lambda_i}$ and $\tau \in (0, \lambda_m - \lambda_{m-1}]$, with $\lambda = \mathrm{rem}(\lambda_m, \tau)$ and $\tilde{m} = \mathrm{quo}(\lambda_m, \tau)$, $q^*(x; \tau) = x^{\lambda_m} + x^{\lambda} \sum_{i=0}^{\tilde{m}-1} q_i(\tau) x^{i\tau}$ is an upper "polynomial" of $q^*(x)$, where $q_i(\tau) = \sum_{(i-1)\tau < \lambda_j - \lambda \leq i\tau} |\frac{q_j}{q_m}|$. It is obvious that $x^{\lambda} \sum_{i=0}^{\tilde{m}-1} q_i(\tau) x^{i\tau} \geq \sum_{i=0}^{m-1} |\frac{q_i}{q_m}| x^{\lambda_i}$ for $x \geq 1$, so we have the following corollary.

Corollary 11. *For a MPS $q^*(x) = \sum_{i=0}^m q_i x^{\lambda_i}$, if $q^*(x;\tau)$ is an upper polynomial with the parameter $\tau \in (0, \lambda_m - \lambda_{m-1}]$, then $U(\tau) = \sqrt[\tau]{1 + \max_i\{q_i(\tau)\}}$ is a complex root bound of $q^*(x)$.*

Proof. $U(\tau)$ is a complex root bound of both $q^*(x;\tau)$ and $q^*(x)$, since

$$\begin{aligned} U(\tau)^{\lambda_m} &= U(\tau)^{\lambda} U(\tau)^{\tilde{m}\tau} \\ &\geq U(\tau)^{\lambda}[(1 + q_{\tilde{m}-1}(\tau))U(\tau)^{(\tilde{m}-1)\tau}] \\ &\geq U(\tau)^{\lambda}[(1 + q_{\tilde{m}-2}(\tau))U(\tau)^{(\tilde{m}-2)\tau} + q_{\tilde{m}-1}(\tau)U(\tau)^{(\tilde{m}-1)\tau}] \\ &\geq \cdots \\ &\geq U(\tau)^{\lambda} \sum_{i=0}^{\tilde{m}-1} q_i(\tau)U(\tau)^{i\tau} \\ &\geq \sum_{i=0}^{m-1} |\frac{q_i}{q_m}| U(\tau)^{\lambda_i}. \end{aligned} \tag{13}$$

The complexity in Corollary 11 is linear, under any given parameter τ. Next step is how to choose the optimal parameter τ so as to the complex root bound $U(\tau)$ is as small as possible. Intuitionally we claim that the optimal parameter achieves only when $\lambda_m - \lambda_i$ is the least feasible multiple of τ for some $i < m$. Thus it has the quadratic complexity.

Example 3. Consider the MPS $q^*(x) = 3 - 10x - 2x^2 + 4x^3 + 2x^{2\pi+1}$ with the parameter τ ranging from 0 to $2\pi - 2$. We only choose four sample points $2\pi - 2$, $\pi - \frac{1}{2}$, π and $\pi + \frac{1}{2}$ to approximate the smallest complex root bound. After computation,

$$\begin{aligned} &U(2\pi - 2) = \sqrt[2\pi-2]{1 + \tfrac{19}{2}} \approx 2.0242, \\ &U(\pi - \tfrac{1}{2}) = \sqrt[\pi-\frac{1}{2}]{1 + \max\{2, \tfrac{15}{2}\}} \approx 2.2482, \\ &U(\pi) = \sqrt[\pi]{1 + \max\{3, \tfrac{13}{2}\}} \approx 1.8991, \\ &U(\pi + \tfrac{1}{2}) = \sqrt[\pi+\frac{1}{2}]{1 + \max\{8, \tfrac{3}{2}\}} \approx 1.8283, \end{aligned}$$

so we regard 1.8283 as the "smallest" complex root bound.

