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Track-to-Track Measurement Fusion Architectures and Correlation Analysis

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Abstract: The purpose of this paper is to address some theoretical issues related to the track-to-track fusion problem when the measurements tracking the same target are inherently correlated by the common process noise of the underlying target. This problem has been intensively investigated using standard Kalman filter with some appealing theoretical results, however such results are no longer valid in case of suboptimality due to either the presence of strong nonlinearity or to the discrete uncertainty pervading the origin of the measurement. This paper reviews several architectures of parallelized blocks of Kalman filters, including the augmented stacked measurement, sequential and data compression architectures. Next, convex combination architecture will be investigated and some theoretical results concerning its extension as well as in case of presence of correlation are investigated. Two special cases of correlation are highlighted. This concerns the case of presence of only two correlated tracks among all tracks and the case of weak correlation. In both cases some original theoretical results are put forward. Finally, links with related fusion architectures is investigated.

Key Words: Tracking, estimation, correlation
Category: I.2.9, I.6.1

1 Introduction

The problem of fusing information issued from disparate sensors has been widely investigated in the literature of information theory, control and engineering as testified by the amount of publication in the field, see extensive review in [Luo and Kay, 1995], text book of [Bar-shalom and Li, 1995], among others. In estimation theory, the fusion problem arises from both the level of redundancy and diversity occurring in the information supplied by the various sensors. For instance, radar provides accurate range but poor bearing while infrared sensor provides accurate angle but poor range data. So fusing information of the different sensors allows us to extract the relevant information on the target(s). It is typically assumed that the outcome of the fusion node provides the "best" global estimate of the target feature given the system constraints, like computational cost, global accuracy, bandwidth capacity, among others. Strictly speaking, a key issue prior to the fusion is the knowledge of the various interconnections among the information supplied by the sensors. This includes the level of dependency or independence of the sensors, the time of availability of sensory information, the accuracy of the outputs and the computational constraints, etc. The fusion architecture is crucial in any fusion problem in the sense that it describes the level and type of interconnections among the various nodes constituting the whole system. Traditional architecture for data fusion is mainly centralized where data issued from the multiple sensors are sent to a single node that processes the outcome, which is then communicated to the user(s). However, with the developments of the computing and communication technologies, more advanced fusion architecture become feasible. This ranges from the hierarchical architecture to the fully distributed architecture. In the former, the lowest level nodes process data and send it to the next higher level node in the hierarchy to be combined and sent again to the next node and so on. While in the latter, each node can communicate to any other node subject to connectivity constraints. This allows for an increase in communication computational and communication costs, fault tolerance and cost effective realizability, which motivates its growing application in many communication and tracking systems. Indeed, in the distributive architecture for target tracking for instance, the sensors require to send processed data to a set of local processors connected by a communication network where the local nodes/processors process the local sensor data and then communicate output tracks to a global processor that computes a global estimate of the targets to be tracked [Fong, 2008]. The research into the optimality of the fusion rules using various fusion architectures is very active since these last decades as testified by the growing publications in the field. See, for instance [Cavalaro, 2007], [Chong et al., 1990], [Hashemipour et al., 1988], [Chang et al., 1997], [Ducan and Sameer, 2006], [Fong, 2008] and references therein. Especially this has given birth to various extensions of Kalman filtering theory under stochastic and observability constraints, and assuming the existence of both the state and measurement models. Nevertheless, it has also been acknowledged that the research in the field of distributed fusion is very much premature as compared to that carried out for centralized fusion. The track association arises from the fact that in multisensory system or distributed architecture, each local system, associated to a given sensory information, has a number of tracks, so the decision whether two tracks from different systems represent the same target is crucial. Ultimately, if the above associations were made incorrectly, then the fused track estimates will potentially be worse than those arising from a single sensor. Although measurement errors resulting from a single sensor are independent of those resulting from other sensors, the track estimates, corresponding to a given target, computed by different local processors are subject to the same common process noise in the sense that the process noise statistics are used by the local processor estimates. This makes the above estimates ultimately correlated. Besides, in practice, target manoeuvre and sensors typically communicate infrequently to save communication bandwidths, which makes other local sensory systems not necessary aware of new updates occurred elsewhere, yielding a systematic correlation Bar-Shalom and Li, 1995; Chang et al., 1997]. This dependence is characterized by the cross-covariance of the local estimation errors. At least, two distinct streams can be distinguished in handling the correlation issue. The first one follows on Bar-Shalom and Campo's approach [Bar-Shalom and Campos, 1986] that makes use of the static linear estimation equation where the prior is mapped into the posterior using the measurement. The approach is therefore usually sensitive to the choice of the prior, or equivalently, the order in which the measurements or local estimates are handled, especially in case of strong correlation. The second one consists in driving a likelihood-ratio based cost function suitable for the use of a multi-dimensional assignment to decide which track should be fused. Especially, the cost function allows simultaneous consideration of tracks corresponding to the same target [Bar-Shalom and Fortmann, 1988], [Kaplan et al., 2008]. This paper reviews the various parallel fusion schemes and investigates the influence of correlation of the inputs on the final outcome. Especially, one considers the situation of target tracking using different sensors. So, the issue of track correlation will be highlighted. Previous results obtained in [Bar-Shalom, 1981] and [Bar-Shalom and Campo, 1988] in case of two measurements scenarios will be extended to several measurements. On the other hand, special cases of correlation including two (out of n) tracks correlation and weak correlation will be investigated. Section 2 of this paper highlights the general models (dynamic and measurement models) and the standard combination architecture using Kalman filter. Section 3 focuses on the convex combination architecture underlying the main results in the case of non-correlation. Section 4 investigates the general case of dependent tracks where two special cases are examined. The first deals with the occurrence of two-out of n dependent tracks while the latter concerns the case of weak correlation.

2 Dynamic and measurement model

Let us consider a target, which is tracked by a set of sensors modeled by the following discrete state space model

$$x_{k+1/k} = A(k).x_{k/k} + \mathit{\Gamma}(k)v(k) \tag{1}$$

And

$$x_{k,i} = H_i(k).x_{k/k} + \varepsilon_i(k) \tag{2}$$

Where x_k is the state vector of the target at time index k. A(k) and $\mathit{\Gamma}(k)$ corresponds to the linearized state transition matrix and noise transition matrix, respectively. $v(k)$ is the state noise assumed be zero-mean Gaussian with known

variance-covariance matrix Q.
$z_{k,i}$ represents the measurement vector at time k issued from the ith sensor. $H_i(k)$ and $z_{k,i}$ stand for the linearized measurement transition matrix and the zero-mean white Gaussian measurement noise with variance-covariance matrix $R_{k,i}$, respectively. It is also assumed that the noise measurement $\varepsilon_i(k)$ and $\varepsilon_j(k)$ for $i \neq j$ are uncorrelated, so is the state noise $v(k)$ with any of measurement noise $\varepsilon_i(k)$. That is,

$$E[v(k)v(l)^T] = Q(k).\delta_{k-l}$$

$$E[\varepsilon_i(k)\varepsilon_i(l)^T] = R(k).\delta_{k-l}$$

$$E[v(k)\varepsilon_i(l)^T] = 0$$

Where δ_{k-l} denotes the Kronecker delta function; that is, $\delta_{k-l} = 1$ if $k = l$ and $\delta_{k-l} = 0$ if $k \neq l$. The notation $x_{k+1|k}$ in (1) denotes the prediction on the value of state vector x at time $k+1$, given its current value $x_{k|k}$ at time k. While measurement equation (2) provides a quantitative link between the current measurement and the current evaluation of the state vector $x_{k|k}$.
The standard fusion methodology consists to stack all the measurements together into a single stacked vector and then perform the standard Kalman filter equation; that is, the measurement vector and the associated variance-covariance matrices are given by

$$Z(k) = [z_{k,1}, z_{k,2}, ., z_{k,n}] \tag{3}$$

$$R(k) = diag[R_{k,1}, R_{k,2}, ., R_{k,n}] \tag{4}$$

More formally, the output in terms of the state vector estimation and its asso-

Figure 1: Augmented measurement vector based fusion

ciated variance-covariance matrices are given by

$$\hat{x}_{k+1/k+1} = \hat{x}_{k+1/k} \sum_{i=1}^{n} K_{k+1,i}(z_{k+1,i} - H_i(k+1).\hat{x}_{k+1/k}) \tag{5}$$

With Filter gain:

$$K_{k+1,i} = P_{k+1/k+1} H_i^T(k+1) R_{k+1,i}^{-1} \tag{6}$$

And variance matrix

$$P_{k+1/k+1}^{-1} = P_{k+1/k}^{-1} + \sum_{i=1}^{n} H_i^T(k+1).R_{k+1,i}^{-1}.H_i(k+1) \tag{7}$$

It is clear that the above has the advantage of using a single Kalman filter model using the stack of all measurements. However, its shortcoming cannot be ignored. This includes mainly

- The dimension of the measurement vector Z as well the associating variance-covariance matrix R in the sense that high dimension vector and matrices makes the computational of inverse matrices computationally expensive and possibly instable.
- The difficulty in getting synchronized measurements in the sense that there is always some delay in the process of gathering measurements which is not systematically included in the .

In order to deal with the above difficulties, An alternative implementation has been suggested by Willner et al. [Willner et al., 1978] in early seventies using sequential implementation of a set of recursive filters in which the i^{th} sensor measurement will be used to update the state estimate outputted by $(i-1)^{th}$ Kalman filter that is updated by the $(i-1)^{th}$ sensor measurement. This is illustrated in Figure 2. More formally, the estimation in case of sequential architecture yields

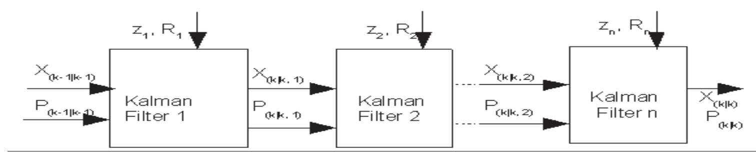

Figure 2: Sequential fusion architecture

$$
\begin{aligned}
\hat{x}_{k+1/k+1} &= \hat{x}_{k+1/k+1,n} \\
&= \hat{x}_{k+1/k,0} + \sum_{i=1}^{n} K_{k+1,i}(z_{k+1,i} - H_i(k+1).\hat{x}_{k+1/k+1,i-1})
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
K_{k+1,i} &= P_{k+1/k+1,i} H_i^T(k+1) R_{k+1,i}^{-1} \qquad i = 1\, to\, n \\
P_{k+1/k+1,i}^{-1} &= P_{k+1/k+1,i-1}^{-1} + H_i^T(k+1).R_{k+1,i}^{-1}.H_i(k+1) \quad i = 1\, to\, n
\end{aligned} \tag{9}
$$

Consequently,

$$
P_{k+1/k+1}^{-1} = P_{k+1/k+1,n}^{-1} = P_{k+1/k+1,0}^{-1} + \sum_{i=1}^{n} H_i^T(k+1).R_{k+1,i}^{-1}.H_i(k+1) \tag{10}
$$

Another reformulating, also investigated in [Willner et al., 1978] consists to compress the initial measurements data zi to a single measurement of the same dimension and then use the latter in the kalman filter equations. In others words, Figure 3 provides an overview of the new Data compression architecture. Especially, the outcome of the data compression block yields the aggregated measurement

$$z(k) = R(k).\sum_{i=1}^{n} R_{k,i}^{-1}.z_{k,i} \tag{11}$$

With

$$R(k) = \sum_{i=1}^{n} R_{k,i}^{-1} \tag{12}$$

And

$$H(k) = \left(\sum_{i=1}^{n} R_i^{-1}\right)^{-1} \sum_{i=1}^{n} R_{k,i}^{-1} H_i(k) \tag{13}$$

Then the state and variance-covariance are obtained by use of standard Kalman filter while updating with measurement (11-12). It is straightforward that the recourse to (11-12) does only make sense if all the measurements are of the same dimension, otherwise the matrix summation does not hold. This clearly adds an extra difficulty in the application of the data compression fusion architecture.

