Journal of Universal Computer Science, vol. 15, no. 6 (2009), 1301-1316
submitted: 3/11/08, accepted: 4/2/09, appeared: 28/3/09 © J.UCS

Topological Complexity of Blowup Problems

Robert Rettinger
(University of Hagen, Germany
Robert.Rettinger@FernUni-Hagen.de)

Klaus Weihrauch
(University of Hagen, Germany
Klaus.Weihrauch@FernUni-Hagen.de)

Ning Zhong
(University of Cincinnati, USA
Ning.Zhong@uc.edu)

Abstract: Consider the initial value problem of the first-order ordinary differential equation

$$\frac{d}{dt}x(t) = f(t, x(t)), \quad x(t_0) = x_0$$

where the locally Lipschitz continuous function $f : \mathbb{R}^{l+1} \to \mathbb{R}^l$ with open domain and the initial datum $(t_0, x_0) \in \mathbb{R}^{l+1}$ are given. It is shown that the solution operator producing the maximal "time" interval of existence and the solution on it is computable. Furthermore, the topological complexity of the blowup problem is studied for functions f defined on the whole space. For each such function f the set Z of initial conditions (t_0, x_0) for which the positive solution does not blow up in finite time is a G_δ-set. There is even a computable operator determining Z from f. For $l \geq 2$ this upper G_δ-complexity bound is sharp. For $l = 1$ the blowup problem is simpler.

Key Words: Type-2 theory, differential equation, blowup
Category: F.2.1

1 Indroduction

Consider an initial value problem of obtaining solutions $x(t)$ to the first-order ODE (*ordinary differential equation*)

$$\begin{cases} \frac{d}{dt}x(t) = f(t, x(t)), \; t \in \mathbb{R}, \; (t, x) \in E \subseteq \mathbb{R}^{l+1} \\ x(t_0) = x_0 \end{cases} \tag{1}$$

where the initial datum $(t_0, x_0) \in E \subseteq \mathbb{R}^{l+1}$ and the (generally non-linear) function $f : E \to \mathbb{R}^l$ are given. In this initial-value problem, x is usually referred to as the space variable and t the time variable. If f is continuous on E and locally Lipschitz continuous in space variable x, then the problem (1) has a unique solution on a maximal time interval (α, β). This result is commonly referred to as Picard-Lindelöf existence and uniqueness theorem. Various versions of the computable Picard-Lindelöf theorem have been studied by several authors, including Aberth [Aberth, 1970, Aberth, 1971], Bishop and Bridges [Bishop, Bridges, 1985], Graça, Zhong and Buescu [Graça, Zhong, Buescu, 2006], Ko [Ko, 1991], Pour-El and Richards [Pour-El, Richards, 1989]. In this paper, we present a fully uniform version of the Picard-Lindelöf theorem.

The Picard-Lindelöf theorem gives a very satisfactory local theory for the existence and uniqueness of solutions to the ODE (1) for locally Lipschitz continuous f. However, there remains a difficult issue: whether the corresponding maximal interval of existence (α, β) is bounded or not for any given initial datum. When β or/and α is finite, the solution $x(t)$ will blow up in finite time in the sense that $||x(t)||$ approaches to infinity as $t \to \beta^-$ or $t \to \alpha^+$. In general, it is difficult to predict whether or not a solution will blow up for a given initial datum, because it often requires extra knowledge on some quantitative estimates and asymptotics of the solution over long period of time, such as whether the solution satisfies a certain "coercive" conservation law. Indeed, it is shown recently in [Graça, Zhong, Buescu, 2006] and [Graça, 2007] that the blowup problem cannot be solved by any algorithm.

In this paper, we study the topological complexity of the blowup problem for functions f defined on the whole space. We shall use the notation CBU_f to denote the set of all initial data at which the solutions to the initial-value problem (1) are global (no blowup). The complement of CBU_f, denoted as BU_f, is then the set of all initial data for which the solutions blow up. We show that the set CBU_f is a G_δ set, i.e a countable intersection of open sets, and there is an algorithm that computes CBU_f from f. Thus the blowup set BU_f has F_σ as an upper complexity bound. Moreover, for every computable G_δ-set G of $\mathbb{R}^{l-1}$ with $l \geq 2$, we show that there exists a computable and effectively locally Lipschitz function $f : \mathbb{R}^l \to \mathbb{R}^l$ such that the solution to the problem "$x'(t) = f(x(t))$, $x(0) = (x_0, 0)$" is global if and only if $x_0 \in G$. In other words, the G_δ-complexity for CBU_f is sharp. It follows that the F_σ-complexity is sharp for the blowup sets.

The paper is organized as follows. Section 2 introduces necessary concepts and results from computable analysis. Section 3 presents a fully uniform version of the computable Picard-Lindelöf theorem. Section 4 contains the theorems on the complexity of the blowup problem.

2 Preliminaries

For studying computability in analysis, in this article we use the representation approach also called type-2 theory of effectivity (TTE) [Weihrauch, 2000]. In this theory computability on finite or infinite sequences, Σ^* or Σ^ω, respectively, over a finite alphabet Σ is defined explicitly by type-2 machines, which are Turing machines with finite or infinite one-way input and output tapes. The elements of Σ^* or Σ^ω are used as "names" of natural, rational or real numbers, of open sets, continuous functions and so on. A *representation* of a set M is a surjective partial function $\delta :\subseteq Y \to M$ ($Y \in \{\Sigma^*, \Sigma^\omega\}$), where p is called a δ-name or a name of $x \in M$ if $\delta(p) = x$. In [Weihrauch, 2000] representations $\delta :\subseteq \Sigma^* \to M$ are called *notations*. A function on represented spaces is computable, if it can be realized by a computable function on the names.

We also use the more general *multi-representations* $\delta : Y \rightrightarrows M$, where $p \in Y$ is considered as a name of each $x \in \delta(p)$ and multi-functions $f : M \rightrightarrows M'$ on represented sets, where $y \in f(x)$ can be interpreted as "y is an acceptable result on input x". For multi-representations $\gamma : Y \rightrightarrows M$ and $\gamma' : Y' \rightrightarrows M'$, a function $h :\subseteq Y \to Y'$ realizes a multi-function $f : M \rightrightarrows M'$, if $h(p)$ is a γ'-name of some $y \in f(x)$ whenever $p \in \Sigma^\omega$ is a γ-name of x (see Figure 1). We call the multi-function f (γ, γ')-computable (-continuous), if it has a computable (continuous) realization [Weihrauch, 2005, Weihrauch, 2008].

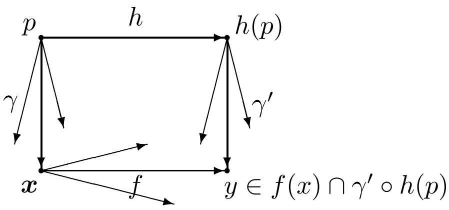

Figure 1: $h(p)$ is a name of some $y \in f(x)$, if p is a name of $x \in \mathrm{dom}(f)$.

