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An Axiomatization of a First-order Branching Time Temporal Logic

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Abstract: We introduce a first-order temporal logic for reasoning about branching time. It is well known that the set of valid formulas is not recursively enumerable and there is no finitary axiomatization. We offer a sound and strongly complete axiomatization for the considered logic.

Key Words: branching time logic, first order logic, strong completeness
Category: F.4.1, I.2.4

1 Introduction

The study of temporal logics started with the Prior's work [Prior 1957]. Temporal logics are designed in order to analyze and reason about the way that systems change over time, and have been shown to be a useful tool in describing behavior of an agent's knowledge base, for specification and verification of programs, hardware, protocols in distributed systems etc., [Emerson 1990, Emerson 1996].

In the early works, classical logic languages are enriched with two temporal operators: the "future" operator F and "past" operator P (for complete axiomatization, see [van Benthem 1982]). A stronger, more expressive language with the new binary operators "until" (U) and "since" (S) is introduced in [Kamp 1968]; the corresponding completeness result is presented in [Burgess 1982].

An important division of temporal logics is into linear (each moment of time has a unique possible future) and branching time (there can be two or more possible futures) [Burgess 1984].

A complete axiomatization of propositional linear time logic with the "next" operator $\bigcirc$ and "until" operator is presented in [Gabbay et al. 1980] and [Lichtenstein and Pnueli 2000], while its first-order extension is given in [Manna and Pnueli 1981]. A strongly complete axiomatization (in the first-order case) is proposed in [Ognjanović 2001], and its probabilistic extension is presented in [Ognjanović 2006]. A translation of first-order linear time formulas into classical formulas with explicit time parameters is considered in [Abadi 1989, Abadi 1990, Andréka et al. 1991], and alternative notions of non-standard completeness (for example, domains that correspond to time parameters need not be countable) are proposed.

Propositional branching time temporal logic with the basic operators $\bigcirc$, U and A (universal path operator), usually called Computation tree logic, is introduced in [Emerson and Halpern 1986, Emerson and Sistla 1984]. Unlike the linear operators $\bigcirc$ and U, the temporal operator A involves path switching. There are only a few complete axiomatizations for propositional Computation tree logic, with respect to different classes of models [Stirling 1992, Kaivola 1996, Reynolds 2001]. To the best of our knowledge, there is no such result for the first order case.

In this paper we present a strongly complete infinitary axiomatization for first-order branching time temporal logic with the operators $\bigcirc$, U and A, with the meaning: $\bigcirc\alpha$ – α holds in the next time moment on the branch, $\alpha U \beta$ – α holds in every time moment (on the branch) until β becomes true, and $A\alpha$ – α holds on every branch which passes trough the current state. It is known that, in the case of first-order linear time logic with $\bigcirc$ and U, the set of valid formulas is not recursively enumerable, and there is no recursive axiomatization of the logic [Abadi 1989, Andréka et al. 1979, Gabbay et al. 1994, Kröger 1990, Szałas and Holenderski 1988]. Consequently, the same holds for the corresponding branching logics. Another proof-theoretical problem, even in the propositional case, is the non-compactness (e.g. consider the set $\{\bigcirc\alpha, \bigcirc\bigcirc\alpha, \bigcirc\bigcirc\bigcirc\alpha, \ldots\} \cup \{\top U \neg\alpha\}$ which is finitely satisfiable, but not satisfiable). Thus, there is no strongly complete ("every consistent set of formulas has a model") finitary axiomatization.

In our logic the term "infinitary" concerns the meta language only, i.e., the object language is countable, and formulas are finite, while only proofs are allowed to be infinite. Similar logics are axiomatized in [Szałas 1987] (the corresponding completeness is proved using an algebraic method). In our paper Deduction theorem is proved and the Henkin construction is used, similarly as in [Ognjanović 2001, Ognjanović 2006, Ognjanović and Rašković 2000].

The rest of the paper is organized as follows. In Section 2 we introduce the first-order branching time temporal logic and define the semantics. Section 3 contains an axiomatization of the logic. In Section 4 it is proved that the axiomatization is sound and strongly complete with respect to the corresponding class of models. We conclude in Section 5.

2 Syntax and semantics

We consider a first order language L which contains:

1. the variables $x, y, z, \ldots$;
2. for every integer $k \geq 0$, k-ary relation symbols $P_0^k, P_1^k, \ldots$, and k-ary function symbols $F_0^k, F_1^k, \ldots$;
3. Boolean connectives $\neg$ and $\wedge$, quantifier $\forall$, comma, parentheses, and
4. temporal operators $\bigcirc$ (next), U (until), and A (universal path quantifier).