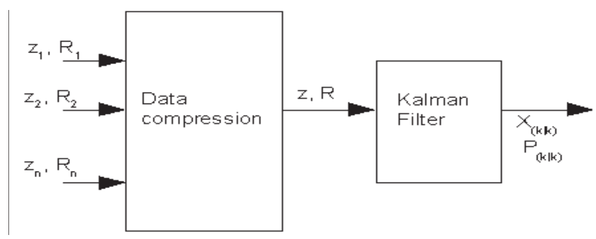

Figure 3: Data compression architecture fusion

The following result has been demonstrated by Willner et al. [Willner et al., 1978], restated here differently

Proposition 1

- In the case of linear system and zero-mean Gaussian uncorrelated state and measurement noise, the outcomes of the parallel and sequential architectures are equivalent.
- If in addition to the above, the measurement equations have identical transition matrices, then the data compression architecture yields the same result as that of the parallel and sequential architectures.

In other words, in the case of linear system (both state and measurement) the outputs of the parallel and the sequential architecture always coincide, while if additionally, the measurement equations have the same transition matrices H_i, then the output of the data compression architecture always coincides with the two previous architectures. This is particularly useful in terms of computational cost where it is straightforward that the data compression architecture yields a lower burden time followed by the sequential architecture and then followed by the parallel architecture. In this respect, despite the presence of several measurements at a given time, the order in which these measurements are processed in the sequential architecture does not matter. In the presence of nonlinearity, the above equivalence between the three architectures is no longer valid. For instance, in case of sequential architecture, the order in which these measurements are processed is important. In this respect, it is recommended to start with the measurement with the lowest variance-covariance matrix R_i -the most accurate one-, in order to reduce subsequent linearization errors. Besides, data compression architecture does not perform well in general since it does not use the full rank of the measurements -all measurements are aggregated to a single measurement-. On the other hand, as far as the track-to-track tracking is concerned, the independence assumption of the noise is no longer valid since the measurements are intuitively correlated by the same noise process of the target. Therefore the equivalence between the three architectures is also no longer valid. The rest of the paper will investigate this issue.

3 Convex combination architecture

A fourth architecture advocated by Singer and Kanyuck (Singer and Kanyuck, 1971] consists of running a set of independent Kalman filters, each using one measurement together with the same state process, and then combine the outcomes of the n filer blocks using an optimal convex combination rule. Figure 4 illustrates the block diagram of such fusion architecture. More formally, the rationale behind the optimality as suggested in [Bar-Shalom and Campo, 1986; Bar-Shalom and Li, 1995] consists of assuming that one of the output is a prior while the other one as a measurement, thereby, used for the update. Although the authors in the above references have restricted the reasoning to a two-measurement model, none extension exists as far as authors' knowledge. In the two-measurements scenario, the outputs reads as

$$X_{k/k} = P_{k/k,1}(P_{k/k,1} + P_{k/k,2})^{-1} X_{k/k,2} + P_{k/k,2}(P_{k/k,1} + P_{k/k,2})^{-1} X_{k/k,1} \tag{14}$$

And

$$P_{k/k} = P_{k/k,1}(P_{k/k,1} + P_{k/k,2})^{-1} P_{k/k,2} \tag{15}$$

In the following, one shall investigate the extension of (14-15) to the case of several measurements; that is, given, at time t, a set of local estimators $(X_{k/k,i}, P_{k/k,i})$ for $i = 1$ to n of X, how to determine the underlying global estimator $(X_{k/k}, P_{k/k})$. In this course, following the approach in [Bar-Shalom

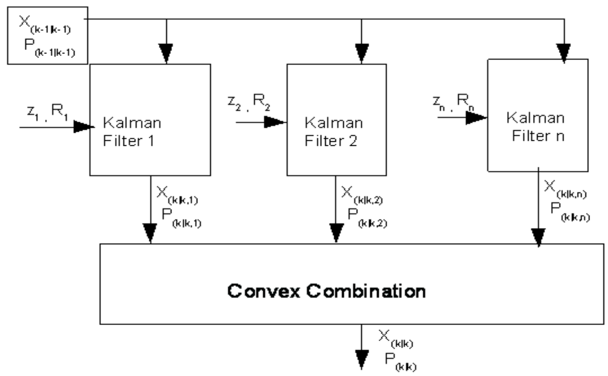

Figure 4: Parallel-Convex architecture

and Campo, 1986] and [Ducan and Sameer, 2006], one assumes one estimate, say, $(X_{k/k,i})$ to act as a prior information denoted D_1, so that

$$P(X/D_1) = N(X; X_{k/k,1}, P_{k/k,1}) \tag{16}$$

Where notations of right hand side of (16) stands for Gaussian distribution with mean $X_{k/k,1}$ and variance-covariance matrix $P_{k/k,1}$ of state vector X. Similarly, the estimate $X_{k/k,i}$, $i \neq 1$, which now acts as a measurement yielding information D_i is such that

$$\bar{X}_{k/k,l} = E[X_{k/k,i}|D_1] = X_{k/k,1} \quad i = 1\, to\, n \tag{17}$$

So, the global estimate $X_{k/k}$ will be estimated from the posterior probability $p(X|X_{k/k,2}, X_{k/k,3},, X_{k/k,n})$ by using for instance, the minimum mean square estimate where the solution is given in terms of conditional mean $X\ \hat{(X)} = E[X|Z] = \int xp(x|Z)dx$, here Z is the vector of all measurements. Consider a stacked vector $y = [x\, z_1\, z_2 \cdots z_n]^T$, with $p(x) = N(x; \bar{x}, P_x x)$ and $p(z_i) = N(z_i; \bar{z}_i, P_{z_i z_i})$, $i = 1$ to n. Let us also assume that x, z_1, z_2, ., z_n are jointly Gaussian, *i.e.*, the stacked vector y is also Gaussian:

$$p(x, z_1, z_2, ., z_n) = p(y) = N(y; \bar{y}, P_y y)$$

Then the conditional mean is given by the following.

Proposition 2

$$\hat{x} = E[x|z_1, z_2, ., z_n] = \bar{x} - T_{xx}^{-1} \sum_{i=1}^{n} T_{xz_i}(z_i - \bar{z}_l) \tag{18}$$

with

$$P_{xx|z_1,z_2,.,z_n} = T_{xx}^{-1} \tag{19}$$

$$T_{xx}^{-1} = P_{xx} - (P_{xz_1} P_{xz_2} . P_{xz_n}) \begin{pmatrix} P_{z_1 z_1} & P_{z_1 z_2} & . & P_{z_1 z_n} \\ P_{z_2 z_1} & P_{z_2 z_2} & . & P_{z_2 z_n} \\ & . & & \\ P_{z_n z_1} & P_{z_n z_2} & . & P_{z_n z_n} \end{pmatrix}^{-1} \begin{pmatrix} P_{xz_1} \\ . \\ P_{xz_n} \end{pmatrix} \tag{20}$$

$$T_{xx}^{-1}(T_{xz_1}T_{xz_2}.T_{xz_n}) = -(P_{xz_1}P_{xz_2}\cdots P_{xz_n})\begin{pmatrix} P_{z_1z_1} & P_{z_1z_2} & . & P_{z_1z_n} \\ P_{z_2z_1} & P_{z_2z_2} & . & P_{z_2z_n} \\ & . & & \\ P_{z_nz_1} & P_{z_nz_2} & . & P_{z_nz_n} \end{pmatrix}^{-1} \tag{21}$$

Proof Let us use the notations $\zeta_x = x - \bar{x}$, $\zeta_i = z_i - \bar{z}_l$, $i = 1\,to\,n$

Using Bayes'theorem,we have

$$P(x|z) = p(x|z_1, z_2, ., z_n) = \frac{p(x, z_1, z_2, ., z_n)}{p(z_1, z_2, ., z_n)} = \frac{p(y)}{p(Z)} \tag{22}$$

With

$$P(y) = p(y; \bar{y}, P_{yy}) = |2\pi|P_{yy}|^{1/2} exp(-(y-\bar{y})P_{yy}^{-1}(y-\bar{y})/2) \tag{23}$$

Where $\bar{y} = [\bar{x}\,\bar{z}_1\,\bar{z}_2\,.\,\bar{z}_n]^T$ and $P_{yy} = \begin{pmatrix} P_{xx} & P_{xz_1} & . & P_{xz_n} \\ P_{z_1x} & P_{z_1z_1} & . & P_{z_1z_n} \\ & . & & \\ P_{z_nx} & P_{z_nz_1} & . & P_{z_nz_n} \end{pmatrix}$

Similarly,

$$P(Z) = p(Z; \bar{Z}, P_{zz}) = |2\pi|P_{zz}|^{1/2} exp(-(Z-\bar{Z})P_{zz}^{-1}(Z-\bar{Z})/2) \tag{24}$$

By substituting into expression of $p(x|Z)$, the exponent of the exp function becomes

$$q = \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ . \\ \zeta_n \end{pmatrix} \begin{pmatrix} P_{xx} & P_{xz_1} & . & P_{xz_n} \\ P_{z_1x} & P_{z_1z_1} & . & P_{z_1z_n} \\ & . & & \\ P_{z_nx} & P_{z_nz_1} & . & P_{z_nz_n} \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ . \\ \zeta_n \end{pmatrix} - \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ . \\ \zeta_n \end{pmatrix}^{-1} \begin{pmatrix} P_{z_1z_1} & P_{z_nz_1} \\ P_{z_1z_2} & P_{z_nz_2} \\ . & \\ P_{z_1z_n} & P_{z_nz_n} \end{pmatrix}^T \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ . \\ . \\ . \\ \zeta_n \end{pmatrix} \tag{25}$$