Corollary 12. *For a MPS $q^*(x) = \sum_{i=0}^m q_i x^{\lambda_i}$, with $\tau = \lambda_m - \lambda_{m-1}$, $U = \max\{1, \sqrt[\tau]{\sum_{i=0}^{m-1} |\frac{q_i}{q_m}|}\}$ is a complex root bound.*

Proof. U is a complex root bound, since

$$\begin{cases} U^{\tau} \geq \sum_{i=0}^{m-1} |\frac{q_i}{q_m}| \text{ implies } U^{\lambda_m} \geq \sum_{i=0}^{m-1} |\frac{q_i}{q_m}| U^{\lambda_{m-1}}, \\ U \geq 1 \text{ implies } \sum_{i=0}^{m-1} |\frac{q_i}{q_m}| U^{\lambda_{m-1}} \geq \sum_{i=0}^{m-1} |\frac{q_i}{q_m}| U^{\lambda_i}. \end{cases} \tag{14}$$

Although we have known that a positive number u is a positive root upper bound of $q^*(x)$ if $\mathcal{V}([\mathrm{PDS}(q^*)]_{x=u}) = \mathcal{V}([\mathrm{PDS}(q^*)]_{x=+\infty})$ from Theorem 7, we will bring some more computable positive root upper bounds under the assumption $\mathcal{V}(q^*) > 0$ (otherwise $q^*(x)$ has no positive root).

Given a MPS $q^*(x) = \sum_{i=0}^{m} q_i x^{\lambda_i}$, we partition it as an alternating sum of sub-MPSs, i.e. $q^*(x) = \sum_{j=0}^{l} (-1)^j r_j^*(x)$, satisfying:

1. $\mathrm{he}(r_{j_1}^*(x)) < \mathrm{te}(r_{j_2}^*(x))$ if $j_1 < j_2$,
2. $r_0^*(x) = 0$ if $q_0 q_m > 0$,
3. all coefficients in every $r_j^*(x)$ are positive except when $r_0^*(x)$ is zero.

We focus on the property that the j_2-th sub-MPS $(-1)^{j_2} r_{j_2}^*(x)$ is asymptotically dominant to the j_1-th sub-MPS $(-1)^{j_1} r_{j_1}^*(x)$ as x increases if $j_1 < j_2$. Hence, matching the positive coefficients with the negative coefficients by different combination and partition strategy yields different positive root upper bounds (cf. [Ştefănescu 2005, Akritas and Vigklas 2007, Akritas 2009]).

Corollary 13. *For a MPS $q^*(x) = \sum_{i=0}^{m} q_i x^{\lambda_i} = \sum_{j=0}^{l} (-1)^j r_j^*(x)$, a positive number u is a positive root upper bound if*

$$\bigwedge_{0 \le j < l \wedge 2 \nmid (l-j)} r_{j+1}^*(u) \ge r_j^*(u). \tag{15}$$

If the MPS $r_{j+1}^*(x) - r_j^*(x)$ is nonnegative at a positive number u_j, then u_j is a positive root upper bound of it, since it has only one positive root by Theorem 8. However we can compute the root bounds of $\mathrm{hc}(r_{l+1}^*(x)) x^{\mathrm{he}(r_{j+1}^*(x))} - r_j^*(x)$ and $r_{j+1}^*(1) x^{\mathrm{te}(r_{j+1}^*(x))} - r_j^*(x)$ by Corollary 11–12 instead of u_j because of $r_{j+1}^*(x) \ge \mathrm{hc}(r_{l+1}^*(x)) x^{\mathrm{he}(r_{j+1}^*(x))}$ for $x > 0$ and $r_{j+1}^*(x) \ge r_{j+1}^*(1) x^{\mathrm{te}(r_{j+1}^*(x))}$ for $x > 1$. Then (15) is not more than the quadratic complexity in total.

Example 4. Consider the MPS $q^*(x) = 3 - 10x - 2x^2 + 4x^3 + 2x^{2\pi+1}$ with the partitions $r_3^*(x) = 4x^3 + 2x^{2\pi+1}$, $r_2^*(x) = 10x + 2x^2$, $r_1^*(x) = 3$ and $r_0^*(x) = 0$. The root bounds of $2x^{2\pi+1} - r_2^*(x)$ is $\sqrt[2\pi-1]{6}$ while that of $6x^3 - r_2^*(x)$ is 2 by Corollary 12. Hence the positive root upper bound $\sqrt[2\pi-1]{6}$ is better.