The function symbols of arity 0 are called constant symbols. *Terms* and *atomic formulas* are defined as usual. The set of *formulas* is the smallest set containing all atomic formulas that is closed under the formation rules: if α and β are formulas, then $\neg\alpha$, $\alpha \wedge \beta$, $(\forall x)\alpha$, $\bigcirc\alpha$, $\alpha U \beta$ and $A\alpha$ are also formulas. The other Boolean connectives ($\vee$, $\rightarrow$,$\leftrightarrow$) and existential quantifier ($\exists$) can be defined as usual, while $\top$ and $\bot$ are the notations for $\alpha \vee \neg\alpha$ and $\alpha \wedge \neg\alpha$, respectively. The temporal operators F (sometime), G (always) and E (existential path quantifier) are defined as follows: $F\alpha$ denotes $\top U \alpha$, $G\alpha$ denotes $\neg F \neg\alpha$ and $E\alpha$ denotes $\neg A \neg \alpha$. *Sentences* (i.e., formulas without free variables) are defined as usual.

An example of a formula is $E\bigcirc P_1^1(F_6^0) \rightarrow (\forall x)A(\exists y)(P_0^2(y,x)\ U\ P_0^2(F_0^0, y))$. If T is a set of formulas, then $\bigcirc T$ denotes $\{\bigcirc\alpha | \alpha \in T\}$, while AT denotes $\{A\alpha | \alpha \in T\}$. Furthermore, for $k \in \omega$, $\bigcirc^{k+1}\alpha$ is an abbreviation for $\bigcirc(\bigcirc^k \alpha)$.

We define the notion of a model as a special kind of Kripke model. Namely, a *model* M is a tuple $\langle S, R, \Sigma, D, I \rangle$ where:

- S is a non-empty set (of *states*),
- R is a total binary relation on S, i.e., for every $s \in S$ there is $t \in S$ such that sRt,
- Σ is a set of ω-sequences $\sigma = s_0, s_1, s_2, \ldots$ of states from S such that $s_i R s_{i+1}$, for all $i \in \omega$. A *path* is an element of Σ. We assume that Σ is suffix-closed, i.e., if $\sigma = s_0, s_1, s_2, \ldots$ is a path and $i \in \omega$, the sequence $s_i, s_{i+1}, s_{i+2}, \ldots$ is also a path.
- D is a non empty domain, and
- I associates an interpretation $I(s)$ with every state $s \in S$ such that for all j and k:

1. $I(s)(F_j^k)$ is a function from D^k to D,
2. for every $t \in S$, $I(s)(F_j^k) = I(t)(F_j^k)$, and
3. $I(s)(P_j^k)$ is a k-ary relation on D.

Note that we use fixed domain models with rigid function symbols. To simplify the notation, we introduce the following convention: if $\sigma = s_0, s_1, s_2, \ldots$, we write σ_i for the state s_i and $\sigma_{\geq i}$ for the path $s_i, s_{i+1}, s_{i+2}, \ldots$

Let $M = \langle S, R, \Sigma, D, I \rangle$ be a model.

A *variable valuation* v assigns some element of the domain to every state s and every variable x, i.e., $v(s)(x) \in D$. If $s \in S$, $d \in D$, and v is a valuation, then $v[d/x]_s$ is a valuation identical to v with the exception that $v[d/x]_s(s)(x) = d$.

The value of a term t in a state s with respect to v (denoted by $I(s)(t)_v$) is recursively defined as follows:

- if t is a variable x, then $I(s)(x)_v = v(s)(x)$, and
- if $t = F_j^k(t_1, \ldots, t_k)$, then $I(s)(t)_v = I(s)(F_j^k)(I(s)(t_1)_v, \ldots, I(s)(t_k)_v)$.

We define what it means for a formula α to be satisfied at a path σ in a model M under a valuation v, denoted by $(M, \sigma, v) \models \alpha$, as follows:

- $(M, \sigma, v) \models P_j^k(t_1, \ldots, t_k)$ iff $\langle I(\sigma_0)(t_1)_v, \ldots, I(\sigma_0)(t_k)_v \rangle \in I(\sigma_0)(P_j^k)$,
- $(M, \sigma, v) \models \neg\alpha$ iff $(M, \sigma, v) \not\models \alpha$,
- $(M, \sigma, v) \models \alpha \wedge \beta$ iff $(M, \sigma, v) \models \alpha$ and $(M, \sigma, v) \models \beta$,
- $(M, \sigma, v) \models \bigcirc\alpha$ iff $(M, \sigma_{\geq 1}, v) \models \alpha$,
- $(M, \sigma, v) \models \alpha U \beta$ iff there is some $i \in \omega$ such that $(M, \sigma_{\geq i}, v) \models \beta$ and for each $j \in \omega$, if $0 \leq j < i$ then $(M, \sigma_{\geq j}, v) \models \alpha$,
- $(M, \sigma, v) \models (\forall x)\alpha$ iff for every $d \in D$ $(M, \sigma, v[d/x]_{\sigma_0}) \models \alpha$,
- $(M, \sigma, v) \models A\alpha$ iff for every path π, if $\sigma_0 = \pi_0$ then $(M, \pi, v) \models \alpha$.