Rewriting the inverse matrix, using the matrix inversion lemma as follows:

$$\begin{pmatrix} P_{xx} & P_{xz_1} & . & P_{xz_n} \\ P_{z_1x} & P_{z_1z_1} & . & P_{z_1z_n} \\ & . & & \\ P_{z_nx} & P_{z_nz_1} & . & P_{z_nz_n} \end{pmatrix}^{-1} = \left(\begin{array}{c|ccc} P_{xx} & P_{xz_1} & . & P_{xz_n} \\ -- & -- & -- & -- \\ P_{z_1x} & P_{z_1z_1} & . & P_{z_1z_n} \\ & . & & \\ P_{z_nx} & P_{z_nz_1} & . & P_{z_nz_n} \end{array}\right)^{-1} = \left(\begin{array}{c|ccc} T_{xx} & T_{xz_1} & . & T_{xz_n} \\ -- & -- & -- & -- \\ T_{z_1x} & T_{z_1z_1} & . & T_{z_1z_n} \\ & . & & \\ T_{z_nx} & T_{z_nz_1} & . & T_{z_nz_n} \end{array}\right) \tag{26}$$

$$T_{xx}^{-1} = P_{xx} - \begin{pmatrix} P_{xz_1} & P_{xz_2} & . & P_{xz_n} \end{pmatrix} \begin{pmatrix} P_{z_1z_1} & P_{z_1z_2} & . & P_{z_1z_n} \\ & & . & \\ P_{z_nz_1} & P_{z_nz_2} & . & P_{z_nz_n} \end{pmatrix}^{-1} \begin{pmatrix} P_{xz_1} \\ P_{xz_2} \\ . \\ P_{xz_n} \end{pmatrix} \tag{27}$$

$$\begin{pmatrix} P_{z_1z_1} & P_{z_1z_2} & . & P_{z_1z_n} \\ & . & & \\ P_{z_nz_1} & P_{z_nz_2} & . & P_{z_nz_n} \end{pmatrix}^{-1} = \begin{pmatrix} T_{z_1z_1} & T_{z_1z_2} & . & T_{z_1z_n} \\ T_{z_2z_1} & T_{z_2z_2} & . & T_{z_2z_n} \\ & . & & \\ T_{z_nz_1} & T_{z_nz_2} & . & T_{z_nz_n} \end{pmatrix} - \begin{pmatrix} T_{z_1x} \\ T_{z_2x} \\ . \\ T_{z_nx} \end{pmatrix} T_{xx}^{-1} \begin{pmatrix} T_{xz_1} \\ T_{xz_2} \\ . \\ T_{xz_n} \end{pmatrix}^T \tag{28}$$

$$T_{xx}^{-1}\begin{pmatrix} T_{xz_1} & T_{xz_2} & . & T_{xz_n} \end{pmatrix} = \begin{pmatrix} P_{xz_1} & P_{xz_2} & . & P_{xz_n} \end{pmatrix} \begin{pmatrix} P_{z_1z_1} & P_{z_1z_2} & . & P_{z_1z_n} \\ P_{z_2z_1} & P_{z_2z_2} & . & P_{z_2z_n} \\ & . & & \\ P_{z_nz_1} & P_{z_nz_2} & . & P_{z_nz_n} \end{pmatrix}^{-1} \tag{29}$$

Let us denote for simplicity

$$\begin{pmatrix} P_{z_1z_1} & P_{z_1z_2} & . & P_{z_1z_n} \\ P_{z_2z_1} & P_{z_2z_2} & . & P_{z_2z_n} \\ & . & & \\ P_{z_nz_1} & P_{z_nz_2} & . & P_{z_nz_n} \end{pmatrix} = P_{zz} \quad \text{and} \quad \begin{pmatrix} T_{z_1z_1} & T_{z_1z_2} & . & T_{z_1z_n} \\ T_{z_2z_1} & T_{z_2z_2} & . & T_{z_2z_n} \\ & . & & \\ T_{z_nz_1} & T_{z_nz_2} & . & T_{z_nz_n} \end{pmatrix} = T_{zz}$$

Rewriting the expression of q, after substituting (28) and (26) into (25) leads to

$$
\begin{aligned}
q = & \begin{pmatrix} \zeta_x \\ \zeta_1 \\ . \\ \zeta_n \end{pmatrix}^T \begin{pmatrix} T_{xx} & | & T_{xz_1} & . & T_{xz_n} \\ -- & -- & -- & -- & -- \\ T_{z_1x} & | & T_{z_1z_1} & . & T_{z_1z_n} \\ & | & . & & \\ T_{z_nx} & | & T_{z_nz_1} & . & T_{z_nz_n} \end{pmatrix} \begin{pmatrix} \zeta_x \\ \zeta_1 \\ . \\ \zeta_n \end{pmatrix} \\
& - \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ . \\ \zeta_n \end{pmatrix}^T \begin{pmatrix} T_{z_1z_1} - T_{z_1x}T_{xx}^{-1}T_{xz_1} & . & T_{z_1z_n} - T_{z_1x}T_{xx}^{-1}T_{xz_n} \\ . & \ddots & . \\ T_{z_nz_1} - T_{z_nx}T_{xx}^{-1}T_{xz_1} & . & T_{z_1z_n} - T_{z_nx}T_{xx}^{-1}T_{xz_n} \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ . \\ \zeta_n \end{pmatrix}
\end{aligned} \tag{30}
$$

After some manipulations, (30) is equivalent to

$$
\begin{aligned}
q = & \zeta_x^T T_{xx}\zeta_x + \zeta_x^T \sum_{i=1}^{n} T_{xz_i}\zeta_i + \left(\sum_{i=1}^{n} \zeta_i^T T_{z_ix}\right)\zeta_x + \sum_{i=1}^{n}\sum_{k=1}^{n} \zeta_i^T T_{z_iz_k} \\
& - \left(\sum_{i=1}^{n}\sum_{k=1}^{n} \zeta_i^T T_{z_iz_k} - \sum_{i=1}^{n} \zeta_i^T T_{z_ix}T_{xx}^{-1}\sum_{k=1}^{n} \zeta_k^T T_{xz_k}\right)
\end{aligned} \tag{31}
$$

$i.e.$,

$$
q = \zeta_x^T T_{xx}\zeta_x + \zeta_x^T \sum_{i=1}^{n} T_{xz_i}\zeta_i + \left(\sum_{i=1}^{n} \zeta_i^T T_{z_ix}\right)\zeta_x + \sum_{i=1}^{n} \zeta_i^T T_{z_ix}T_{xx}^{-1}\left[\sum_{k=1}^{n} T_{xz_k}\right]\zeta_i \tag{32}
$$

After some manipulations this yields, using the fact that $T_{xz_i}^T = T_{xz_i}$ and $T_{xx}^{-1}.T_{xx} = T_{xx}.T_{xx}^{-1} = I$

$$
q = \left(\zeta_x + T_{xx}^{-1}\sum_{i=1}^{n} T_{xz_i}\zeta_i\right)^T T_{xx} \left(\zeta_x + T_{xx}^{-1}\sum_{i=1}^{n} T_{xz_i}\zeta_i\right) \tag{33}
$$

Which is a quadratic form in x, therefore, the conditional mean is achieved when

$$
\zeta_x + T_{xx}^{-1}\sum_{i=1}^{n} T_{xz_i}\zeta_i = 0 \tag{34}
$$

$i.e.$,

$$
(x-\bar{x}) + T_{xx}^{-1}\sum_{i=1}^{n} T_{xz_i}(z-\bar{z}) = 0 \Longrightarrow E[x|Z] = \hat{x} = \bar{x} - T_{xx}^{-1}\sum_{i=1}^{n} T_{xz_i}(z-\bar{z}) \tag{35}
$$

And

$$
P_{xx|z} = T_{xx}^{-1} = P_{xx} - \begin{pmatrix} P_{xz_1} & . & P_{xz_n} \end{pmatrix} \begin{pmatrix} P_{z_1z_1} & . & P_{z_1z_n} \\ & . & \\ P_{z_nz_1} & . & P_{z_nz_n} \end{pmatrix}^{-1} \begin{pmatrix} P_{xz_1} \\ . \\ P_{xz_n} \end{pmatrix} \tag{36}
$$

which completes the proof.

The following result concerns the matrix inversion and will be used in the later development. It is adapted from [8], therefore, the reader should refer the above citation for full proof.

Lemma 1 If A is an arbitrary non-singular square matrix $n \times n$ and let U, V be $n \times k$ matrices with $k \leq n$, then

$$\left(A + UV^T\right) = A^{-1} - A^{-1}.U.S^{-1}V^T.A^{-1}$$

With $S = I + V^T A^{-1} U$

The above follows straightforwardly from Woodbury formula of inversion of sum of matrices, see, for instance [Dahquist and Bjorck, 1974; Halmos, 1958], for detailed calculus in this respect. Although to perform the calculus S requires to be non-singular matrix. The preceding provides an efficient tool to determine inverse of matrix as a function of the non-singular matrix A whose inverse is known or easily processed.

Proposition 3

Given a set of local estimators $(X_{k/k,i}, P_{k/k,i})$ for $i = 1$ to n of X, the global estimator $(X_{k/k}, P_{k/k})$ obtained by considering one local estimator as prior information and the others as measurements and assuming independence of the local estimators, is given by

$$X_{k/k} = \zeta \sum_{i=1}^{n} P_{k/k,i}^{-1} X_{k/k,i} \tag{37}$$

$$P_{k/k} = \left[\sum_{i=1}^{n} P_{k/k,i}^{-1}\right]^{-1} \tag{38}$$

Proof

Using the notations of stacked vector y in Proposition 1, and given the assumption (16) and let us assume that the prior is constituted of the first estimate $X_{k/k,1}$, the key is to find the counterpart of the various parameters of y

- $\bar{x}$ corresponds to $E\left[X/D_1\right] = X_{k/k,1}$
- z_i corresponds to $X_{k/k,i}$
- P_{xx} corresponds to $E\left[(X - X_{k/k,1})(X - X_{k/k,1})^T\right] = P_{k/k,1}$
- $P_{z_i z_i}$ corresponds to $E\left[(X_{k/k,i} - X_{k/k,1})(X_{k/k,i} - X_{k/k,1})^T\right]$
$= E\left[\left((X - X_{k/k,1}) - (X - X_{k/k,i})\right)\left((X - X_{k/k,1}) - (X - X_{k/k,i})\right)^T\right]$
$= P_{k/k,i} + P_{k/k,1}$
- $P_{z_i z_j}$ corresponds to $E\left[(X_{k/k,i} - X_{k/k,1})(X_{k/k,j} - X_{k/k,1})^T\right]$, so

$$P_{z_i z_j} = E\left[\left((X - X_{k/k,1}) - (X - X_{k/k,i})\right)\left((X - X_{k/k,1}) - (X - X_{k/k,j})\right)^T\right] \tag{39}$$
$$= P_{k/k,1}$$

- P_{xz_i} corresponds to $E\left[(X - X_{k/k,1})(X_{k/k,i} - X_{k/k,1})^T\right]$, so
$= E\left[(X - X_{k/k,1}))\left((X - X_{k/k,1}) - (X - X_{k/k,i})\right)^T\right] = P_{k/k,1}$