The extension to multi-functions $f : M_1 \times \ldots \times M_k \rightrightarrows M'$ is straightforward. For multi-representations $\gamma : Y \rightrightarrows M$ and $\gamma' : Y' \rightrightarrows M'$, $\gamma \leq \gamma'$ (γ is reducible to γ'), if there is a computable function $h :\subseteq Y \to Y'$ such that $\gamma(p) \subseteq \gamma' \circ h(p)$ for all $p \in \mathrm{dom}(\gamma)$. The representations are equivalent, if they are reducible to each other. Equivalent representations induce the same computability and relative continuity on the represented set.

If multi-functions on represented sets have realizations, then their composition is realized by the composition of the realizations. Therefore, the computable multi-functions on represented sets are closed under composition. Much more generally, the computable multi-functions on represented sets are closed under flowchart programming with indirect addressing [Weihrauch, 2005, Weihrauch, 2008]. We will apply this result repeatedly, which allows convenient informal constructions of new computable functions on multi-represented sets from given ones.

Let $\gamma : Y \rightrightarrows M$ and $\gamma' : Y' \rightrightarrows M'$ be multi-representations. By means of computable standard pairing and tupling functions on Σ^* and Σ^ω, all of which we denote by $\langle\ \rangle$ [Weihrauch, 2000], multi-representations of products can be defined: $[\gamma, \gamma']\langle y, y'\rangle := \gamma(y) \times \gamma'(y')$ and $\gamma^\omega\langle y_0, y_1, \ldots\rangle := \gamma(y_0) \times \gamma(y_1) \times \ldots$.

In [Weihrauch, 2008] a multi-representation $[\gamma \rightrightarrows \gamma']$ of the (γ, γ')-continuous multi-functions $f : M \rightrightarrows M'$ is defined by $f \in [\gamma \rightrightarrows \gamma'](p)$, if η_p realizes f ($\eta_p = h$ in Figure 1). Here η is the canonical representation of the continuous functions $h :\subseteq Y \to Y'$ with open domain (for $Y' = \Sigma^*$) or G_δ-domain (for $Y' = \Sigma^\omega$) [Weihrauch, 2000]. Their restrictions to the partial functions and total functions are called $[\gamma \to_p \gamma']$ and $[\gamma \to \gamma']$, respectively.

Let $\gamma_0 :\subseteq Y_0 \rightrightarrows M_0$ be another multi-representation. For a multi-function $f : M_0 \times M \rightrightarrows M'$ define $Tf(x)(y) := f(x, y)$. Then T is $([[\gamma_0, \gamma] \rightrightarrows \gamma'], [\gamma_0 \to [\gamma \rightrightarrows \gamma']])$-computable and its inverse is $([\gamma_0 \to [\gamma \rightrightarrows \gamma'], [[\gamma_0, \gamma] \rightrightarrows \gamma'])$-computable. As corollaries,

$$f \text{ is } (\gamma_0, \gamma, \gamma')\text{-computable} \iff Tf \text{ is } (\gamma_0, [\gamma \rightrightarrows \gamma'])\text{-computable}, \tag{2}$$

and for every multi-representation δ of multi-functions $h : M \rightrightarrows M'$, the evaluation $(h, x) \mapsto h(x)$ is $(\delta, \gamma, \gamma')$-computable iff $\delta \le [\gamma \rightrightarrows \gamma']$ [Weihrauch, 2008] (cf. the special case for single-valued representations and total functions [Weihrauch, 2000, Theorem 3.3.15]).

Let $\nu_\mathbb{N}$ and $\nu_\mathbb{Q}$ be standard notations of the natural numbers and the rational numbers, respectively. For single-valued representations $\gamma :\subseteq Y \to M$, $\gamma^\omega \equiv [\nu_\mathbb{N} \to \gamma]$ (representation of sequences on M).

On the space $\mathbb{R}^n$ we use the maximum norm

$$\|(x_1, \ldots, x_n)\| := \max\{|x_1|, \ldots, |x_n|\}.$$

For $x \in \mathbb{R}^n$ and $r > 0$ let $B(x, r) := \{y \in \mathbb{R}^n \mid \|x - y\| < r\}$ be the open ball or cube with center x and radius r. Let I^n be a natural notation of the set of all rational open balls $\mathrm{RB}^n := \{B(x, r) \mid x \in \mathbb{Q}^n,\ \ r \in \mathbb{Q},\ \ r > 0\}$ in $\mathbb{R}^n$. Let $\rho^n :\subseteq \Sigma^\omega \to \mathbb{R}^n$ be the representation defined by $\rho^n(p) = x$, iff p is a list of *all* open balls $J \in \mathrm{RB}^n$ (encoded by I^n) such that $x \in J$. Then $\rho := \rho^1$ is equivalent to the Cauchy representation of the real numbers [Weihrauch, 2000]. For the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$, the "lower representation" $\overline{\rho}_< : \Sigma^\omega \to \mathbb{R}$ and the "upper representation" $\overline{\rho}_> : \Sigma^\omega \to \mathbb{R}$ are defined by $\overline{\rho}_<(p) = \sup\{r \in \mathbb{Q} \mid r \text{ is listed by } p\}$. and $\overline{\rho}_<(p) = \inf\{r \in \mathbb{Q} \mid r \text{ is listed by } p\}$.

For the set $O(\mathbb{R}^n)$ of open subsets and the set $G_\delta(\mathbb{R}^n)$ of the G_δ-subsets (the countable intersections of open subsets) of $\mathbb{R}^n$ we use the representations θ^n and δ^n_G defined by $\theta(p) = U$ iff p is a list $J_0, J_1, \ldots$ of open balls from RB^n (encoded by I^n) such that $U = \bigcup_i J_i$ and $\delta^n_G\langle p_0, p_1, \ldots\rangle = \bigcap_j \theta^n(p_j)$ [Weihrauch, 2000, Weihrauch, 1993]. The θ^n-computable sets are called r.e.-open.

For the space $\mathrm{CP}(\mathbb{R}^m, \mathbb{R}^n)$ of the partial (topologically) continuous functions $f :\subseteq \mathbb{R}^m \to \mathbb{R}^n$, we use the multi-representation $\delta_{m,n}$ defined as follows: $f \in \delta_{m,n}(p)$ iff p is (encodes) a list $(J_i, K_i)_{i\in\mathbb{N}}$, ($J_i \in \mathrm{RB}^m$, $K_i \in \mathrm{RB}^n$), such that

$$f^{-1}L = \mathrm{dom}(f) \cap \bigcup\{J_i \mid K_i = L\} \qquad \text{for all } L \in \mathrm{RB}^n\,. \tag{3}$$

This representation is equivalent to $[\rho^m \to_p \rho^n]$ [Grubba, Weihrauch, Xu, 2007]. Therefore by (2), evaluation $(f, x) \mapsto f(x)$ is $(\delta_{m,n}, \rho^m, \rho^n)$-computable.