We write $(M, \sigma) \models \alpha$ if for every valuation v, $(M, \sigma, v) \models \alpha$ holds. A sentence α is *satisfiable* if there is a path σ in a model M such that $(M, \sigma) \models \alpha$. A set T of sentences is satisfiable if there is a path σ in a model M such that for every $\alpha \in T$, $(M, \sigma) \models \alpha$.

Notice that in the above definition the future includes the present, so that:

- $(M, \sigma) \models F\alpha$ if there is $j \geq 0$ such that $(M, \sigma_{\geq j}) \models \alpha$, and
- $(M, \sigma) \models G\alpha$ if for every $j \geq 0$, $(M, \sigma_{\geq j}) \models \alpha$.

3 Axiomatization

In this section we present an axiomatization with the following axiom schemas and inference rules:

Axiom schemas

A1. all the axioms of the classical propositional logic

A2. $(\forall x)(\alpha \to \beta) \to (\alpha \to (\forall x)\beta)$, where x is not free in α

A3. $(\forall x)\alpha(x) \to \alpha(t/x)$, where $\alpha(t/x)$ is obtained by substituting all free occurrences of x in $\alpha(x)$ by the term t which is free for x in $\alpha(x)$

A4. $\bigcirc(\alpha \to \beta) \to (\bigcirc\alpha \to \bigcirc\beta)$

A5. $\neg \bigcirc \alpha \leftrightarrow \bigcirc\neg\alpha$

A6. $\alpha U\beta \leftrightarrow \beta \vee (\alpha \wedge \bigcirc(\alpha U\beta))$

A7. $\alpha U\beta \to F\beta$

A8. $\alpha \to A\alpha$, where α is an atomic formula

A9. $E\alpha \to \alpha$, where α is an atomic formula

A10. $A\alpha \to \alpha$

A11. $A(\alpha \to \beta) \to (A\alpha \to A\beta)$

A12. $A\alpha \to AA\alpha$

A13. $E\alpha \to AE\alpha$

A14. $(\forall x) \bigcirc \alpha(x) \to \bigcirc(\forall x)\alpha(x)$

A15. $(\forall x)A\alpha(x) \to A(\forall x)\alpha(x)$

Inference rules

R1. from $\{\alpha, \alpha \to \beta\}$ infer β

R2. from α infer $(\forall x)\alpha$

R3. from α infer $\bigcirc\alpha$

R4. from α infer $A\alpha$

R5. from $\{\beta \to \bigcirc^i\alpha\}$ for all $i \geq 0$, infer $\beta \to G\alpha$

Let us briefly discuss some of the above axioms and rules. Note that the axiom system can be divided into three parts. The first two parts deal with first-order and temporal reasoning, respectively, while the last two axioms concern mixing of both of them.

The classical first-order logic is a sublogic of the presented logic (by the axioms A1. – A3. and the rules R1. and R2.). The axioms A4. and A5. are the usual axioms for the next operator $\bigcirc$, as well as the axioms A6. and A7. for the until operator [Lichtenstein and Pnueli 2000]. The axioms A8. – A13. concern the non-linear aspect of the temporal logic [Stirling 1992]. The axioms A14. and A15. are variants of the well known Barcan formula.

The rules R1. and R2. are Modus Ponens and Generalization, respectively, while the rules R3. and R4. are two forms of modal Necessitation. The only infinitary inference rule R5. characterizes the always operator.

A formula α is *deducible* from a set T of formulas ($T \vdash \alpha$) if there is an at most countable sequence of formulas $\alpha_0, \alpha_1, \ldots, \alpha$, such that every α_i is an axiom or a formula from T, or it is derived from the preceding formulas by an inference rule, with the exception that the inference rules R2. and R3. can be applied to theorems only. The sequence α_0, α_1, $\ldots$, α is the *proof* of $T \vdash \alpha$ (observe that the length of inference may be any successor ordinal lesser than the first uncountable ordinal ω_1). We say that α is a *theorem* of the deductive system, also denoted by $\vdash \alpha$, if it is deducible from the empty set. A set T of sentences is *consistent* if there is at least one formula which is not deducible from T, otherwise T is *inconsistent*.