Now substituting these values in (18-21) yields (for simplicity of notations, the time subscript is removed from X and P formulations):

$$T_{xx}^{-1} = P_{xx|z} = P_1 - \begin{pmatrix} P_1 \\ P_1 \\ . \\ P_1 \end{pmatrix}^T \begin{pmatrix} P_1 + P_2 & P_1 & . & P_1 \\ P_1 & P_1 + P_3 & . & P_1 \\ & & . & \\ P_1 & P_1 & . & P_1 + P_{n-1} \end{pmatrix}^{-1} \begin{pmatrix} P_1 \\ P_1 \\ . \\ P_1 \end{pmatrix} \tag{40}$$

$$T_{xx}^{-1}\begin{pmatrix} T_{xz_1} & T_{xz_2} & . & T_{xz_n}\end{pmatrix} = -\begin{pmatrix} P_1+P_2 & P_1 & . & P_1 \\ P_1 & P_1+P_3 & . & P_1 \\ & & . & \\ P_1 & P_1 & . & P_1+P_{n-1}\end{pmatrix}^{-1} \tag{41}$$

$$x_{k/k} = x_{k/k,1} + \sum_{i=2}^{n-1} T_{xx}^{-1} T_{xz_i}(x_{k/k,i} - x_{k/k,1}) \tag{42}$$

Now using Lemma 1, one can write the matrix to be inverted into

$$\begin{pmatrix} P_1+P_2 & P_1 & . & P_1 \\ P_1 & P_1+P_3 & . & P_1 \\ & & . & \\ P_1 & P_1 & . & P_1+P_{n-1}\end{pmatrix} = \left[\begin{pmatrix} P_2 & 0 & . & 0 \\ 0 & P_3 & . & 0 \\ & & . & \\ 0 & 0 & . & P_{n-1}\end{pmatrix} + \begin{pmatrix} P_1 & P_1 & . & P_1 \\ P_1 & P_1 & . & P_1 \\ & & . & \\ P_1 & P_1 & . & P_1\end{pmatrix}\right]$$

$$= \left[\begin{pmatrix} P_2 & 0 & . & 0 \\ 0 & P_3 & . & 0 \\ & & . & \\ 0 & 0 & . & P_{n-1}\end{pmatrix} + \begin{pmatrix} I_p \\ I_p \\ . \\ I_p\end{pmatrix}\begin{pmatrix} P_1 & P_1 & . & P_1\end{pmatrix}\right]^{-1} \tag{43}$$

Where I_P is identity matrix of the same as P_i.

Using Woodbury's inversion formula of Lemma 1 and denoting

$$\Gamma = I_p + \begin{pmatrix} P_1 & P_1 & . & P_1\end{pmatrix}\begin{pmatrix} P_2 & 0 & . & 0 \\ 0 & P_3 & . & 0 \\ & & . & \\ 0 & 0 & . & P_{n-1}\end{pmatrix}^{-1}\begin{pmatrix} I_p \\ I_p \\ . \\ I_p\end{pmatrix} \tag{44}$$

Noticing that for a diagonal matrix, the inverse coincides with inverse of its diagonal elements, its holds that

$$\begin{pmatrix} P_2 & 0 & . & 0 \\ 0 & P_3 & . & 0 \\ & & . & \\ 0 & 0 & . & P_{n-1}\end{pmatrix}^{-1} = \begin{pmatrix} P_2^{-1} & 0 & . & 0 \\ 0 & P_3^{-1} & . & 0 \\ & & . & \\ 0 & 0 & . & P_{n-1}^{-1}\end{pmatrix} \tag{45}$$

Therefore,

$$\Gamma = I_p + \begin{pmatrix} P_1 & P_1 & . & P_1\end{pmatrix}\begin{pmatrix} P_2^{-1} & 0 & . & 0 \\ 0 & P_3^{-1} & . & 0 \\ & & . & \\ 0 & 0 & . & P_{n-1}^{-1}\end{pmatrix}\begin{pmatrix} I_p \\ I_p \\ \vdots \\ I_p\end{pmatrix}$$

$$= I_p + P_1P_2^{-1} + P_1P_3^{-1} + . + P_1P_n^{-1}$$

$$= P_1P_1^{-1} + P_1P_2^{-1} + P_1P_3^{-1} + . + P_1P_n^{-1} = P_1\sum_{i=1}^{n} P_i^{-1} \tag{46}$$

i.e.,

$$\Gamma = P_1\sum_{i=1}^{n} P_i^{-1} = P_1\Sigma \tag{47}$$

with $\Sigma = \sum_{i=1}^{n} P_i^{-1}$. So $\Gamma^{-1} = \Sigma^{-1} P_1^{-1}$

Now applying Lemma 1 leads to

$$\begin{pmatrix} P_1 + P_2 & P_1 & . & P_1 \\ P_1 & P_1 + P_3 & . & P_1 \\ & & . & \\ P_1 & P_1 & . & P_1 + P_{n-1} \end{pmatrix}^{-1}$$

$$= \left[\begin{pmatrix} P_2 & 0 & . & 0 \\ 0 & P_3 & . & 0 \\ & & . & \\ 0 & 0 & . & P_{n-1} \end{pmatrix} + \begin{pmatrix} I_p \\ I_p \\ . \\ I_p \end{pmatrix} \begin{pmatrix} P_1 \\ P_1 \\ . \\ P_1 \end{pmatrix}^{-1} \right]^{-1} = \begin{pmatrix} P_2^{-1} & 0 & . & 0 \\ 0 & P_3^{-1} & . & 0 \\ & & . & \\ 0 & 0 & . & P_{n-1}^{-1} \end{pmatrix} \tag{48}$$

$$- \begin{pmatrix} P_2^{-1} & 0 & . & 0 \\ 0 & P_3^{-1} & . & 0 \\ & & . & \\ 0 & 0 & . & P_{n-1}^{-1} \end{pmatrix} \begin{pmatrix} I_p \\ I_p \\ . \\ I_p \end{pmatrix} .^{-1} P_1^{-1} \begin{pmatrix} P_1 \\ P_1 \\ . \\ P_1 \end{pmatrix}^{-1} \begin{pmatrix} P_2^{-1} & 0 & . & 0 \\ 0 & P_3^{-1} & . & 0 \\ & & . & \\ 0 & 0 & . & P_{n-1}^{-1} \end{pmatrix}$$

Let us denote by A the quantity

$$\begin{pmatrix} P_2^{-1} & 0 & . & 0 \\ 0 & P_3^{-1} & . & 0 \\ & & . & \\ 0 & 0 & . & P_{n-1}^{-1} \end{pmatrix} \begin{pmatrix} I_p \\ I_p \\ . \\ I_p \end{pmatrix} \Sigma^{-1} P_1^{-1} \begin{pmatrix} P_1 & P_1 & . & P_1 \end{pmatrix} \begin{pmatrix} P_2^{-1} & 0 & . & 0 \\ 0 & P_3^{-1} & . & 0 \\ & & . & \\ 0 & 0 & . & P_{n-1}^{-1} \end{pmatrix}$$

Since

$$\begin{pmatrix} I_p \\ I_p \\ . \\ I_p \end{pmatrix} \Sigma^{-1} P_1^{-1} \begin{pmatrix} P_1 & P_1 & . & P_1 \end{pmatrix} = \begin{pmatrix} \Sigma^{-1} & \Sigma^{-1} & . & \Sigma^{-1} \\ \Sigma^{-1} & \Sigma^{-1} & . & \Sigma^{-1} \\ & & . & \\ \Sigma^{-1} & \Sigma^{-1} & . & \Sigma^{-1} \end{pmatrix} \tag{49}$$

we have

$$A = \begin{pmatrix} P_2^{-1} & 0 & . & 0 \\ 0 & P_3^{-1} & . & 0 \\ & & . & \\ 0 & 0 & . & P_{n-1}^{-1} \end{pmatrix} \begin{pmatrix} \Sigma^{-1} & \Sigma^{-1} & . & \Sigma^{-1} \\ \Sigma^{-1} & \Sigma^{-1} & . & \Sigma^{-1} \\ & & . & \\ \Sigma^{-1} & \Sigma^{-1} & . & \Sigma^{-1} \end{pmatrix} \begin{pmatrix} P_2^{-1} & 0 & . & 0 \\ 0 & P_3^{-1} & . & 0 \\ & & . & \\ 0 & 0 & . & P_{n-1}^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} P_2^{-1}\Sigma^{-1}P_2^{-1} & P_2^{-1}\Sigma^{-1}P_3^{-1} & . & P_2^{-1}\Sigma^{-1}P_n^{-1} \\ P_3^{-1}\Sigma^{-1}P_2^{-1} & P_3^{-1}\Sigma^{-1}P_3^{-1} & . & P_3^{-1}\Sigma^{-1}P_n^{-1} \\ & & . & \\ P_n^{-1}\Sigma^{-1}P_2^{-1} & P_n^{-1}\Sigma^{-1}P_3^{-1} & . & P_n^{-1}\Sigma^{-1}P_n^{-1} \end{pmatrix} \tag{50}$$

$$\begin{pmatrix} P_1 & P_2 & . & P_n \end{pmatrix} \begin{pmatrix} P_1 + P_2 & P_1 & . & P_1 \\ P_1 & P_1 + P_3 & . & P_1 \\ & & . & \\ P_1 & P_1 & . & P_1 + P_n \end{pmatrix}^{-1} \begin{pmatrix} P_1 \\ P_2 . \\ P_n \end{pmatrix} =$$

$$\begin{pmatrix} P_1 & P_1 & . & P_1 \end{pmatrix} \begin{pmatrix} P_2^{-1} & 0 & . & 0 \\ 0 & P_3^{-1} & . & 0 \\ & & . & \\ 0 & 0 & . & P_{n-1}^{-1} \end{pmatrix} \begin{pmatrix} P_1 \\ P_1 \\ . \\ P_1 \end{pmatrix} - \begin{pmatrix} P_1 & P_1 & . & P_1 \end{pmatrix} A \begin{pmatrix} P_1 \\ P_1 \\ . \\ P_1 \end{pmatrix} \tag{51}$$

After some manipulations, we have

$$\begin{pmatrix} P_1 & P_1 & . & P_1 \end{pmatrix} \begin{pmatrix} P_2^{-1} & 0 & . & 0 \\ 0 & P_3^{-1} & . & 0 \\ & & . & \\ 0 & 0 & \cdots & P_{n-1}^{-1} \end{pmatrix} \begin{pmatrix} P_1 \\ P_1 \\ . \\ P_1 \end{pmatrix} = P_1 \left(\sum_{i=2}^{n} P_i^{-1} \right) P_1 \tag{52}$$