Corollary 14. *A positive number l is a positive root lower bound of $q^*(x)$ if and only if $\frac{1}{l}$ is a positive root upper bound of $q^*(\frac{1}{x})$.*

6 Isolation algorithm

Definition 15. Given a function f, let $\alpha_1, \alpha_2, \cdots, \alpha_m$ be all complex roots of f, the minimum root separation is

$$\mathrm{sep}(f) = \min_{1 \le i < j \le m} |\alpha_i - \alpha_j|,$$

with the convention that $\mathrm{sep}(f) = +\infty$ in case f has only one root and $\mathrm{sep}(f) = 0$ in case f has multiple roots.

Here we give an efficient method for isolating all q^*'s positive roots based on generalized Fourier's theorem under the assumption $\mathrm{sep}(q^*) > \epsilon$.

$$L \leftarrow \mathrm{ISOL}(q^*(x), \epsilon).$$

Input: $q^*(x)$ is a nonzero MPS without any multiple roots and $\epsilon \in \mathbb{Q}^+$.

Output: $L = \{(a_1, b_1), (a_2, b_2), \cdots, (a_k, b_k)\}$ is a list of disjoint open intervals with rational endpoints, satisfying:

(a) k is the number of distinct positive roots of q^*;

(b) each (a_i, b_i) contains one distinct positive root of q^*.

S1 (Initialization) Compute $\mathrm{PDS}(q^*) := [G_0, G_1, \cdots, G_m]$.

S2 (Bound) Compute the scope of all positive roots (l, u) with positive rational endpoints. Let $L' := \{(l, u)\}$ and $L, L'' := \emptyset$.

S3 (Refinement) For each $I = (a, b) = (\frac{a_1}{a_2}, \frac{b_1}{b_2}) \in L'$:

(a) If $\mathcal{V}([\mathrm{PDS}(q^*)]_{x=a}) - \mathcal{V}([\mathrm{PDS}(q^*)]_{x=b}) = 1$, then $L := L \cup \{I\}$.

(b) If $\mathcal{V}([\mathrm{PDS}(q^*)]_{x=a}) - \mathcal{V}([\mathrm{PDS}(q^*)]_{x=b}) > 1$, then the median $c := \frac{a_1+b_1}{a_2+b_2}$.

i. If $q^*(c) = 0$, then $L := L \cup \{(\max\{a, c - \epsilon\}, \min\{c + \epsilon, b\})\}$.

A. If $a < c - \epsilon$, then $L'' := L'' \cup \{(a, c - \epsilon)\}$.

B. If $c + \epsilon < b$, then $L'' := L'' \cup \{(c + \epsilon, b)\}$.

ii. Otherwise $L'' := L'' \cup \{(a, c), (c, b)\}$.

Finally set $L' := L' \setminus \{I\}$.

S4 (Reduction) For each $I = (a, b) \in L''$:

(a) If $\| I \| \leq \epsilon$ and $q^*(a)q^*(b) < 0$, then $L := L \cup \{I\}$.

(b) If $\| I \| > \epsilon$, then $L' := L' \cup \{I\}$.

Finally set $L'' := L'' \setminus \{I\}$.

S5 (Recursion) If $L' = \emptyset$, rearrange L and RETURN it; else GOTO S3.

Here we use the median $\frac{a_1+b_1}{a_2+b_2}$ instead of the average $\frac{a+b}{2}$ to bisect the interval. The bisection by average has the optimal complexity of at most $\frac{\mathcal{V}(q^*)}{2} \log_2(\frac{u-l}{\epsilon})$ theoretically. However the bisection by median works more efficiently in practice.

Remark. For a MPS with rational coefficients q^*, let $\tilde{q}^*$ be the greatest square-free factor of it by factorization. Under Conjecture 5, the only possible multiple root of $\tilde{q}^*$ is 1. Thus set $L' := \{(l, \min\{1 - \epsilon, u\}), (\max\{1 + \epsilon, l\}, u)\}$ in S2 if $\tilde{q}^*(1) = 0$.