A set T of sentences is said to be *maximal* if for every sentence α, either $\alpha \in T$ or $\neg\alpha \in T$. A set T of sentences is *saturated* if it is consistent and maximal and satisfies:

if $\neg(\forall x)\alpha(x) \in T$, then for some term t, $\neg\alpha(t) \in T$.

Lemma 1. *The above axiomatization is sound with respect to the class of models defined in Section 2.*

Proof. Using a straightforward induction on the length of the inference. For example, consider the axiom A15.

Suppose that σ is a path in a model $M = \langle S, R, \Sigma, D, I\rangle$, and $(M, \sigma) \models (\forall x)A\alpha(x)$, i.e., for every valuation v, $(M, \sigma, v) \models (\forall x)A\alpha(x)$. It follows that for every valuation v and every $d \in D$, $(M, \sigma, v[d/x]_{\sigma_0}) \models A\alpha(x)$. Consequently, for every v and d, and every path π in M, if $\sigma_0 = \pi_0$ then $(M, \pi, v[d/x]_{\pi_0}) \models \alpha(x)$. Thus, for every v and π, if $\sigma_0 = \pi_0$ then $(M, \pi, v) \models (\forall x)\alpha(x)$. Finally, for every valuation v, $(M, \sigma, v) \models A(\forall x)\alpha(x)$, i.e., $(M, \sigma) \models A(\forall x)\alpha(x)$. $\Box$

4 Completeness

From the above definition of deducibility from the set of formulas, the deduction theorem follows.

Theorem 2 (Deduction theorem). *If T is a set of formulas, α is a sentence, and $T, \alpha \vdash \beta$, then $T \vdash \alpha \to \beta$.*

Proof. We use the transfinite induction on the length of the inference. The cases when $\vdash \beta$ or $\beta = \alpha$ are standard, as well as the cases when β is obtained by the inference rules R1. and R2.

Assume that $T, \alpha \vdash \beta$, where $\beta = A\gamma$ is obtained by the inference rule R4. Since R4. can be applied to the theorems only, we have $\vdash \gamma$. Thus, $\vdash \beta$, and $T \vdash \alpha \to \beta$. The case concerning the rule R3. follows similarly.

Suppose that $T, \alpha \vdash \beta \to G\gamma$ is obtained by the inference rule R5. Then $T, \alpha \vdash \beta \to \bigcirc^n\gamma$, for all $n \in \omega$. By the induction hypothesis, we obtain $T \vdash \alpha \to (\beta \to \bigcirc^n\gamma)$, or, equivalently, $T \vdash (\alpha \wedge \beta) \to \bigcirc^n\gamma$, for all $n \in \omega$. Hence, using R5. we obtain $T \vdash (\alpha \wedge \beta) \to G\gamma$, i.e.,$T \vdash \alpha \to (\beta \to G\gamma)$. $\square$

The following lemma contains some auxiliary statements.

Lemma 3. *Let α, β be formulas.*

1. $\vdash G\alpha \leftrightarrow \alpha \wedge \bigcirc G\alpha$,
2. $\vdash G \bigcirc \alpha \leftrightarrow \bigcirc G\alpha$,
3. $\vdash (\bigcirc\alpha \to \bigcirc\beta) \to \bigcirc(\alpha \to \beta)$,
4. $\vdash \bigcirc(\alpha \wedge \beta) \leftrightarrow (\bigcirc\alpha \wedge \bigcirc\beta)$,
5. $\vdash \bigcirc(\alpha \vee \beta) \leftrightarrow (\bigcirc\alpha \vee \bigcirc\beta)$,
6. $G\alpha \vdash \bigcirc^i\alpha$ *for every* $i \geq 0$,
7. *if* $\vdash \alpha$, *then* $\vdash G\alpha$,
8. *for* $j \geq 0$, $\bigcirc^j\beta, \bigcirc^0\alpha, \ldots, \bigcirc^{j-1}\alpha \vdash \alpha U \beta$,
9. *if* T *is a set of formulas and* $T \vdash \alpha$, *then* $\bigcirc T \vdash \bigcirc\alpha$,
10. *if* T *is a set of formulas and* $T \vdash \alpha$, *then* $AT \vdash A\alpha$.