Similarly,

$$\left(P_1 \; P_1 \; . \; P_1 \right) \begin{pmatrix} P_2^{-1}\Sigma^{-1}P_2^{-1} & P_2^{-1}\Sigma^{-1}P_3^{-1} & . & P_2^{-1}\Sigma^{-1}P_n^{-1} \\ P_3^{-1}\Sigma^{-1}P_2^{-1} & P_3^{-1}\Sigma^{-1}P_3^{-1} & . & P_3^{-1}\Sigma^{-1}P_n^{-1} \\ & & . & \\ P_n^{-1}\Sigma^{-1}P_2^{-1} & P_n^{-1}\Sigma^{-1}P_3^{-1} & . & P_n^{-1}\Sigma^{-1}P_n^{-1} \end{pmatrix} \begin{pmatrix} P_1 \\ P_1 \\ . \\ P_1 \end{pmatrix}$$

$$= P_1 \left(\sum_{i=2,nj=2,n} P_i^{-1}\Sigma^{-1}P_j^{-1} \right) = P_1 \left(\left[\sum_{i=2}^{n} P_i^{-1} \right] \Sigma^{-1} \left[\sum_{i=2}^{n} P_i^{-1} \right] \right) \tag{53}$$

Substituting these entities in expression of T_{xx}^{-1} in (41), yields

$$T_{xx}^{-1} = P_{k/k} = P_1 - P_1 \left(\sum_{i=2}^{n} P_i^{-1} \right) P_1 + P_1 \left(\left[\sum_{i=2}^{n} P_i^{-1} \right] \Sigma^{-1} \left[\sum_{i=2}^{n} P_i^{-1} \right] \right) P_1 \tag{54}$$

Noticing that $\sum_{i=2}^{n} P_i^{-1} = \sum_{i=1}^{n} P_i^{-1} - P_1^{-1} = \Sigma - P_1^{-1}$, we have

$T_{xx}^{-1} = P_{k/k} = P_1 - P_1 \left(\Sigma - P_1^{-1} \right) P_1 + P_1 \left(\left(\Sigma - P_1^{-1} \right) \Sigma^{-1} \left(\Sigma - P_1^{-1} \right) \right) P_1$

To prove the result for X, one shall first determine the quantity $T_{xx}^{-1}\, T_{xz_i}$. After some manipulations, we have

$$-\left(P_1 \; P_1 \; . \; P_1 \right) \begin{pmatrix} P_1 + P_2 & P_1 & . & P_1 \\ P_1 & P_1 + P_3 & . & P_1 \\ & & . & \\ P_1 & P_1 & . & P_1 + P_{n-1} \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} -P_1 P_2^{-1} \\ -P_1 P_3^{-1} \\ \vdots \\ -P_1 P_n^{-1} \end{pmatrix} - \begin{pmatrix} P_1 [\sum_{i=2}^{n} P_i^{-1}] \Sigma^{-1} P_2^{-1} \\ P_1 [\sum_{i=2}^{n} P_i^{-1}] \Sigma^{-1} P_3^{-1} \\ \vdots \\ P_1 [\sum_{i=2}^{n} P_i^{-1}] \Sigma^{-1} P_n^{-1} \end{pmatrix} \tag{55}$$

Consequently,

$$\begin{aligned} T_{xx}^{-1} T_{xz_i} &= -P_1 P_i^{-1} + P_1 \left[\sum_{i=2}^{n} P_i^{-1} \right] \Sigma^{-1} P_i^{-1} \\ &= -P_1 P_i^{-1} + P_1 \left(\Sigma - P_1^{-1} \right) \Sigma^{-1} P_i^{-1} = -\Sigma^{-1} P_i^{-1} \end{aligned} \tag{56}$$

Therefore expression of X becomes,

$$\begin{aligned} X_{k/k} &= X_{k/k,1} - \sum_{i=2}^{n} \left(-\Sigma^{-1} P_i^{-1} \right) \left(X_{k/k,i} - X_{k/k,1} \right) \\ &= X_{k/k,1} - \Sigma^{-1} \sum_{i=2}^{n} P_i^{-1} X_{k/k,i} - \Sigma^{-1} \sum_{i=2}^{n} P_i^{-1} X_{k/k,1} \\ &= X_{k/k,1} - \Sigma^{-1} \sum_{i=2}^{n} P_i^{-1} X_{k/k,i} - \Sigma^{-1} (\Sigma - P_1^{-1}) X_{k/k,1} = \Sigma^{-1} \sum_{i=1}^{n} P_i^{-1} X_{k/k,i} \end{aligned} \tag{57}$$

This completes the proof.

Especially, the result pointed out in Proposition 3 indicates that the outcome of the fusion rule does not depend on the specific choice of the prior. In other words, the global estimate (38-39) is still valid if one chooses as a prior another $(X_{k/k,i}, P_{k/k,i})$ with $i \neq 1$. This arises from the symmetry of the expressions (38-39) with respect to individual local estimates $(X_{k|k,i}, P_{k|k,i})$. An interesting issue consists to compare the quality of the estimate as supplied by the associated variance-covariance matrix compared to that of individual local estimates in the special case where the variance-covariance matrix has dominant diagonal elements (e.g., $P_{ss} >> P_{jj}$ for all integers i, j, s within $[0, n]$ and $i \neq j$). In this course, the following holds.

Proposition 4

If the individual variance-covariance matrices $P_{k/k,i}$ $(i = 1, n)$ have dominant diagonal elements, then it holds that

$$Determinant(P_{k/k}) \leq Determinant P_{k/k,i}), \quad for\, all\ i = 1, n \tag{58}$$

Proof

First, one shall notice that the condition of dominant diagonal elements allows us to stipulate that the determinant of the sum of matrices is always greater than the determinant of any individual matrix. To see it, assume without loss of generality, that matrices A and B are 2x2. Therefore, using the notation |.| for determinant, we have

$$\begin{aligned}
|A+B| &= (a_{11}+b_{11})(a_{22}+b_{22}) - (a_{21}+b_{21})(a_{12}+b_{12}) \\
&= (a_{11}a_{22} - a_{21}a_{12}) + (b_{11}b_{22} - b_{21}b_{12}) + (a_{11}b_{22} - b_{11}a_{22} - a_{21}b_{12} - a_{12}b_{21}) \\
&= |A| + |B| + (a_{11}b_{22} - b_{11}a_{22} - a_{21}b_{12} - a_{12}b_{21})
\end{aligned}$$

Clearly, when diagonal parts are dominant the quantity under brackets is positive, and, therefore, $|A+B| \geq |A|+|B|$. Therefore $|A+B| \geq |A|$ and $|A+B| \geq |B|$ On the other hand, from the product rule of determinant, i.e., $|A.B| = |A|.|B|$, it turns out since $A.A^{-1} = I$, that $|A^{-1}| = \frac{1}{|A|}$ for any non-singular matrix A. Consequently,

$$\left| \left[\sum_{i=1}^{n} P_{k/k,i}^{-1} \right]^{-1} \right| = \frac{1}{\left| \left[\sum_{i=1}^{n} P_{k/k,i}^{-1} \right] \right|} \tag{59}$$

Using the assumption of dominant diagonal elements, we have

$$\left| \sum_{i=1}^{n} P_{k/k,i}^{-1} \right| \geq \left| P_{k/k,i}^{-1} \right| = \frac{1}{\left| P_{k/k,i} \right|}, \quad for\, all\ i = 1, n \tag{60}$$

Consequently,

$$\frac{1}{\left| \left[\sum_{i=1}^{n} P_{k/k,i}^{-1} \right] \right|} \leq \left| P_{k/k,i} \right| \tag{61}$$

Which completes the proof.

Proposition 4 stipulates that when the accuracy is understood in terms of the determinant of the variance-covariance matrix, then the global estimate resulting from the fusion of individual local estimates induces a smaller variance-covariance matrix, indicating an improvement of the accuracy with respect to the best accuracy of individual estimator.
The preceding shows that the use of this fusion rule always guarantees a reduction of the variance-covariance matrix in the sense of its determinant. As special case, when the variance-covariance matrices are real valued, in which case it holds that $|P_{k/k,i}| = P_{k/k,i}$, therefore, Proposition 4 entails that

$$P_{k/k} \leq min_i P_{k/k,i} \tag{62}$$

On the other hand, one notices the following

- The assumption of dominant diagonal elements in local variance-covariance matrices pointed out in Proposition 3 is widely realistic when the components of the state vector are uncorrelated. However in case of strong correlation, such condition may no longer be valid. This also occurs when the local estimator is the result of a large number of iterations using (extended) Kalman filter, which due to lack of convergence may lead to highly correlated variance estimators.

- In terms of computational complexity of the calculus of the global estimate, one shall notice that, assuming the variance-covariance matrices are m x m matrices, the determination of $P_{k/k}$ involves $(n+1)$ matrix inversion, which yields a complexity $O((n+1)m^3) = O(n.m^3)$ when using standard Gaussian elimination approach for inverse matrix calculus. While this complexity reduces to $O(n.m^{2.376})$ in case of Coppersmith-Winograd type algorithm. Similarly, the calculus of the global state vector estimate involves $(n+1)$ matrix inversions for the calculus of expression ζ plus n matrix multiplications of $(m \times m) \times (m \times 1)$ matrices, so with complexity $O(n.m^2)$. Therefore the complexity of the calculus of $X_{k/k})$ yields $O(n.m^3) + O(n.m^2)$, which entails O(n.m3) in case of Gaussian elimination approach and $O(n.m^{2.376})$ in case of Coppersmith-Winograd type algorithm. This indicates that both global state vector and variance-covariance matrix of the fusion rule do have equal computational complexity of $O(n.m^3)$.

- Expression (38) indicates that the global state vector estimate is given as a weighted average of local state vector estimates, where the weights are inversely related to the associated variance-covariance matrix. This results in a biased estimation where the global estimate is driven by the local estimate with smaller variance-covariance matrix. Indeed, smaller variance-covariance $P_{k/k,i}$, indicating a more accurate and reliable local estimation, yields a larger value of $P^{-1}_{k/k,i}$. Therefore the contribution of the underlying $X_{k/k,i}$ to the sum becomes important.

– In case of equally state vectors estimates, say, $X_{k/k,i} = X_0$ for all i, then using (38), it is easy to see that $X_{k/k} = \left[\sum_{i=1}^{n} P_{k/k,i}^{-1}\right]^{-1} . \sum_{i=1}^{n} P_{k/k,i}^{-1}.X_0 = X_0$. In other words, if all local estimators provide the same outcome in terms of state vector $X_{k/k,i}$, then regardless the associated variance-covariance estimates, the fusion rule will also yield the same state vector estimate.

– Similarly, in case of equally valued variance-covariance matrices, $i.e.$, $P_{k/k,i} = P_0$, then the use of (30-40) yields $P_{k/k} = [n.P_0^{-1}]^{-1} = \frac{1}{n}P_0$. In other words, regardless the estimate of the local state vectors, the global variance-covariance matrix is proportional to local estimate variance-covariance matrix P_0.

– The preceding indicates a complete decoupling between the estimates of state vector and variance-covariance estimates.

– One shall notice that the fusion rule corresponding to the parallel convex combination was thought originally to be optimal in the sense of minimum mean square error but has been discovered recently that it is optimal only in maximum likelihood sense. This also applies to the extensions developed throughout this section.