If the representations of the sets under consideration are fixed, we will simply say "computable" instead of "(γ, δ)-computable" etc.

3 The Solution Operator Is Computable

By the Picard-Lindelöf theorem, unique local solutions of the initial value problem (1) exist. The following version is from [Heuser, 1981] slightly adjusted for our purposes. For $f :\subseteq \mathbb{R}\times\mathbb{R}^l \to \mathbb{R}^l$ and $Z\subseteq\mathrm{dom}(f)$ we will call $M \in \mathbb{R}$ an upper bound of f on Z if $\|f(z)\| \le M$ for all $z \in Z$, and we will call $L \in \mathbb{R}$ a Lipschitz constant of f on Z, if $\|f(t,x) - f(t,y)\| \le L\|x - y\|$ for all $(t,x),(t,y) \in Z$.

Theorem 3.1 *[Picard-Lindelöf] Let $f : \overline{B}((t_0, x_0), r) \to \mathbb{R}^l$, $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^l$, $0 < r \le 1$, be continuous. Let $L > 0$ be a Lipschitz constant and let $M \ge 1$ be an upper bound of f (on $\mathrm{dom}(f)$). Then the initial value problem (1) has a unique solution h on $[t_0 - b; t_0 + b]$ for $b = \min(r/M, 1/(2L))$ (see Figure 2).*

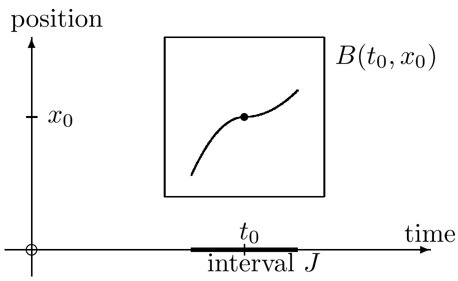

Figure 2: A local solution h of the initial value problem (1).

We outline a classical proof [Heuser, 1981], which already shows a way how to "compute" the local solution. Let $C(J)$ be the Banach space of continuous functions $f : J \to \mathbb{R}^l$, $J := [t_0 - b; t_0 + b]$, with maximum norm $\|\,\|_\infty$. Then $C_0 := \{g \in C(J) \mid \|g(t) - x_0\| \le r$ for all $t \in J\}$ is a closed subset of $C(J)$ and the operator $A : C(J) \to C(J)$, defined by

$$A(g)(t) := x_0 + \int_{t_0}^{t} f(\tau, g(\tau))\, d\tau\,, \tag{4}$$

maps C_0 into itself and is contracting on C_0, that is, $\|A(g_1) - A(g_2)\|_\infty \leq \frac{1}{2}\|g_1 - g_2\|_\infty$ for $g_1, g_2 \in C_0$. By the Banach fixed point theorem the operator A has a unique fixed point, and this function is the local solution $h : [t_0 - b; t_0 + b] \to \mathbb{R}^l$ of our initial value problem [Heuser, 1981]. The sequence $h_0, h_1, \ldots \in C_0$ defined by $h_0(t) := x_0$, $h_{n+1} := A(h_n)$, converges to the fixed point h of the operator A. Since $\|h_1 - h_0\|_\infty \leq r \leq 1$, $\|h_{n+1} - h_n\|_\infty \leq 2^{-n}$, and therefore, $\|h_k - h_n\|_\infty \leq 2^{-n+1}$ for $k > n$ and

$$\|h - h_n\|_\infty = \|h - A^n(g_0)\|_\infty \leq 2^{-n+1}\,. \tag{5}$$

Effectivizing this idea we get a fully uniform computable version of the Picard-Lindelöf theorem. For convenience we consider only positive integer bounds L and M.

Lemma 3.2 *[Computable Picard-Lindelöf] There is a $(\delta_{l+1,l}, \rho, \rho^l, \delta_{1,l})$-computable operator $T : (f, t_0, x_0) \mapsto h$ mapping each continuous function $f :\subseteq \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^l$, each $t_0 \in \mathbb{R}$ and each $x_0 \in \mathbb{R}^l$ to some $h :\subseteq \mathbb{R} \to \mathbb{R}^l$ such that the restriction of h to the interval $[t_0 - b\,; t_0 + b]$ is a local solution of (1), if for some r, $0 < r \leq 1$, and some natural numbers $M, L \geq 1$,*

1. *$\overline{B}((t_0, x_0), r) \subseteq \mathrm{dom}(f)$,*
2. *L is a Lipschitz constant and M is an upper bound of f on $\overline{B}((t_0, x_0), r)$,*
3. *$b = \min(r/M, 1/(2L))$.*

Proof. During the proof we consider the representations $[\rho^m \mapsto_p \rho^n]$ which are equivalent to the $\delta_{m,n}$ such that type conversion (2) can be applied easily. Since composition $(f, g) \mapsto (\tau \mapsto f(\tau, g(\tau)))$ for continuous $f :\subseteq \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^l$ and continuous $g :\subseteq \mathbb{R} \to \mathbb{R}^l$ is computable and since integration $(h, a, b) \mapsto \int_a^b h(\tau)\, d\tau$ for continuous $h :\subseteq \mathbb{R} \to \mathbb{R}$ (with defined value, if $[a; b] \subseteq \mathrm{dom}(h)$) is computable [Weihrauch, 2000], the function

$$(f, t_0, x_0, g, t) \mapsto x_0 + \int_{t_0}^{t} f(\tau, g(\tau))\, d\tau \quad \text{is computable.} \tag{6}$$

Applying type conversion (2) twice we obtain a computable operator $T_1 : (f, t_0, x_0) \mapsto B$ where $B(g)(t) = x_0 + \int_{t_0}^{t} f(\tau, g(\tau))\, d\tau$. For $x \in \mathbb{R}^l$ let $g_x(t) := x$ for all $t \in \mathbb{N}$. For operators D of the functional type of B, the function $(D, x, t, n) \mapsto D^n(g_x)(t) \in \mathbb{R}^l$ (for $t \in \mathbb{R}$) is computable

[Weihrauch, 2005, Weihrauch, 2008], hence the operator $T_2 : (D, x, t) \mapsto (n \mapsto D^n(g_x)(t))$ into the sequences in $\mathbb{R}^l$ is computable. Finally the limit operator $\lim_l : (z_i)_{i\in\mathbb{N}} \mapsto \lim_i z_i$ on $\mathbb{R}^l$ which is defined if $\|z_i - z_j\| \leq 2^{-i+1}$ for $j > i$ is computable [Weihrauch, 2000]. Let $T_3(f, t_0, x_0, t) = \lim_l \circ T_2(T_1(f, t_0, x_0), x_0, t)$. Then T_3 is computable and the operator T obtained from T_3 by type conversion (2), $T(f, t_0, x_0)(t) = T_3(f, t_0, x_0, t)$, is computable.