Proof. The proofs are easy consequences of the temporal part of the axiomatization. We will consider the statement 10. We will use the induction on the depth of the derivation of α from T. Suppose that $T \vdash (\forall x)\alpha$ is obtained from $T \vdash \alpha$ by the inference rule R2. Then we have

$T \vdash \alpha$,

$AT \vdash A\alpha$ (by the induction hypothesis),

$AT \vdash (\forall x)A\alpha$ (by the inference rule R2.),

$AT \vdash A(\forall x)\alpha$ (by the axiom A15).

The other cases can be solved in a similar way. $\square$

Theorem 4. *(1) Let T be a consistent set of sentences in the language L and C a countably infinite set of constants such that $L \cap C = \emptyset$. Then T can be extended to a saturated set $\mathcal{T}$ in the language $L \cup C$.*

(2) If $\mathcal{T}_1$ is a saturated set of sentences then the set $\mathcal{T}_2 = \{\alpha | \bigcirc \alpha \in \mathcal{T}\}$ is also saturated.

Proof. (1) Let $\alpha_0, \alpha_1, \ldots$ be an enumeration of all sentences in L. We define a completion $\mathcal{T}$ of T recursively:

1. $T_0 = T$.
2. For every $i \geq 0$,
   (a) If $T_i \cup \{\alpha_i\}$ is consistent, then $T_{i+1} = T_i \cup \{\alpha_i\}$.
   (b) Otherwise, if α_i is of the form $\gamma \rightarrow G\beta$, then $T_{i+1} = T_i \cup \{\gamma \rightarrow \neg \bigcirc^{j_0} \beta\}$ for some $j_0 \geq 0$ such that T_{i+1} is consistent (the existence of such j_0 is provided by Deduction theorem; if we suppose that $T_i \cup \{\gamma \rightarrow \neg \bigcirc^{j_0} \beta\}$ is inconsistent for all j_0, we can conclude that $T_i \vdash \neg(\gamma \rightarrow \neg \bigcirc^{j_0} \beta)$, for all j_0. Using propositional axioms, we obtain
   $$T_i \vdash \gamma \rightarrow \bigcirc^n \beta,$$
   for all n, so, by R5., $T_i \vdash \gamma \rightarrow G\beta$, which contradicts the assumption).
   (c) Otherwise, if α_i is of the form $\neg(\forall x)\beta(x)$, then $T_{i+1} = T_i \cup \{\neg\beta(c)\}$ for some $c \in C$ such that T_{i+1} is consistent (the proof that such c exists is standard).
   (d) Otherwise, $T_{i+1} = T_i$.
3. $\mathcal{T} = \bigcup_{i=0}^{\infty} T_i$.

Obviously, each T_i is consistent.

Let us prove that $\mathcal{T}$ is maximal, i.e., for each sentence α, either $\alpha \in \mathcal{T}$ or $\neg\alpha \in \mathcal{T}$. Let $\alpha = \alpha_i$ and $\neg\alpha = \alpha_j$. If both $\alpha \notin \mathcal{T}$ and $\neg\alpha \notin \mathcal{T}$, then, by construction of $\mathcal{T}$ and Deduction theorem we obtain $T_i \vdash \neg\alpha$ and $T_j \vdash \alpha$. If n is positive integer such that $n > i, j$, then $T_n \vdash \alpha \wedge \neg\alpha$, so T_n would be inconsistent; a contradiction.

Next, we will show that $\mathcal{T}$ is deductively closed, i.e., that $\mathcal{T} \vdash \alpha$ implies $\alpha \in \mathcal{T}$. Since any axiom is consistent with any consistent set, each instance of any axiom is in $\mathcal{T}$, so it is enough to prove that $\mathcal{T}$ is closed under the inference rules R1.–R5.

In the cases of the finitary inference rules R1.–R4. the proof is standard. For example, let $\mathcal{T} \vdash A\alpha$ be obtained by the rule R4. Then $\vdash \alpha$ (the rule R4. can be applied on theorems only), and $\vdash A\alpha$. Let $\alpha = \alpha_i$. It follows that $T_i \vdash A\alpha$ and $A\alpha \in T_{i+1}$.