4 Case of dependent tracks

4.1 General case

In the case of dependent tracks; that is, the local estimators $(X_{k/k,i}, P_{k/k,i})$, $i = 1$ to n, are no longer independent, which often justified by the fact that the estimators are linked to the same process noise, therefore, they are correlated, at least from this perspective. In this course, using the same reasoning as that carried out in the proof of Proposition 3, assuming, without loss of generality that the first local estimator $(X_{k/k,1}, P_{k/k,1})$ acts as a prior estimator, then using the same notations as in Proposition 1, the counterpart of the various parameters of the stacked vector y are:

– $\bar{x}$ corresponds to $E[X/D_1] = X_{k/k,1}$

– z_i corresponds to $X_{k/k,i}$

– $\bar{z}_i$ corresponds to $E\left[X_{k/k,i}/D_1\right] = X_{k/k,1}$

– P_{xx} corresponds to $E\left[(X - X_{k/k,1})(X - X_{k/k,1})^T\right] = P_{k/k,1}$

– $P_{z_iz_i}$ corresponds to $E\left[(X_{k/k,i} - X_{k/k,1})(X_{k/k,i} - X_{k/k,1})^T\right]$,i.e.,
$P_{z_iz_i} = E\left[\left((X - X_{k/k,1}) - (X - X_{k/k,i})\right)\left((X - X_{k/k,1}) - (X - X_{k/k,i})\right)^T\right]$
$P_{z_iz_i} = P_{k/k,i} + P_{k/k,1} - P_{k/k,1i} - P_{k/k,i1}$

- $P_{z_i z_j}$ $(i \neq j)$ corresponds to $E\left[(X_{k/k,i} - X_{k/k,1})(X_{k/k,j} - X_{k/k,1})^T\right]$, so
  $P_{z_i z_j} = E\left[\left((X - X_{k/k,1}) - (X - X_{k/k,i})\right)\left((X - X_{k/k,1}) - (X - X_{k/k,j})\right)^T\right]$ $P_{z_i z_j} = P_{k/k,1} - P_{k/k,1i} - P_{k/k,1j} + P_{k/k,ij}$
- P_{xz_i} corresponds to $E\left[(X - X_{k/k,1})(X_{k/k,i} - X_{k/k,1})^T\right]$, so
  $P_{xz_i} = E\left[(X - X_{k/k,1}))\left((X - X_{k/k,1}) - (X - X_{k/k,i})\right)^T\right] = P_{k/k,1} - P_{k/k,1i}$

Consequently using result of Proposition 2, we have

$$T_{xx}^{-1} = P_1 - (P_1 - P_{12}\ .\ P_1 - P_{1n}) \times$$

$$\begin{pmatrix} P_1 + P_{22} - P_{12} - P_{21} & . & P_1 + P_{2n} - P_{1n} - P_{21} \\ P_1 + P_{32} - P_{12} - P_{31} & . & P_1 + P_{3n} - P_{1n} - P_{31} \\ & . & \\ P_1 + P_{n2} - P_{12} - P_{n1} & . & P_1 + P_{nn} - P_{1n} - P_{n1} \end{pmatrix}^{-1} \begin{pmatrix} P_1 - P_{21} \\ P_1 - P_{31} \\ . \\ P_1 - P_{n1} \end{pmatrix}$$

i.e.,

$$P_{xx|z} = P_1 - (\ P_1 - P_{12}\ .\ P_1 - P_{1n}\ ) \times$$

$$\left[\begin{pmatrix} P_{22} - P_{12} - P_{21} & . & P_{2n} - P_{1n} - P_{21} \\ P_{32} - P_{12} - P_{31} & . & P_{3n} - P_{1n} - P_{31} \\ & . & \\ P_{n2} - P_{12} - P_{n1} & . & P_{nn} - P_{1n} - P_{n1} \end{pmatrix} + \begin{pmatrix} P_1 & . & P_1 \\ & . & \\ P_1 & . & P_1 \end{pmatrix}\right]^{-1} \begin{pmatrix} P_1 - P_{21} \\ P_1 - P_{31} \\ . \\ P_1 - P_{n1} \end{pmatrix} \tag{63}$$

It should be noted that unlike the case uncorrelated track, the matrix inversion in the above expression can be simplified further using Woodbury formulae, consequently, one cannot drive straightforwardly a simplified expression for T_{xx}^{-1} or, equivalently, $P_{XX|Z}$. Similarly, using Proposition 2, the state vector of the fusion rule is given by

$$X_{k/k} = X_{k/k,1} - \sum_{i=1}^{n} P_{xx|Z} T_{xz_i} (X_{k/k,i} - X_{k/k,1}) \tag{64}$$

with

$$(P_{xx|Z}.T_{xz_1}\ .\ P_{xx|Z}.T_{xz_n}\ ) = -\ (\ P_1 - P_{12}\ .\ P_1 - P_{1n}\ ) \times$$

$$\left[\begin{pmatrix} P_{22} - P_{12} - P_{21} & . & P_{2n} - P_{1n} - P_{21} \\ P_{32} - P_{12} - P_{31} & . & P_{3n} - P_{1n} - P_{31} \\ & . & \\ P_{n2} - P_{12} - P_{n1} & . & P_{nn} - P_{1n} - P_{n1} \end{pmatrix} + \begin{pmatrix} P_1 & . & P_1 \\ & . & \\ P_1 & . & P_1 \end{pmatrix}\right]^{-1} \tag{65}$$

Strictly speaking, the previous expressions testify the complexity of the calculus of the quantities $P_{xx|Z}$ and $X_{k/k}$. Such complexity raises to $O((n-1)^3.m^3)$ in case of variance-covariance matrices $P_{k/k,i}$ of dimension m x m. Especially, the result (64-65) is very much dependent on the choice of the prior. In other words, the result of the fusion rule is no longer symmetric with respect to the choice of the prior datum Di. Nevertheless, it is worth investigating some special cases of the correlation.

4.2 Case of two-correlated tracks

A special case arises when there are only two correlated tracks among the set of n tracks. For instance, assume that tracks 2 and 3 are correlated. Therefore, the counterpart of the expression (63) will be

$$P_{xx|Z} = P_1 - (\, P_1 - P_{12} \;.\; P_1 - P_{1n} \,) \times$$

$$\left[\begin{pmatrix} P_{22} & P_{23} & 0_m & 0_m & . & 0_m & 0_m \\ P_{32} & P_{33} & 0_m & 0_m & . & 0_m & 0_m \\ 0_m & 0_m & P_{44} & 0_m & . & 0_m & 0_m \\ & & & . & & & \\ 0_m & 0_m & 0_m & 0_m & . & 0_m & P_{nn} \end{pmatrix} + \begin{pmatrix} I_p \\ I_p \\ . \\ I_p \end{pmatrix} (\, P_1 \; P_1 \;.\; P_1 \,) \right]^{-1} \begin{pmatrix} P_1 \\ P_1 \\ . \\ P_1 \end{pmatrix} \tag{66}$$

Given that

$$\begin{pmatrix} P_{22} & P_{23} & 0 & 0 & . & 0 & 0 \\ P_{32} & P_{33} & 0 & 0 & . & 0 & 0 \\ 0 & 0 & P_{44} & 0 & . & 0 & 0 \\ & & & . & & & \\ 0 & 0 & 0 & 0 & . & 0 & P_{nn} \end{pmatrix}^{-1} = \begin{pmatrix} \begin{pmatrix} P_{22} & P_{23} \\ P_{32} & P_{33} \end{pmatrix}^{-1} & 0_{2m} & . & 0_{2m} \\ 0_m & P_{44}^{-1} \; 0_m & . & 0_m \\ & & . & \\ 0_m & & . & 0_m \; P_{nn} \end{pmatrix} \tag{67}$$

Let $Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} = \begin{pmatrix} P_{22} & P_{23} \\ P_{32} & P_{33} \end{pmatrix}^{-1}$

And let

$$\triangle_i = \begin{cases} (Q_{11} + Q_{12})^{-1} & if \;\; i = 2 \\ (Q_{21} + Q_{22})^{-1} & if \;\; i = 3 \\ P_i & Otherwise \end{cases}$$

Using the same development as that performed in the proof of Proposition 3, we have

$$\Gamma = I_p + (P_1 \; P_1 \;.\; P_1) \begin{pmatrix} \begin{pmatrix} P_{22} & P_{23} \\ P_{32} & P_{33} \end{pmatrix}^{-1} & 0_{2m} & . & 0_{2m} \\ 0_m & P_{44}^{-1} \; 0_m & . & 0_m \\ & & . & \\ 0_m & & . & 0_m \; P_{nn} \end{pmatrix} \begin{pmatrix} I_p \\ I_p \\ . \\ I_p \end{pmatrix}$$

$$\Gamma = I_p + \Delta_1 \Delta_2^{-1} + \Delta_1 \Delta_3^{-1} + . + \Delta_1 \Delta_n^{-1} = \Delta_1 \sum_{i=1}^{n} \Delta_i^{-1} = \Delta_1 \Sigma$$

Using Woodbury formula and after some manipulations in the same spirit as that carried out in (48-50), we have

$$\left[\begin{pmatrix} P_2 & P_{23} & 0 & 0 & . & 0 \\ P_{32} & P_3 & 0 & 0 & . & 0 \\ 0 & 0 & P_4 & 0 & . & 0 \\ & & & . & & \\ 0 & 0 & 0 & 0 & . & P_n \end{pmatrix} + \begin{pmatrix} P_1 & . & P_1 \\ . & . & . \\ P_1 & . & P_1 \end{pmatrix} \right]^{-1} = \begin{pmatrix} Q_{11} & Q_{21} & 0_m & 0_m & . & 0_m \\ Q_{21} & Q_{22} & 0_m & 0_m & . & 0_m \\ 0_m & 0_m & \Delta_4^{-1} & 0_m & . & 0_m \\ & & & . & & \\ 0_m & 0_m & 0_m & 0_m & . & \Delta_{nn}^{-1} \end{pmatrix}$$

$$- \begin{pmatrix} \Delta_2^{-1} \Sigma^{-1} \Delta_2^{-1} & \Delta_2^{-1} \Sigma^{-1} \Delta_3^{-1} & . & \Delta_2^{-1} \Sigma^{-1} \Delta_n^{-1} \\ \Delta_3^{-1} \Sigma^{-1} \Delta_2^{-1} & \Delta_3^{-1} \Sigma^{-1} \Delta_3^{-1} & . & \Delta_3^{-1} \Sigma^{-1} \Delta_n^{-1} \\ & & . & \\ \Delta_n^{-1} \Sigma^{-1} \Delta_2^{-1} & \Delta_n^{-1} \Sigma^{-1} \Delta_3^{-1} & . & \Delta_n^{-1} \Sigma^{-1} \Delta_n^{-1} \end{pmatrix}$$

Substituting in (63) yields after some manipulations, using $\Sigma = \sum_{i=2}^{n} \Delta_i^{-1}$

$$\begin{aligned} T_{xx}^{-1} = P_{k/k} = \Delta_1 - \Delta_1 \left(\sum_{i=2}^{n} \Delta_i^{-1} \right) \Delta_1 \\ + \Delta_1 \left[\left(\sum_{i=2}^{n} \Delta_i^{-1} \right) \Sigma^{-1} \left(\sum_{i=2}^{n} \Delta_i^{-1} \right) \right] \Delta_1 = \Sigma^{-1} \end{aligned} \tag{68}$$