Example 5. Consider the MPS $\tilde{q}^*(x) = 7 - 7x - 2x^{\sqrt{2}} - x^3 + x^{2\pi}$, its pseudo-derivative sequence PDS($\tilde{q}^*$) is:

$$G_0 = 7 - 7x - 2x^{\sqrt{2}} - x^3 + x^{2\pi},$$

$$G_1 = -7 - 2\sqrt{2}x^{\sqrt{2}-1} - 3x^2 + 2\pi x^{2\pi-1},$$

$$G_2 = (2\sqrt{2} - 4) - 6x^{2-\sqrt{2}} + (4\pi^2 - 2\pi)x^{2\pi-\sqrt{2}},$$

$$G_3 = (6\sqrt{2} - 12) + (8\pi^3 - 4\sqrt{2}\pi^2 - 4\pi^2 + 2\sqrt{2}\pi)x^{2\pi-3},$$

$$G_4 = 16\pi^4 - 32\pi^3 - 8\sqrt{2}\pi^3 + 16\sqrt{2}\pi^2 + 12\pi^2 - 6\sqrt{2}\pi \approx 530.6502 > 0.$$

Then $u = \sqrt[2\pi-3]{10} \approx 2.0164 < \frac{121}{60}$ is a positive root upper bound while $l = \frac{7}{10}$ is a positive root lower bound by Corollary 11–14. Finally the isolation list is computed as:

Table 1: Isolation process.

$I \in L'$	$\mathcal{V}([\text{PDS}(\tilde{q}^*)])$	median	L
$(\frac{7}{10}, \frac{121}{60})$	$2 - 0 = 2$	$\frac{128}{70} = \frac{64}{35}$	$\{\}$
$(\frac{7}{10}, \frac{64}{35})$	$2 - 0 = 2$	$\frac{71}{45}$	$\{\}$
$(\frac{64}{35}, \frac{121}{60})$	$0 - 0 = 0$	N/A	$\{\}$
$(\frac{7}{10}, \frac{71}{45})$	$2 - 0 = 2$	$\frac{78}{55}$	$\{\}$
$(\frac{71}{45}, \frac{64}{35})$	$0 - 0 = 0$	N/A	$\{\}$
$(\frac{7}{10}, \frac{78}{55})$	$2 - 1 = 1$	N/A	$\{(\frac{7}{10}, \frac{78}{55})\}$
$(\frac{78}{55}, \frac{71}{45})$	$1 - 0 = 1$	N/A	$\{(\frac{7}{10}, \frac{78}{55}), (\frac{78}{55}, \frac{71}{45})\}$

7 Conclusion

In this paper we extend a series of techniques used to handle polynomials, such as factorization, Fourier's theorem and Descartes' sign rule, and analyze the positive roots of multiple power sums. Thus we offer some effective formulas for estimating root bounds and present an efficient algorithm for isolating positive roots under the given minimum root separation.

For future work, it is a challenging job to estimate a nontrivial (positive) lower bound for the minimum root separation $\text{sep}(q^*)$. A promising approach, inherited [Collins and Horowitz 1974], is to utilize a proper definition of the resultant $\text{res}_x(p^*, q^*)$ about two MPSs $p^*(x), q^*(x)$ satisfying:

$$\text{res}_x(p^*, q^*) = K \prod_{i,j} (\alpha_i - \beta_j)^{n_{ij}}, \tag{16}$$

where $K \in \mathbb{R} \setminus \{0\}$, α_i, β_j are distinct complex roots of p^*, q^* respectively and n_{ij} is the product of α_i's multiplicity and β_j's multiplicity. With $\mathrm{PDS}(q^*) = [G_0, G_1, \cdots, G_m]$ $(m > 1)$ of the MPS q^* without any multiple roots, we have that (assume $\mathrm{sep}(q^*) = |\alpha_1 - \alpha_2|$ and $U \geq |\alpha_i|$)

$$\begin{aligned} 0 < |\mathrm{res}_x(G_0, G_1)| &= |K \prod_{i<j} (\alpha_i - \alpha_j)^2| \\ &= |K| \prod_{i<j} (\alpha_i - \alpha_j)^2 \\ &= |K|(\mathrm{sep}(q^*))^2 \prod_{i<j \wedge (i,j) \neq (1,2)} (\alpha_i - \alpha_j)^2 \\ &\leq |K|(\mathrm{sep}(q^*))^2 (2U)^{\sum_{i<j \wedge (i,j) \neq (1,2)} 2} \\ &\leq |K|(\mathrm{sep}(q^*))^2 (2U)^{(m-2)(m+1)}. \end{aligned} \tag{17}$$

Then we can obtain

$$\mathrm{sep}(q^*) \geq \sqrt{\frac{|\mathrm{res}_x(G_0, G_1)|}{|K|(2U)^{(m-2)(m+1)}}} > 0. \tag{18}$$

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