In the proof of the classical theorem above, the operator A corresponds to $T_1(f, t_0, x_0)$, and $h_n(t)$ corresponds to $A^n(g_{x_0})(t)$. Suppose that 1.-3. above are true. Then for $h := T(f, t_0, x_0)$, $h(t)$ exists for all $t \in [t_0 - b; t_0 + b]$, and the restriction of h to the interval $[t_0 - b; t_0 + b]$ is a local solution of (1). $\square$

In the following we will compute the global solution of the initial value problem (1) for locally Lipschitz bounded continuous functions $f :\subseteq \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^l$ with open domain. Let h_0 be the local solution computed in Lemma 3.2 with initial point (t_0, x_0). Since $t_1 := t_0 + b \in \mathrm{dom}(f)$ we can extend the partial solution h_0 by a partial solution h_1 obtained by Lemma 3.2 with initial point (t_1, x_1) for $x_1 := h_0(t_1)$. This process can be iterated. For each of the points $(t_i, x_i) \in \mathrm{dom}(f)$ we need a neighbourhood ball with Lipschitz constant L_i and upper bound M_i. We consider a representation $\overline{\delta}$ such that a name of a function f contains data for evaluation (a $\delta_{l+1,l}$-name) and information about its open domain and local Lipschitz data. Local upper bounds M can be computed from these data (Lemma 3.4).

Definition 3.3 *Define a representation $\overline{\delta}$ of the locally Lipschitz bounded continuous functions $f :\subseteq \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^l$ with open domain as follows: $f \in \overline{\delta}\langle p, q\rangle$ iff $f \in \delta_{l+1,l}(p)$ and $q \in \Sigma^\omega$ is (a code of) a sequence $((B_0, L_0), (B_1, L_1), \ldots)$, such that $B_i \in \mathrm{RB}^{l+1}$, $L_i \in \mathbb{N}$, $\overline{B}_i \subseteq \mathrm{dom}(f)$ and L_i is a Lipschitz constant of f on $\overline{B}_i$ for all $i \in \mathbb{N}$, and $\mathrm{dom}(f) = \bigcup_{i\in\mathbb{N}} B_i$.*

Obviously, $\overline{\delta} \leq \delta_{l+1,l}$, hence evaluation $(f, z) \mapsto f(z)$ is $(\overline{\delta}, \rho^{l+1}, \rho^l)$-computable. For applying Lemma 3.2 to $(t_0, x_0) \in \mathrm{dom}(f)$ we want to find a radius r' and constants L, M from the input data such that $\overline{B}((t_0, x_0), r') \subseteq \mathrm{dom}(f)$ and L is a Lipschitz constant and M is an upper bound of f on this closed ball. Since the sequence $(B_i, L_i)_i$ is not suitable for this purpose, we introduce another representation $\tilde{\delta}$ that is equivalent to $\overline{\delta}$. Let $B(x, r)/4 := B(x, r/4)$. Suppose, L is a Lipschitz constant and M is an upper bound of f on $\overline{B} \subseteq \mathrm{dom}(f)$. If $(t_0, x_0) \in B/4$ then L is a Lipschitz constant and M is an upper bound of f also on $\overline{B}((t_0, x_0), r/4) \in \mathrm{dom}(f)$. Therefore, a name of the new representation should also contain an upper bound of f on $\overline{B}_i \subseteq \mathrm{dom}(f)$ for each i and should satisfy the stronger condition $\mathrm{dom}(f) = \bigcup_{i\in\mathbb{N}} B_i/4$.

Lemma 3.4 *Define a representation $\tilde{\delta}$ of the locally Lipschitz bounded continuous functions $f :\subseteq \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^l$ with open domain as follows: $f \in \overline{\delta}\langle p, q\rangle$ iff $f \in$*

$\delta_{l+1,l}(p)$ *and* $q \in \Sigma^\omega$ *is (a code of) a sequence* $((C_0, K_0, M_0), (C_1, K_1, M_1), \ldots)$ *of triples such that for all* i*,* $C_i \in \mathrm{RB}^{l+1}$ *has radius* ≤ 1*,* $\overline{C}_i \subseteq \mathrm{dom}(f)$*,* $K_i, M_i \in \mathbb{N} \setminus \{0\}$*, and* K_i *is a Lipschitz constant and* M_i *is an upper bound of* f *on* $\overline{C}_i$*, and such that* $\mathrm{dom}(f) = \bigcup_i C_i/4$*. Then* $\overline{\delta} \equiv \tilde{\delta}$*.*

Proof: Obviously, $\tilde{\delta} \leq \overline{\delta}$.
For proving the other direction we first compute upper bounds N_i of f on the $\overline{B}_i$. We observe that $(f, z) \mapsto \|f(z)\|$ is computable. By [Weihrauch, 2000, Corollary 6.2.5], which holds for partial continuous functions f as well if $K \in \mathrm{dom}(f)$, the exact maximum $\max\{\|f(z)\| \mid z \in \overline{B}\}$ can be computed, hence also an integer upper bound can be computed from f and the compact set $\overline{B}$. Therefore, a sequence of upper bounds N_i can be computed.

Next, we show that in general

$$B(x, r) = \bigcup \big\{ B(y, s) \mid y \in \mathbb{Q}^{l+1},\ \|x - y\| < r,\ s \in \mathbb{Q},\ s \leq 1,\ s < (r - \|x - y\|)/5 \big\}\,. \tag{7}$$

For such (y, s) let $z \in B(y, 4s)$. Then
$\|z - x\| \leq \|z - y\| + \|y - x\| \leq 4(r - \|x - y\|)/5 + \|y - x\| < r$. Therefore,

$$B(y, 4s) \subseteq B(x, r) \quad \text{for all balls} \quad B(y, s) \quad \text{in (7).} \tag{8}$$

Consequently, $B(x, r) \supseteq \bigcup\{B(y, s) \mid \ldots\}$ in (7). For showing "$\subseteq$" let $z \in B(x, r)$. Choose $t \in \mathbb{Q}$ such that $0 < t \leq 1$ and $t < (r - \|x - z\|)/6$, and choose $y \in \mathbb{Q}^{l+1}$ such that $\|z - y\| < t$. Then $\|y - x\| \leq \|y - z\| + \|z - y\| \leq t + \|z - y\| \leq (r - \|x - z\|)/6 + \|z - y\| < r$ and hence $z \in B(y, t)$. For (7) it remains to show $t \leq (r - \|x - y\|)/5$. Since $-\|x - z\| \leq -\|x - y\| + \|z - y\| \leq -\|x - y\| + t$, $t < (r - \|x - z\|)/6 \leq (r - \|x - y\| + t)/6$ and hence $t \leq (r - \|x - y\|)/5$. Therefore (7) is true.