Similarly,

$$T_{xx}^{-1}T_{xz_i} = P_{xx/Z}T_{xx/z_i} =$$
$$-\Delta_1\Delta_i + \Delta_1 \underbrace{\left[\sum_{i=2}^{n}\Delta_i^{-1}\right]}_{=\Sigma^{-1}} \Sigma^{-1}\Delta_i^{-1} = -\Sigma^{-1}\Delta_i^{-1} \tag{69}$$

Substituting in (64) and using similar developments as in (57) yields

$$X_{k/k} = X_{k/k,1} - \sum_{i=2}^{n}(-\Sigma^{-1}\Delta_i^{-1})(X_{k/k,i} - X_{k/k,1}) = \Sigma^{-1}\sum_{i=1}^{n}\Delta_i^{-1}X_{k/k,i} \tag{70}$$

4.3 Discussion

Clearly the above results have been established assuming that only tracks 2 and 3 were correlated but can be extended to any two tracks among the n tracks. However cautious should be made when dealing with the prior information. Indeed, given track one was used as a prior, two cases should be distinguished:
Case 1: track i and j are correlated where $i \neq 1$ and $j \neq 1$, the results pointed out earlier are valid under the following refinements (assuming $j > i$)

$$Q = \begin{pmatrix} Q_{11} & Q_{12} & . & Q_{1l} \\ Q_{21} & Q_{22} & . & Q_{2l} \\ & & . & \\ Q_{l1} & Q_{l2} & . & Q_{ll} \end{pmatrix} = \begin{pmatrix} P_i & 0 & . & 0 & P_{ij} \\ 0 & P_{i+1} & . & 0 & 0 \\ & & . & & \\ P_{ij} & 0 & . & 0 & P_j \end{pmatrix}^{-1} \quad , \; l = j - i \tag{71}$$

$$\Delta_k = \begin{cases} (Q_{k1} + Q_{k2} + . + Q_{kl})^{-1} \;, & if\, k \in [i,j] \\ P_k & otherwise \end{cases} \tag{72}$$

Under the above changes, expressions (68) and (70) are still valid for the calculus of the global state and variance-covariance matrix of the fusion rule.
Case 2: track 1 and track j are correlated In this case, the development carried for the determination of the block inversion matrix is no longer held as the block diagonal cannot be straightforwardly constituted. Indeed, the counterpart of the inverse matrix in (69) will be

$$Q = \left[\begin{pmatrix} P_2 & 0 & . & P_{1j} & 0 & . & 0 \\ 0 & P_3 & 0 & . & P_{1j} & . & 0 \\ & & & . & & & \\ -P_{j1} & -P_{j1} & . & -P_1 & -P_{j1} & . & -P_{j1} \\ & & & . & & & \\ 0 & 0 & . & -P_{j1} & 0 & . & P_n \end{pmatrix} + \begin{pmatrix} P_1 & . & P_1 \\ . & . & . \\ P_1 & . & P_1 \end{pmatrix}\right] = U + \begin{pmatrix} P_1 & . & P_1 \\ . & . & . \\ P_1 & . & P_1 \end{pmatrix} \tag{73}$$

In order to apply Woodbury formula, one would require to determine the inverse of the first matrix in the above expression.
For this purpose, a simple approach consists of use of Gauss elimination method. After some manipulations, the detail is omitted here, by taking

$$\zeta = P_j - P_{1j} - P_{j1} - \sum_{i=2\,,\,i\neq j}^{n} P_{1j}P_i^{-1}P_{j1} \tag{74}$$

The final result is given by:

$$U^{-1} = \left\{ \begin{array}{cccc} (I + P_2^{-1} P_{j1} \zeta^{-1} P_{j1}) P_2^{-1} & P_3 \zeta^{-1} P_{j1} P_2^{-1} & . & \zeta^{-1} P_{j1} P_2^{-1} \\ P_2^{-1} P_{j1} \zeta^{-1} P_{j1} P_3^{-1} & (I + P_3^{-1} P_{j1} \zeta^{-1} P_{j1}) P_3^{-1} & . & \zeta^{-1} P_{j1} P_3^{-1} \\ & & . & \\ P_2^{-1} P_{j1} \zeta^{-1} & P_3^{-1} P_{j1} \zeta^{-1} & . & \zeta^{-1} \\ & & . & \\ P_2^{-1} P_{j1} \zeta^{-1} P_{j1} P_n^{-1} & P_3^{-1} P_{j1} \zeta^{-1} P_{j1} P_n^{-1} & . & \zeta^{-1} P_{j1} P_n^{-1} \end{array} \right.$$

$$\left. \begin{array}{ccc} P_{j+1} \zeta^{-1} P_{j1} P_2^{-1} & . & P_n \zeta^{-1} P_{j1} P_2^{-1} \\ P_{j+1} \zeta^{-1} P_{j1} P_3^{-1} & . & P_n \zeta^{-1} P_{j1} P_3^{-1} \\ & . & \\ P_{j+1}^{-1} P_{j1} \zeta^{-1} & . & P_n^{-1} P_{j1} \zeta^{-1} \\ & . & \\ P_{j+1}^{-1} P_{j1} \zeta^{-1} P_{j1} P_n^{-1} & . & (I + P_n^{-1} P_{j1} \zeta^{-1} P_{j1}) P_n^{-1} \end{array} \right\}$$

Now using Woodbury formulae, the counterpart of (45) will by

$$\Gamma = I_p + (\, P_1 \;\; P_1 \;\; . \;\; P_1 \,)\, U^{-1} \begin{pmatrix} I_p \\ I_p \\ . \\ I_p \end{pmatrix}$$

$$\Gamma = P_1 \left[I + \sum_{i=2,\, i \neq j} P_i^{-1} P_{j1} \zeta^{-1} P_{1j} \sum_{i=2,\, i \neq j} P_i^{-1} + \sum_{i=2,\, i \neq j} P_i^{-1} \right.$$
$$\left. + \sum_{i=2,\, i \neq j} P_i^{-1} P_{j1} \zeta^{-1} + \sum_{i=2,\, i \neq j} \zeta^{-1} P_{1j} P_i^{-1} + \zeta^{-1} \right]$$

i.e.,

$$\Gamma = P_1 \left[\left(P_{j1}^{-1} + \sum_{i=2,\, i \neq j} P_i^{-1} \right) P_{j1} \zeta^{-1} P_{1j} \left(\sum_{i=2,\, i \neq j} P_i^{-1} + P_{1j}^{-1} \right) + \sum_{i=2,\, i \neq j} P_i^{-1} + I \right] = P_1 . \Lambda \tag{75}$$

Similarly to (50), we have

$$U^{-1} \begin{pmatrix} \Lambda^{-1} & \Lambda^{-1} & . & \Lambda^{-1} \\ & & . & \\ \Lambda^{-1} & \Lambda^{-1} & . & \Lambda^{-1} \end{pmatrix} U^{-1} \begin{pmatrix} M_2 \Lambda^{-1} N_2 & M_2 \Lambda^{-1} N_3 & . & M_2 \Lambda^{-1} N_n \\ & & . & \\ M_n \Lambda^{-1} N_2 & M_n \Lambda^{-1} N_3 & . & M_n \Lambda^{-1} N_n \end{pmatrix} \tag{76}$$

With

$$M_i = \left\{ \begin{array}{ll} P_{j1}^{-1} \left(I + P_{j1}^{-1} \left(P_{j1}^{-1} + \sum_{i=2\; i \neq j}^{n} P_k^{-1} P_{j1} \zeta^{-1} P_{j1} \right) \right) P_i^{-1} \;\; , & \begin{array}{c} if \;\; i = \; 2, \, n \\ and \; i \neq j \end{array} \\ \left(P_{j1}^{-1} + \sum_{i=2\; i \neq j}^{n} P_k^{-1} \right) P_{j1} \zeta^{-1} \;\; , \;\; if \;\; i \; = \; j & \end{array} \right. \tag{77}$$

$$N_i = \left\{ \begin{array}{ll} \left(I + P_{j1}^{-1} \zeta^{-1} P_{1j} \left(P_{1j}^{-1} + \sum_{i=2\; i \neq j}^{n} P_k^{-1} \right) \right) & , \;\; if \;\; i \, = \, 2, \, n \; and \; i \neq j \\ \zeta^{-1} P_{1j} \left(P_{1j}^{-1} + \sum_{i=2\; i \neq j}^{n} P_k^{-1} \right) & , \;\; if \;\; i \; = \; j \end{array} \right. \tag{78}$$

This yields after some manipulations:

$$P_{k/k} = T_{xx}^{-1} = P_1 - F_1 - F_2 \tag{79}$$

$$T_{xx}^{-1}T_{xz} = -\begin{pmatrix} P_1 \\ P_1 \\ . \\ P_1 - P_{1j} \\ P_1 \\ . \\ P_1 \end{pmatrix}^T \left[U^{-1} - \begin{pmatrix} M_2\Lambda^{-1}N_2 & M_2\Lambda^{-1}N_3 & . & M_2\Lambda^{-1}N_n \\ M_3\Lambda^{-1}N_2 & M_3\Lambda^{-1}N_3 & . & M_3\Lambda^{-1}N_n \\ & & . & \\ M_n\Lambda^{-1}N_2 & M_n\Lambda^{-1}N_3 & . & M_n\Lambda^{-1}N_n \end{pmatrix} \right] \tag{80}$$

Or for the i^{th} component:

$$\begin{aligned} T_{xx}^{-1}T_{xz_i} = -P_1 &\left[P_i^{-1} + P_i^{-1}P_{j1}\zeta^{-1}P_{1j}\left(\sum_{i=2\ i\neq j}^{n} P_k^{-1} + I\right) - \sum_{i=2}^{n} M_k\Lambda^{-1}M_i \right] \\ &-P_{1j}\left(P_i^{-1}P_{j1}\zeta^{-1} - M_j\Lambda^{-1}N_i\right) \end{aligned} \tag{81}$$

with

$$\begin{aligned} F_1 = P_1 &\left(P_{j1}^{-1}\sum_{i=2\ i\neq j}^{n} P_i^{-1}\right) P_{j1}\zeta^{-1}P_{1j}\left(P_{1j}^{-1}\sum_{i=2\ i\neq j}^{n} P_i^{-1}\right) P_1 + P_1 \sum_{i=2\ i\neq j}^{n} P_i^{-1}P_1 \\ -P_{1j} &\sum_{i=2\ i\neq j}^{n} P_i^{-1}P_{1j}\zeta^{-1}P_1 - P_1\zeta^{-1}\left(I + P_{1j}\sum_{i=2\ i\neq j}^{n} P_i\right) P_{j1} + P_{1j}\zeta^{-1}P_{j1} \end{aligned} \tag{82}$$

$$\begin{aligned} F_2 = P_1 \sum_{i=2\ i\neq j}^{n} M_i\Lambda^{-1} \sum_{i=2\ i\neq j}^{n} N_iP_1 - P_{1j}M_k\Lambda^{-1}\left(\sum_{i=2\ i\neq j}^{n} N_iP_1 - P_{j1}\right) \\ -P_1M_k\Lambda^{-1}P_{j1} \end{aligned} \tag{83}$$

Therefore,

$$X_{k/k} = X_{k/k,1} - \sum_{i=2}^{n} T_{xx}^{-1}T_{xz_i}\left(X_{k/k,i} - X_{k/k,1}\right) \tag{84}$$

As it can be seen from expressions (79-84), the case where the correlation occurs between the prior yields highly coupled outcomes of Pi as compared to the first scenario where the correlation does not involve the prior track. Intuitively, in the absence of the symmetry feature, this is mainly intuitive as the correlated prior would ultimately affect all other tracks.