Let $\gamma \rightarrow G\beta$ be obtained from $\mathcal{T} \vdash \gamma \rightarrow \bigcirc^j \beta$ for every $j \geq 0$ by the inference rule R5. Suppose that $\gamma \rightarrow G\beta \notin \mathcal{T}$, which is equivalent to $\neg(\gamma \rightarrow G\beta) \in \mathcal{T}$ by maximality of $\mathcal{T}$. Then there are $i, j_0 \geq 0$ such that $\neg(\gamma \to G\beta), \gamma \to \neg \bigcirc^{j_0} \beta \in T_i$. By the induction hypothesis, for every $j \geq 0$, $\gamma \to \bigcirc^j \beta \in \mathcal{T}$. So there is $i' \geq i$ such that $\gamma \to \neg \bigcirc^{j_0} \beta$, $\gamma \to \bigcirc^{j_0}\beta$, and $\neg(\gamma \to G\beta) \in T_{i'}$. So $T_{i'} \vdash \neg(\gamma \to G\beta)$ and $T_{i'} \vdash \gamma \wedge \neg G\beta$. Since $T_{i'} \vdash \gamma \to \neg \bigcirc^{j_0} \beta$ and $T_{i'} \vdash \gamma \to \bigcirc^{j_0}\beta$, we get $T_{i'} \vdash \neg \bigcirc^{j_0} \beta \wedge \bigcirc^{j_0}\beta$ which is in contradiction with consistency of $T_{i'}$.

It follows that $\mathcal{T}$ is consistent. If $\mathcal{T} \vdash \bot$, then, by deductive closeness of $\mathcal{T}$, $\bot \in \mathcal{T}$, so $\bot \in T_i$, for some $i \in \omega$ which is in contradiction with consistency of T_i. Finally, the step 2c of the construction guarantees that $\mathcal{T}$ is saturated.

(2) We will prove that $\mathcal{T}_2$ is consistent and maximal and satisfies: if $\neg(\forall x)\alpha(x) \in \mathcal{T}_2$, then for some term t, $\neg\alpha(t) \in \mathcal{T}_2$.

Suppose that $\mathcal{T}_2$ is not consistent, i.e. $\mathcal{T}_2 \vdash \alpha \wedge \neg\alpha$, for any sentence α. By Lemma 3.8, $\mathcal{T}_1 \vdash \bigcirc(\alpha \wedge \neg\alpha)$. By Lemma 3.4, and and Axiom A5., we have $\mathcal{T}_1 \vdash \bigcirc\alpha \wedge \neg \bigcirc \alpha$ which is in contradiction with consistency of $\mathcal{T}_1$.

Suppose that $\mathcal{T}_2$ is not maximal. There is a formula α such that $\alpha \notin \mathcal{T}_2$ and $\neg\alpha \notin \mathcal{T}_2$. Consequently, $\bigcirc\alpha \notin \mathcal{T}_1$ and $\neg \bigcirc \alpha \notin \mathcal{T}_1$ (by the axiom A.5) which is in contradiction with the maximality of $\mathcal{T}_1$.

Suppose that there is a sentence $\neg(\forall x)\alpha(x) \in \mathcal{T}_2$ such that for every variable-free term t in L, $\neg\alpha(t) \notin \mathcal{T}_2$. Thus, $\bigcirc\neg(\forall x)\alpha(x) \in \mathcal{T}_1$, and for every t, $\bigcirc\neg\alpha(t) \notin \mathcal{T}_1$. Using the axioms A.15 and A.5, we obtain $\neg(\forall x) \bigcirc \alpha(x) \in \mathcal{T}_1$, and for every term t in L, $\neg \bigcirc \alpha(t) \notin \mathcal{T}_1$, which is in contradiction with the assumption that $\mathcal{T}_1$ is saturated. □

A sentence is a *state* sentence if it is a boolean combination of variable-free basic formulas and sentences of the form $A\alpha$. We denote the set of all state sentences by St. The equivalence relation $\sim$ on the set of saturated sets of sentences is defined as follows:

$\mathcal{T}_1 \sim \mathcal{T}_2$ iff $\mathcal{T}_1 \cap St = \mathcal{T}_2 \cap St$.

The equivalence class of $\mathcal{T}$ is $[\mathcal{T}] = \{\mathcal{T}' | \mathcal{T}' \sim \mathcal{T}\}$.

Lemma 5. *If $\mathcal{T}$ is a saturated set of sentences and $A\alpha \notin \mathcal{T}$, then there exists $\mathcal{T}' \in [\mathcal{T}]$ such that $\alpha \notin \mathcal{T}'$.*

Proof. Let $T = \mathcal{T} \cap St$. If $T \cup \{\neg\alpha\}$ is consistent then, by Theorem 4(1), it can be extended to a saturated set $\mathcal{T}'$. Since $T = \mathcal{T}' \cap St$, then $\mathcal{T}' \in [\mathcal{T}]$.