Now suppose $f \in \overline{\delta}\langle p, q\rangle$ such that q encodes the list $(B_0, L_0), (B_1, L_1), \ldots$. As we have shown this list can be extended to a list $(B_0, L_0, N_0), (B_1, L_1, N_1), \ldots$ containing also upper bounds N_i of f on the $\overline{B}_i$. By (7,8), for every ball B_i we can find a list $(B_{ij})_{j \in \mathbb{N}}$ of balls $B_{ij} \in \mathrm{RB}^{l+1}$ $(j \in \mathbb{N})$ such that $B_i = \bigcup_j B_{ij}/4$ and $B_{ij} \subseteq B_i$ and the radius B_{ij} is not greater 1 for all j (take the $B(y, 4s)$).

Therefore, from the list $(B_i, L_i, N_i)_{i \in \mathbb{N}}$ we can compute a list $(C_i, K_i, M_i)_{i \in \mathbb{N}}$ consisting of all such (B_{ij}, L_i, N_i) $(i, j \in \mathbb{N})$. Since $\langle p, q'\rangle$ such that q' is a name of this new list is a $\tilde{\delta}$-name of f, we have shown $\overline{\delta} \leq \tilde{\delta}$. □

In the following lemma, we consider the behaviour of the global solution for for $t \geq t_0$. The case $t \leq t_0$ can be analysed similarly. Let T be the operator from Lemma 3.2 for computing local solutions.

Lemma 3.5 *Let* $f \in \tilde{\delta}\langle p, q\rangle$ *where* q *is a name of the list* $(C_i, K_i, M_i)_{i \in \mathbb{N}}$*. Let* h *be the global solution for the initial condition* (t_0, x_0) *and suppose* $t_0 < \overline{t}$*. Then*

$h(t)$ exists for $t_0 \le t \le \overline{t}$, if, and only if, there are $n \ge 0$ and triples (D_k, L_k, N_k) $(k \le n)$ such that

$$(D_0, L_0, N_0), \ldots, (D_n, L_n, N_n) \in \{(C_i, K_i, M_i) \mid i \in \mathbb{N}\}\,, \tag{9}$$

$$(\forall k \le n)\,(t_k, x_k) \in D_k/4 \quad \text{and} \tag{10}$$

$$\overline{t} < t_{n+1} \tag{11}$$

where the (t_k, x_k), $1 \le k \le n+1$, are determined as follows:

$$B(a_k, r_k) = D_k, \qquad d_k = \min\Big(\frac{r_k}{4N_k}, \frac{1}{2L_k}\Big), \tag{12}$$

$$t_{k+1} = t_k + d_k, \qquad x_{k+1} = T(f, t_k, x_k)(t_{k+1}). \tag{13}$$

Proof: Suppose the right hand side of the equivalence is true. We show by induction that for all $k \le n+1$,

$$h(t) \ \text{ exists for all } \ t_0 \le t \le t_k\,. \tag{14}$$

This is true for $k = 0$. Suppose (14) is true for $k \le n$. Since $(t_k, x_k) \in D_k/4$ by (10), the cube $\overline{B}((t_k, x_k), r_k/4)$ is contained in D_k and therefore, N_k is a bound and L_k is a Lipschitz constant of f on it. By Lemma 3.2, $h(t) = T(f, t_k, x_k)(t)$ exists also for $t_k \le t \le t_k + d_k = t_{k+1}$. This ends the induction. Finally, $h(t)$ exists for $t_0 \le t \le \overline{t}$ since $t_0 \le \overline{t} \le t_{n+1}$.

On the other hand suppose $h(t)$ exists for $t_0 \le t \le \overline{t}$. Since h is continuous, the graph $G := \{(t, h(t)) \mid t_0 \le t \le \overline{t}\}$ is compact. Therefore, there is a finite set $Q \subseteq \{(C_i, K_i, M_i) \mid i \in \mathbb{N}\}$ such that $G \subseteq \bigcup\{B/4 \mid (B, K, M) \in Q$. From this set we select a sequence $(D_k, L_k, N_k)_{0 \le k \le n}$ such that (9-11) holds.

Suppose for some k, $(t_k, x_k) \in G$, hence $x_k = h(t_k)$. Then $(t_k, x_k) \in B/4$ for some $(B, K, M) \in Q$, where $B =: B(a_k, r_k)$ for some a_k and r_k. Then $B((t_k, x_k), r_k/4) \subseteq B$ and so K is a Lipschitz constant and M is a bound of f on $B((t_k, x_k), r_k/4)$ as well. By Lemma 3.2, $h(t)$ exists for $t_k \le t \le t_k + d_k$ where $d_k := \min(r_k/4M, 1/2K)$. Let $(L_k, D_k, N_k) := (B, K, M)$, $t_{k+1} := t_k + d$ and $x_{k+1} := h(t_{k+1})$. Since $(t_0, x_0) \in G$, starting from (t_0, x_0) we can define (L_k, d_k, N_k) and t_k for $k = 0, 1, \ldots$ as long as $t_k \le \overline{t}$. Since the set Q is finite, the d_k have a common non-zero lower bound. Therefore, there is a first n such that $t_{n+1} > \overline{t}$. □

For representations $\delta_i \ :\subseteq \ Y_i \ \to \ M_i$ a set $A \subseteq M_1 \times \ldots \times M_n$ is $(\delta_1, \ldots, \delta_n)$-r.e. (recursively enumerable) iff there is a Type-2 machine, which for any input $(y_1, \ldots, y_n) \in \mathrm{dom}(\delta_1, \ldots, \delta_n)$ halts iff $(\delta_1(y_1), \ldots, \delta_n(y_n)) \in A$ [Weihrauch, 2000]. If the representations are fixed, we shall say "relatively r.e." or simly "r.e.". Let $\gamma : Y \rightrightarrows M$ be a multi-representation and let $\nu :\subseteq \Sigma^* \to N$ be a representation with recursive domain. Let $Q \subseteq M \times N$ be (γ, ν)-r.e.. We state without proofs:

$$\text{the multi-function } x \rightrightarrows y \text{ such that } (x,y) \in Q \text{ is computable,} \tag{15}$$

$$\text{the projection } \{x \in M \mid (\exists y)\,(x,y) \in Q\} \text{ is } \gamma\text{-r.e.}, \tag{16}$$

$$\text{the multi-function } x \rightrightarrows (y_i)_{i\in\mathbb{N}} \text{ such that } (y_i)_{i\in\mathbb{N}} \text{ is a list of all } y \text{ such that } (x,y) \in Q \text{ is } (\gamma, \nu^\omega)\text{-computable.} \tag{17}$$

By applying the above result we are now able to prove the following main result of this section.

Theorem 3.6 *1. The solution operator $S : (f, t_0, x_0) \mapsto h$ where $h :\subseteq \mathbb{R} \to \mathbb{R}^l$ is the maximal solution of the initial value problem (1) is $(\overline{\delta}, \rho, \rho^l, \delta_{1,l})$-computable.*

2. The function $F : (f, t_0, x_0) \mapsto U$ where U is the domain of the maximal solution of the initial value problem (1) is $(\overline{\delta}, \rho, \rho^l, \theta^1)$-computable.