4.4 Case of weak correlation

Another interesting case worth considering is the situation in which the correlation among any track is quite weak. In other words, the diagonal elements any joint variance-covariance matrix are dominant with respect to off-diagonal elements. More formally, it holds that
$E\left[(X - X_i)(X - X_j)^T\right] \ll E\left[(X - X_i)(X - X_i)^T\right]$ and
$E\left[(X - X_i)(X - X_j)^T\right] \ll E\left[(X - X_j)(X - X_j)^T\right]$ for all i, j.
Using the assumptions and notation of Section 4, and denoting the matrix in

expression (63) by QQ; that is,

$$QQ = \begin{pmatrix} P_{22} - P_{12} - P_{21} & . & P_{2n} - P_{1n} - P_{21} \\ P_{32} - P_{12} - P_{31} & . & P_{3n} - P_{1n} - P_{31} \\ & . & \\ P_{n2} - P_{12} - P_{n1} & . & P_{nn} - P_{1n} - P_{n1} \end{pmatrix} + \begin{pmatrix} P_1 & . & P_1 \\ . & . & . \\ P_1 & . & P_1 \end{pmatrix} = Q_1 Q_{P_1}$$

Again the hint consists of how to find appropriate inverse for matrix $Q1$. For this purpose, one uses the following Lemma [Dahlquist and Bjorck, 1974].

Lemma 2 [Neumann Series]
If P is a square matrix and $|P| < 1$,

$$(I - P)^{-1} \approx I + P + P^2 + . + P^n \tag{85}$$

Consequently, rewriting the matrix Q_1 as the sum of a diagonal matrix (Q_D) and another matrix containing the off-diagonal elements of Q_1 and zero-diagonal, denoted Δ, *i.e.*, $Q_1 = Q_D + \Delta$. Therefore applying Lemma 2 leads to

$$Q_1^{-1} = Q_D^{-1}(I - \Delta Q_D^{-1} + \Delta Q_D^{-1} \Delta Q_D^{-1} + ...) \tag{86}$$

Consequently, if one stops at the second term of the Neumann expansion, one has the approximation

$$Q_1^{-1} = Q_D^{-1}(I - \Delta Q_D^{-1}) = Q_D^{-1} - Q_D^{-1} Q_D^{-1} \tag{87}$$

While if one restricts to the first term of the series expansion, one gets $Q_1^{-1} = Q_D^{-1}$, *i.e.*, the inverse coincides to the inverse of the diagonal matrix of Q_1. Propositions 5 and 6 below are provided without proof because of its similarity with previous ones.

Proposition 5

Using the first order approximation of Neumann series and assuming the first track $(X_{k|k,1}, P_{k|k,1})$ as the prior, the weakly correlated track yields

$$\begin{aligned} P_{XX|Z} &= P_1 - \sum_{i=2}^{n} (P_1 - P_{1i})(P_i - P_{i1} - P_{1i})^{-1} \\ &\times \left(\zeta^{-1} \sum_{j=2}^{n} (P_j - P_{j1} - P_{1j})^{-1}(P_1 - P_{j1}) - (P_1 - P_{i1}) \right) \end{aligned} \tag{88}$$

$$\begin{aligned} T_{xx}^{-1} T_{xz_i} &= -(P_1 - P_{1i})(P_i - P_{i1} - P_{1i})^{-1} \\ &+ \sum_{j=2}^{n} (P_1 - P_{1j})(P_j - P_{j1} - P_{1j})^{-1} \zeta^{-1} (P_i - P_{i1} - P_{1i})^{-1} \end{aligned} \tag{89}$$

With

$$\zeta = \sum_{j=1}^{n} (P_j - P_{j1} - P_{1j})^{-1} \tag{90}$$

Proposition 6

If second order Neumann approximation series was used and assuming the first track $(X_{k/k,1}, P_{k/k,1})$ as the prior, the weakly correlated track yields

$$P_{XX|Z} = P_1 - \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}(P_1 - P_{1,i+1})\left(\Phi_{ij} - \sum_{k=1}^{n-1}\Phi_{ik}\Upsilon^{-1}\sum_{k=1}^{n-1}\Phi_{kj}\right)(P_1 - P_{j+1,1}) \quad (91)$$

$$T_{xx}^{-1}T_{xz_i} = -\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}(P_1 - P_{1,i+1})\left(\Phi_{ij} - \sum_{k=1}^{n-1}\Phi_{ik}\Upsilon^{-1}\sum_{k=1}^{n-1}\Phi_{kj}\right) \quad (92)$$

With

$$\Upsilon = P_1^{-1} + \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\Phi_{ij} \quad (93)$$

$$\Theta_{ij} = \begin{cases} (P_{i+1} + P_{1,i+1} + P_{i+1,1})^{-1} & i = j \\ -(P_{i+1} + P_{1,i+1} + P_{i+1,1})^{-1} \times (P_{i+1,j+1} + P_{1,j+1} + P_{i+1,1}) & \\ \times(P_{j+1} + P_{1,j+1} + P_{j+1,1})^{-1} & otherwise \end{cases} \quad (94)$$

Note that the expression of $X_{k/k}$ is omitted in Proposition 5 and Proposition 6 because of its similarity to the general expression (64), so only parameters $T_{xx}^{-1}T_{xz_i}$ involved in $X_{k/k}$ expression is detailed.

5 Conclusion

This paper reviewed some of the fusion architectures used in literature of stochastic estimation theory. Especially, the augmented measurement vector based fusion, sequential and data compression architecture have been reviewed. Next the convex combination put forward by Bar-Shalom and Campo, where some track were used as a prior and others as measurements in the static linear estimation equation, has been investigated and its extension to several sources has been investigated. The paper has also investigated the effect of the correlation among the tracks extending the BarShalom's and Campos's-based approach. Especially, two types of correlations have been investigated. The former consists on the existence of only two-out of the total n tracks, which are correlated. The theoretical results show that the outcome can be straightforwardly interchangeable with respect to track annotation as far as the track used as the prior is not concerned. Otherwise, more complex analytical expressions are entailed. The second situation consists of the presence of only weak correlation. In this course, the use of Neumann Series approximation has been employed to establish results in case of first and second approximation. Finally, some links to distributed and hierarchical fusion architecture pointed out by Chong has been established. So far this study shown the ultimate importance of the prior track employed in the fusion architecture. Indeed, except the case of zero-correlation or specific distribution of the correlation, the final outcome is ultimately influenced by the choice of the track used as a prior. Intuitively, this order-dependence can be avoided by simultaneous consideration of all tuples at the cost of increased complexity.

References

[Bar-Shalom, 1981] Bar-Shalom, Y. (1981). On the track-to-track correlation problem. IEEE Transactions on Automatic Control, 26(2), 571-572.

[Bar-Shalom and Campo, 1886] Bar-Shalom, Y., and Campo, L. (1986). The effect of the common process noise on the two-sensor fused-track covariance. IEEE Transactions on Aerospace and Electronic Systems, 22(6), 803-805.

[Bar-Shalom and Fortmann, 1988] Bar-Shalom, Y., and Fortmann, T. E. (1988). Tracking and Data Association. New York: Academic Press.

[Y. Bar-Shalom and X. R. Li, 1995] Bar-Shalom, Y. and Li, X. R. (1995). Multitarget-Multisensor Tracking: Principles and Techniques. Storrs, CT: YBS Publishing.

[Cavalaro, 2007] Cavalaro A. (2007), Multi-Sensor Object detection and tracking, Special issue of Signal, Image and Video Processing, 1(2).

[Chong 1979] Chong, C. Y. (1979). Hierarchical Estimation, in Proc. MITIONR Workshop on C3, Monterey, CA.

[Chong et al., 1990] Chong, C.Y., Mori S., and Chang, K. C. (1990). Distributed Multitarget Multisensor Tracking, in: Multitarget Multisensor Tracking: Advanced Applications, Y. Bar Shalom, ed., Norwood, MA Artech House, 247-295.

[Chang et al., 1997] Chang, K. C., Saha R. K., and Bar-Shalom Y. (1997), On Optimal Track-to-Track Fusion, IEEE Transactions on Aerospace, Electronic Systems, 33(4), 1271-1276.

[Dahlquist and Bjorck, 1974] Dahlquist, G. and Bjorck A. (1974), Numerical methods, Prentice Hall, Englewood-Cliffs, NJ.

[Duncan and Sameer, 2006] Duncan, S and Sameer, S. (2006), Approaches to Multisensor data fusion in target tracking: A survey. IEEE Transactions on Knowledge and Data Engineering, 18(12), 1696-1710

[Fong, 2008] Fong, LW. (2008), Multi-sensor track-to-track fusion via linear minimum variance sense estimators, Asian Journal of Control, 10(3) p. 277-290, 2008.

[Halmos 1958] Halmos, P. R. (1958), Finite Dimensional Vector Space, Van Nostrand, Princeton, 1958.

[Hashemipour et al., 1988] Hashemipour, H. R., Roy, S. and Laub, A.J. (1988), Decentralized Structures for Parallel Kalman Filtering, IEEE Transactions on Automatic and Control, 33(1), pp. 88-93.

[Luo and Kay, 1995)] Luo, L.C., and Kay, M. J. (1995). Multisensor Integration and Fusion for Intelligent Machines and Systems, Ablex Publishing Corporation,U.S.; Reissue edition

[Kaplan et al., 2008] Kaplan, LM., Bar-Shalom, Y. and Blair, WD. (2008). Assignment costs for multiple sensor track-to-track association, IEEE Transactions on Aerospace and Electronic Systems 44 (2), 655-677.

[Singer and Kanyuck, 1971] Singer R. A. and Kanyuck A. J. (1971). Computer Control of Multiple Site Track Correlation . Automatica, 7, 455-462.

[Li et al., 2003] Li, X. R., Zhu, Y. M., and Han C. Z. (2003). Unified Optimal Linear Estimation Fusion-Part I: Unified Models and Fusion Rules, IEEE Transactions on Information Theory, 49(3), 2192-2208

[Willner et al., 1978] Willner D., C.B. Chang, and Dunn K. P. (1978). Kalman Filter Algorithms for a Multi-Sensor Systems. In: Proceedings of the IEEE Conference on Decision and Control, 570-574.