If $T \cup \{\neg\alpha\}$ is inconsistent then $T \vdash \alpha$. By Lemma 3.10, $AT \vdash A\alpha$. Since $T \subseteq St$, by the axioms A8., A12. and A13., $T \vdash A\alpha$. Thus, $A\alpha \in \mathcal{T}$ which contradicts the assumption. □

We define a model $\mathcal{M} = \langle S, R, \Sigma, D, I \rangle$ as follows:

- $S = \{[\mathcal{T}] | \mathcal{T} \text{ is saturated}\}$;

- sRt if there exist $\mathcal{T}_1 \in s$, $\mathcal{T}_2 \in t$ such that $\mathcal{T}_2 = \{\alpha | \bigcirc \alpha \in \mathcal{T}_1\}$;
- Σ is the set of paths $[\mathcal{T}_0]$, $[\mathcal{T}_1]$, $[\mathcal{T}_2]$,... such that $\mathcal{T}_{i+1} = \{\alpha | \bigcirc \alpha \in \mathcal{T}_i\}$, for all $i \in \omega$;
- D is the set of all variable-free terms in L;
- for $s \in S$, $I(s)$ is an interpretation such that:
  - for every function symbol F_j^k, $I(s)(F_j^k)$ is a function from D^k to D such that for all variable-free terms $t_1, \dots, t_k$ in L, $I(s)(F_j^k) : \langle t_1, \dots, t_k \rangle \mapsto F_j^k(t_1, \dots, t_k)$, and
  - for every relation symbol P_j^k, $I(s)(P_j^k) = \{\langle t_1, \dots, t_k \rangle : t_1, \dots, t_k$ are variable-free terms in $L, P_j^k(t_1, \dots, t_k) \in \mathcal{T}\}$, for any $\mathcal{T} \in s$.

Observe that the total relation R is well defined. Namely, by Theorem 4(2), if the set $\mathcal{T}_1$ is saturated, the same holds for $\mathcal{T}_2 = \{\alpha | \bigcirc \alpha \in \mathcal{T}_1\}$. A path Σ consists of a sequence of the classes of equivalence $[\mathcal{T}_0]$, $[\mathcal{T}_1]$, $[\mathcal{T}_2]$,... representing states. Since each $[\mathcal{T}_i]$ contains many saturated sets, it is possible that a state belongs to several paths. The definition of interpretation I is also correct, since every variable-free formula $P_j^k(t_1, \dots, t_k)$ is a state sentence, and it belongs to a saturated set $\mathcal{T}$ if and only if it belongs to any other saturated set from $[\mathcal{T}]$.

If the sequence of saturated sets $\{\mathcal{T}_i\}_{i\in\omega}$ determines a path σ, we will write $\sigma(i)$ for $\mathcal{T}_i$.

Theorem 6 (Strong completeness theorem). *A set T of sentences is consistent if and only if it is satisfiable.*

Proof. The ($\Leftarrow$)-direction follows from Lemma 1. In order to prove the ($\Rightarrow$)-direction we construct $\mathcal{M}$ as above, and show that for every sentence α, $(\mathcal{M}, \sigma) \models \alpha$ iff $\alpha \in \sigma(0)$.

If α is an atomic sentence, by the definitions of I and Σ, $(\mathcal{M}, \sigma) \models \alpha$ iff $\alpha \in \sigma(0)$.

The cases when formulas are negations and conjunctions can be proved as usual.

If $\alpha = (\forall x)\beta \in \sigma(0)$, then, by the axiom A3., $\beta(t) \in \sigma(0)$ for every $t \in D$. By the induction hypothesis $(\mathcal{M}, \sigma) \models \beta(t)$ for every $t \in D$, and $(\mathcal{M}, \sigma) \models (\forall x)\beta$. If $\alpha \notin \sigma(0)$, there is some $t \in D$ such that $(\mathcal{M}, \sigma) \models \neg\beta(t)$, because $\sigma(0)$ is saturated. It follows that $(\mathcal{M}, \sigma) \not\models (\forall x)\beta$.

If $\alpha = \bigcirc\beta$, we have $(\mathcal{M}, \sigma) \models \alpha$ iff $(\mathcal{M}, \sigma_{\geq 1}) \models \beta$ iff $\beta \in \sigma(1)$ iff $\alpha \in \sigma(0)$ (by the construction of $\mathcal{M}$).

Let $\alpha = \beta U \gamma$. Suppose that $(\mathcal{M}, \sigma) \models \beta U \gamma$. There is some $j \geq 0$ such that $(\mathcal{M}, \sigma_{\geq j}) \models \gamma$ and for every k, $0 \leq k < j$, $(\mathcal{M}, \sigma_{\geq k}) \models \beta$. By the induction hypothesis, $\gamma \in \sigma(j)$, and $\beta \in \sigma(k)$, for $j \geq 0$, $0 \leq k < j$. By the construction of $\mathcal{M}$, $\bigcirc^j\gamma \in \sigma(0)$, $\bigcirc^k\beta \in \sigma(0)$, for $j \geq 0$, $0 \leq k < j$. It follows from Lemma 3.8 that $\beta U\gamma \in \sigma(0)$.