Proof: In the proof we use the representations $\tilde{\delta}$, the $\delta_{m,n}$ and natural representations for triples (D, L, N) with $D \in \mathrm{RB}^{l+1}$ and $L, N \in \mathbb{N}$, for the finite sequences and for the infinite sequences of such triples. For the other data we use the representations that we have already introduced. By Lemma 3.4 it suffices to prove computability w.r.t. $\tilde{\delta}$. We use Lemma 3.5.

For 2. There is a flowchart F_1 that on input $\langle p, q\rangle$ such that $f \in \tilde{\delta}\langle p, q\rangle$ where q is a name of the list $(C_i, K_i, M_i)_{i\in\mathbb{N}}$ and further inputs t_0, x_0, $\overline{t} \in \mathbb{Q}$ ($\overline{t} > t_0$) and $(D_0, L_0, N_0), \ldots, (D_n, L_n, N_n)$ halts iff (9-11) are true. It first tries to verify (9) and, if successful, using the operator T from Lemma 3.2 tries to compute the (t_k, x_k) and verify (19) in turn. Finally it tries to verify (11).

By (16), from F_1 a flowchart F_2 can be constructed that on input $\langle p, q\rangle$ such that $f \in \tilde{\delta}\langle p, q\rangle$ where q is a name of the list $(C_i, K_i, M_i)_{i\in\mathbb{N}}$ and further inputs t_0, x_0 and $\overline{t} \in \mathbb{Q}$ ($\overline{t} > t_0$) halts iff (9-11) are true for some $(D_0, L_0, N_0), \ldots, (D_n, L_n, N_n)$. By Lemma 3.5 F_2 halts iff the intrval $[t_0; \overline{t}]$ is contained in $\mathrm{dom}(h)$.

By (17), from F_2 a flowchart F_3 can be constructed that on input $\langle p, q\rangle$ such that $f \in \tilde{\delta}\langle p, q\rangle$ where q is a name of the list $(C_i, K_i, M_i)_{i\in\mathbb{N}}$ and further inputs t_0, x_0 computes a list of all $\overline{t} \in \mathbb{Q}$, $\overline{t} > t_0$, such that $[t_0; \overline{t}]$ is contained in $\mathrm{dom}(h)$.

Correspondingly, a list of all $\overline{t} < t_0$ such that $h(\overline{t})$ exists can be computed. Therefore, F is $(\overline{\delta}, \rho, \rho^l, \theta^1)$-computable.

For 1. There is a flowchart F_3 similar to F_1 that on input $\langle p, q\rangle$ such that $f \in \tilde{\delta}\langle p, q\rangle$ where q is a name of the list $(C_i, K_i, M_i)_{i\in\mathbb{N}}$ and further inputs t_0, x_0 , $\overline{t} \in \mathbb{R}$ ($\overline{t} > t_0$) and $(D_0, L_0, N_0), \ldots, (D_n, L_n, N_n)$ halts iff (9-11) are true.

By (15), from F_3 a flowchart F_4 can be constructed such that on input $\langle p, q\rangle$ such that $f \in \tilde{\delta}\langle p, q\rangle$ where q is a name of the list $(C_i, K_i, M_i)_{i\in\mathbb{N}}$ and further inputs t_0, x_0 and $\overline{t} \in \mathbb{R}$ ($\overline{t} > t_0$) computes a list $(D_0, L_0, N_0), \dots, (D_n, L_n, N_n)$ such that (9-11) are true.

From $(D_0, L_0, N_0), \dots, (D_n, L_n, N_n)$ and the other input data the tuples (t_k, x_k, d_k) such that (9-13) are true can be computed. Since there is some k such that $\overline{t} \in (t_k - d_k; t_k + d_k)$ such a k can be computed. Finally $h(\overline{t}) = T(f, t_k, x_k)(\overline{t})$ can be computed.

We have shown that $(f, t_0, x_0, t) \mapsto h(t)$ for $t > t_0$ can be computed. Correspondingly, $h(t)$ can be computed for $t < t_0$. Therefore, $(f, t_0, x_0, t) \mapsto h(t)$ is $(\tilde{\delta}, \rho, \rho^l, \rho, \rho^l])$-computable. By (2), $(f, t_0, x_0) \mapsto h$ is $(\tilde{\delta}, \rho, \rho^l, [\rho \to_p \rho^l])$-computable, hence $(\overline{\delta}, \rho, \rho^l, \delta_{1l})$-computable. $\square$

For open subsets of the real line, the function $U \mapsto \sup U$ is $(\theta, \overline{\rho}_>)$-computable and the function $U \mapsto \inf U$ is $(\theta, \overline{\rho}_>)$-computable. Therefore, from f and the initial values t_0, x_0 we can compute α from above and β from below such that (α, β) is the maximal interval of existence. If the input data are computable, α is right-r.e. and β is left-r.e.

4 The Complexity of Blowups

We will study the blowup for the initial value problem (1) for locally Lipschitz continuous functions $f : \mathbb{R}^{l+1} \to \mathbb{R}^l$ defined on all of $\mathbb{R}^{l+1}$. We consider only "positive blowup", that is, the behaviour of the solution for $t \geq t_0$. Let BU_f be the set of all initial conditions (t_0, x_0) such that for the maximal solution h of (1), $\sup\{t \mid h(t) \text{ exists}\} < \infty$ (the *blowup points*) and let $\mathrm{CBU}_f := \mathbb{R}^{l+1} \setminus \mathrm{BU}_f$ be the set of initial conditions for which there is no blowup. By the next theorem the set CBU_f is a G_δ-set which can be computed from f.

Theorem 4.1 *The function $B : f \mapsto \mathrm{CBU}_f$ for locally Lipschitz continuous (total) functions $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is $(\overline{\delta}, \delta_G^{l+1})$-computable.*

Proof: Let S and F be the functions from Theorem 3.6. Define a function H_1 by

$$H_1(f, i, t_0, x_0) := \begin{cases} 1 \text{ if } \ i < \sup F(f, t_0, x_0) \\ 0 \text{ otherwise.} \end{cases}$$

This function is $(\overline{\delta}, \nu_{\mathbb{N}}, \rho, \rho^l, \overline{\rho}_<)$-computable: First compute the open set $V := F(f, t_0, x_0)$ and then try to find $i \in \mathbb{N}$ in V. As long as i has not been found print (a $\nu_{\mathbb{Q}}$-code of) 0 on the output tape, as soon as i has been found continue writing 1s. By (2), the function H_2 defined by $H_2(f, i)(t_0, x_0) := H_1(f, i, t_0, x_0)$ is $(\overline{\delta}, \nu_{\mathbb{N}}, [\rho^{l+1} \to \overline{\rho}_<])$-computable. $H_2(f, i)$ is the characteristic function of the set

$$V_i := \{(t_0, x_0) \mid S(f, t_0, x_0)(t) \ \text{ exists for some } \ t > i\}.$$

Define the Sierpinski representation $\delta^n_{\rm Sierpinski}$ of subsets of $\mathbb{R}^n$ by

$$\delta^n_{\rm Sierpinski}(p) = W \ \text{ iff } \ [\rho^n \to \overline{\rho}_<](p) \ \text{ is the characteristic function of } \ W.$$

Then $H_3 : (f, i) \mapsto V_i$ is $(\overline{\delta}, \nu_{\mathbb{N}}, \delta^{l+1}_{\rm Sierpinski})$-computable. Since $\theta^n \equiv \delta^n_{\rm Sierpinski}$ [Brattka, Presser, 2003], the function H_3 is $(\overline{\delta}, \nu_{\mathbb{N}}, \theta^{l+1})$-computable. In particular, all the sets V_i are open. By (2), $H_4 : f \mapsto (V_i)_i$ is $(\overline{\delta}, [\nu_{\mathbb{N}} \to \theta^{l+1}])$-computable and hence $(\overline{\delta}, (\theta^{l+1})^\omega)$-computable. Since $\delta^n_G(p) = \bigcap_i (\theta^n)^\omega(p)(i)$, the function $H_5 : f \mapsto \bigcap_i V_i$ is $(\overline{\delta}, \delta^{l+1}_G)$-computable. It remains to observe that $\mathrm{CBU}_f = \bigcap_i V_i$. □

Therefore, for every locally Lipschitz continuous (total) function $f : \mathbb{R}^{l+1} \to \mathbb{R}^l$, the set CBU_f is a G_δ-set and its complement BU_f is (by definition) an F_σ-set. If, in addition, f is computable (more precisely, $\overline{\delta}$-computable), then the set CBU_f is a computable G_δ-set.

Theorem 4.1 shows that F_σ is an upper complexity bound for the blowup sets. Although not every F_σ-set is a blowup set, for example, if it is bounded; this upper F_σ-complexity bound is sharp for $l \geq 2$. This result is a corollary of the following theorem in which we show that there is indeed a kind of G_δ lower bound of CBU_f for $l \geq 2$, even for time independent systems. We prove a non-uniform version. For a time independent system we may choose $t_0 = 0$.

Theorem 4.2 *Let $l \geq 2$ be given. Then there exists a $(\delta^{l-1}_G, \overline{\delta})$-computable multi-function F so that for every G_δ-set $X \subseteq \mathbb{R}^{l-1}$ the function $f = F(X)$*

1. *is a locally Lipschitz continuous function $f : \mathbb{R}^l \to \mathbb{R}^l$ and*
2. *the solution u of the initial-value problem*

$$u'(t) = f(u(t)), \quad u(0) = (x_0, 0) \tag{18}$$

has a finite blowup for increasing t if and only if $x_0 \notin X$.

Proof. First we consider $l = 2$. For $n \in \mathbb{N}$, let $\mathrm{BI}_n := \{(a \cdot 2^{-n}, (a+2) \cdot 2^{-n}) \mid a \in \mathbb{Z}\}$ and let I be a canonical injective numbering of the set $\mathrm{BI} := \bigcup_n \mathrm{BI}_n$ of "normed binary intervals". For an open real interval $(a; b)$ let $3(a; b) := (a - (b - a); b + (b - a))$. There is a computable function $g_0 :\subseteq \Sigma^\omega \times \mathbb{N}^2 \to \mathbb{N}$ such that for all $w \in \mathrm{dom}(\delta^{l-1}_G)$ we have

$$\delta^{l-1}_G(w) = \bigcap_i \bigcup_j I g_0(w, i, j) \ .$$

As a first step we normalize this representation of $\delta^{l-1}_G(w)$ by an intersection of unions of open intervals.

Lemma 4.3 *There is a computable function* $g :\subseteq \Sigma^\omega \times \mathbb{N}^2 \to \mathbb{N}$ *such that for all* $i, j \in \mathbb{N}$, $w \in \mathrm{dom}(\delta_G^{l-1})$ *and* $x \in \mathbb{R}$,

$$\delta_G^{l-1}(w) = \bigcap_i O_i, \quad O_0 \supseteq O_1 \supseteq O_2 \ldots \quad \textit{for} \quad O_i = \bigcup_j Ig(w,i,j)\,, \tag{19}$$

$$3Ig(w,i,j) \subseteq O_i\,, \tag{20}$$

$$\{j \mid x \in 3Ig(w,i,j)\} \quad \textit{is finite if} \quad x \in O_i\,. \tag{21}$$

Proof. (Lemma 4.3) Let $O_i := \bigcap_{i' \le i} \bigcup_j Ig_0(w,i',j)$. Then

$$O_0 \supseteq O_1 \supseteq O_2 \ldots\,, \quad \delta_G^{l-1}(w) = \bigcap_i O_i \quad \text{and} \quad O_i = \bigcup_j Ig_1(w,i,j)$$

for some computable function g_1. Since every interval $K \in \mathrm{BI}$ is the union of intervals $L \in \mathrm{BI}$ such that $3L \subseteq K$, there is a computable function g_2 such that $O_i = \bigcup_j Ig_2(w,i,j)$ and $3Ig(w,i,j) \subseteq O_i$ for all i, j. Finally, by successively deleting for each i all $g_2(w,i,j)$ such that $Ig_2(w,i,j) \subseteq Ig_2(w,i,j')$ for some $j' < j$ (such intervals $Ig_2(w,i,j)$ are not necessary for generating O_i) we obtain a computable function g such that (19) and (20) and additionally

$$Ig(w,i,j) \not\subseteq Ig(w,i,j') \quad \text{if} \quad j' < j\,. \tag{22}$$

For showing (21) consider $x \in O_i$. Hence $x \in Ig(w,i,j_0) =: (c;d)$ for some j_0 and some c, d. Furthermore, $2^{-n} < \min(x-c, d-x)$ for some number n. Suppose, $x \in 3Ig(w,i,j)$ for infinitely many j. Since for each k there are at most 6 intervals $L \in \mathrm{BI}_k$ such that $x \in 3L$, there must be numbers $j > j_0$ and $m \ge n+3$ such that $x \in 3Ig(w,i,j)$ and $Ig(w,i,j) \in \mathrm{BI}_m$. Then $\mathrm{length}(3Ig(w,i,j)) = 6 \cdot 2^{-m} < 2^{-n}$ and hence $3Ig(w,i,j) \subseteq (c;d) = Ig(w,i,j_0)$ (since $x \in 3Ig(w,i,j)$). But this is false by (22), since $j_0 < j$. $\square$(Lemma 4.3)