Conversely, assume that $\beta U\gamma \in \sigma(0)$. It follows from the axiom A5. that $F\gamma \in \sigma(0)$, i.e., that $\neg F\gamma = G\neg\gamma \notin \sigma(0)$. By the construction of $\mathcal{M}$, for some $j \geq 0$, $\bigcirc^j\gamma \in \sigma(0)$, i.e., $\gamma \in \sigma(j)$. Let $j_0 = \min\{j : \bigcirc^j\gamma \in \sigma(0)\}$. If $j_0 = 0$, $\gamma \in \sigma(0)$, and by the induction hypothesis $(\mathcal{M}, \sigma) \models \gamma$. It follows that $(\mathcal{M}, \sigma) \models \beta U\gamma$. Thus, suppose that $j_0 > 0$. For every j such that $0 \leq j < j_0$, $\bigcirc^j\gamma \notin \sigma(0)$, i.e. $\gamma \notin \sigma(j)$. From the axiom A4., Lemma 3.4, Lemma 3.5, and $\beta U\gamma \in \sigma(0)$ we have $\gamma \vee (\beta \wedge (\bigcirc\gamma \vee (\bigcirc\beta \wedge \ldots \wedge (\bigcirc^{j_0-1}\gamma \vee (\bigcirc^{j_0-1}\beta \wedge \bigcirc^{j_0}(\beta U\gamma) \ldots) \in \sigma(0)$. It follows that for every $j < j_0$, $\bigcirc^j\beta \in \sigma(0)$, $\beta \in \sigma(j)$, $(\mathcal{M}, \sigma_{\geq j}) \models \gamma$, and $(\mathcal{M}, \sigma) \models \beta U\gamma$.

Let $\alpha = A\beta$. Suppose that $(\mathcal{M}, \sigma) \models A\beta$. Then for all π such that $\sigma(0) \sim \pi(0)$, $(\mathcal{M}, \pi) \models \beta$. By the induction hypothesis, for all π such that $\sigma(0) \sim \pi(0)$, $\beta \in \pi(0)$. If $A\beta \notin \sigma(0)$, by Lemma 4 there exists π such that $\sigma(0) \sim \pi(0)$ and $\beta \notin \pi(0)$, a contradiction.

For the other direction, suppose that $(\mathcal{M}, \sigma) \not\models A\beta$. Then there exists π such that $\sigma(0) \sim \pi(0)$ and $(\mathcal{M}, \pi) \models \neg\beta$. By the induction hypothesis, $\neg\beta \in \pi(0)$ and $\beta \notin \pi(0)$. By Axiom A10., $A\beta \notin \pi(0)$. Since $A\beta$ is a state formula, we obtain $A\beta \notin \sigma(0)$.

Finally, By Theorem 4, T can be extended to a saturated set $\mathcal{T}$, and $[\mathcal{T}] \in S$. Since $\mathcal{T}$ is satisfied in $\mathcal{M}$, the same holds for Ts. $\square$

5 Conclusions

We have introduced a first-order branching time temporal logic. Its infinitary axiomatic system has been proved to be complete. Actually, our results establish strong completeness. We believe that it is not only of a theoretical interest to give an infinitary and complete first order proof system, since the set of all valid formulas is not recursively enumerable [Abadi 1989, Gabbay et al. 1994], and no complete finitary axiomatization is possible in this undecidable framework.

The propositional fragment of our logic provides strong completeness for propositional branching time temporal logic. That also cannot be proved using finitary means. As we mentioned above, Compactness theorem does not hold, and it is well known that in this case we cannot hope for the strong completeness having a finitary axiomatic system. Note that there is an unpleasant logical consequence of finitary axiomatization when the lack of compactness is present: there are unsatisfiable sets of formulas that are consistent with respect to the assumed finitary axiomatic system. In spite of that, the paper [Stirling 1992] presents an axiomatization for the propositional fragment of our logic, for which simple completeness is showed. In [Reynolds 2001] that result is extended to the so called full computation tree logic. Up to our knowledge no strongly complete axiomatization for that logic has been proposed.

In the axiomatization and the proof of the completeness we follow the ideas from [Ognjanović 2001, Ognjanović 2006, Ognjanović and Rašković 2000]. In [Ognjanović 2006] a probabilistic extension of first-order linear time temporal logic is presented. So, the question of axiomatization of a probabilistic extension of first-order branching time logic naturally arises as a topic for further research.

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