Let now an δ_G^{l-1}-name ω of a G_δ-set $X \subseteq \mathbb{R}$ be given. As a next step we define the function $F(X) =: f : \mathbb{R}^2 \to \mathbb{R}^2$. For $k \in \mathbb{N}$, let $y_k := 2^{2^{2^k}}$. Then

$$2 \le y_k < y_k + y_k^2 + 2 \le y_{k+1}\,. \tag{23}$$

For $k = \langle i, j \rangle$ let

$$g_k(x,y) := \begin{cases} y^2 - y_k^2 & \text{if } (x,y) \in Ig(w,i,j) \times (y_k; y_k^2) \\ 0 & \text{if } (x,y) \notin 3Ig(w,i,j) \times (y_k - 1; y_k^2 + 1)\,, \end{cases}$$

and for all (x,y) between the two rectangles, $g_k(x,y)$ is defined by linear interpolation such that $(k,x,y) \mapsto g_k(x,y)$ is a computable function that is effectively locally Lipschitz. By (23) $g_k(x,y) \cdot g_{k'}(x,y) = 0$ for $k \ne k'$. Define the function f by $f(x,y) := (f_1(x,y), f_2(x,y))$ where

$$f_1(x,y) := 0\,,$$

$$f_2(x,y) := \begin{cases} 1 & \text{if } y < 1 \\ y^2 - \sum_{k \in \mathbb{N}} g_k(x,y) & \text{else}\,. \end{cases}$$

We study the solution $u : \mathbb{R} \to \mathbb{R}^2$ of the initial-value problem (18). Since $f_1(x,y) = 0$, all trajectories are in y-direction, hence for each initial value $(x,0)$ we have a one-dimensional problem. We observe a particle starting at $(x,0)$ traveling to position (x,y) with the prescribed speed $f_2(x,y)$ in y-direction. Since $f_2(x,y) \geq 1$ for all x,y, the particle will reach every point (x,y) for $y > 0$.

Suppose $x \in X$. By (19), for all i there is some j such that $x \in Ig(w,i,j)$. Therefore there are infinitely many k $(= \langle i,j\rangle)$ such that

$$f_2(x,y) = y^2 - g_k(x,y) = y_k^2 \quad \text{for} \quad y_k \leq y \leq y_k^2.$$

Therefore, if $u(t_k) = (x, y_k)$ then $u(t_k+1) = (x, y_k+y_k^2)$. Hence the particle needs one time unit for traveling from y_k to $y_k + y_k^2$. Since there are infinitely many such intervals, the particle cannot approach infinity in finite time. Therefore, there is no blowup for this initial value $(x,0)$.

Suppose $x \notin X$. By (19) $x \in O_i$ only for finitely many i. By (21) for each of these numbers i, $x \in 3Ig(w,i,j)$ only for finitely many numbers j. Therefore, there are only finitely many $k = \langle i,j\rangle$ such that $g_k(x,y) > 0$ for some y. Hence, for some k, $f_2(x,y) = y^2$ for $y > y_k$. As is well known, in this case the particle will approach infinity in finite time. Therefore, there is a blowup for this initial value $(x,0)$.

For $l > 2$, replace BI_n by $\mathrm{BI}_n^{(l)} := \{J_1 \times \ldots \times J_{l-1} \mid J_1, \ldots, J_{l-1} \in \mathrm{BI}_n\}$, replace the numbering I by a canonical numbering $I^{(l)}$ of $\mathrm{BI}^{(l)} := \bigcup_n \mathrm{BI}_n^{(l)}$, and define $3K$ accordingly for $K \in \mathbb{N}$. The rest of the proof remains unchanged. □

In the one-dimensional case, we can say even more if we restrict ourselves to functions f which do not depend on t. In this case the blowup sets do solely depend on the zeroes of f.

Theorem 4.4 *Let $f : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function. Then the "positive" blowup set BU_f of the initial value problem "$x'(t) = f(x(t)),\ \ x(0) = x_0$" is the union of two intervals $(-\infty, a)$ and (b, ∞) for some $a, b \in \overline{\mathbb{R}}$. If the function f is computable, then the constant a can be chosen to be $\overline{\rho}_<$-computable and the constant b to be $\overline{\rho}_>$-computable.*

Proof: If f has no zero, then there is a blowup either for all x_0 or for no x_0. In the first case let $a := -\infty$ and $b := \infty$, in the second case let $a := b :=:= \infty$.

Suppose that f has a zero. We observe that $x_0 \notin \mathrm{BU}_f$ if there are x_1, x_2 such that $f(x_1) = f(x_2) = 0$ and $x_1 \leq x_0 \leq x_2$.

If f has no greatest zero then let $b := \infty$. Suppose, f has a greatest zero β. Then there is a blowup either for all $x_0 > \beta$ or for no $x_0 > \beta$. In the first case let $b := \beta$, in the second case let $b := \infty$.

Correspondingly, if f has no smallest zero then let $a := -\infty$. Suppose, f has a smallest zero α. Then there is a blowup either for all $x_0 < \alpha$ or for no $x_0 < \alpha$. In the first case let $a := \alpha$, in the second case let $a := -\infty$.

By [Weihrauch, 2000, Theorem 6.3.4] the smallest zero of a computable function (if it exists) is $\overline{\rho}_<$-computable and the greatest zero of a computable function (if it exists) is $\overline{\rho}_>$-computable. Furthermore, $-\infty$ and ∞ are $\overline{\rho}_<$-computable and $\overline{\rho}_>$-computable. □

References

[Aberth, 1970] Aberth, O.: Computable analysis and differential equations. *Journal of Symbolic Logic*, 40 (1):84, 1975.

[Aberth, 1971] Aberth, O.: The failure in computable analysis of a classical existence theorem for differential equations. *Proceedings of the American Mathematical Society*, 30:151–156, 1971.

[Aberth, 1980] Aberth, O.:. Computable Analysis. McGraw-Hill, 1980.

[Birkhoff, Rota, 1962] Birkhoff, G., Rota, G.: Ordinary Differential Equations. Ginn and Company, 1962.

[Bishop, Bridges, 1985] Bishop, E., Bridges D. S.: Constructive Analysis. Springer, 1985

[Brattka, Presser, 2003] Brattka, V., Presser, G.: Computability on subsets of metric spaces. *Theoretical Computer Science*, 305:43–76, 2003.

[Graça, 2007] Graça, D. S.: Computability with polynomial differential equations. Ph.D. thesis, Instituto Superior Técnico/Universidade Técnica de Lisboa, 2007.

[Grzegorczyk, 1955] Grzegorczyk, A.: Computable functionals. *Fundamenta Mathematicae*, 42:168-202, 1955.

[Grzegorczyk, 1957] Grzegorczyk, A.: On the definitions of computable real continuous functions. *Fundamenta Mathematicae*, 44:61-71, 1957.

[Grubba, Weihrauch, Xu, 2007] Grubba, T., Weihrauch, K., Xu, Y.: Effectivity on continuous functions in topological spaces. In Ruth Dillhage, Tanja Grubba, Andrea Sorbi, Klaus Weihrauch, and Ning Zhong, editors, *CCA 2007, Fourth International Conference on Computability and Complexity in Analysis*, volume 338 of *Informatik Berichte*, pages 137–154. FernUniversität in Hagen, June 2007. CCA 2007, Siena, Italy, June 16–18, 2007.

[Graça, Zhong, Buescu, 2006] Graça, D. S., Zhong, N., Buescu, J.: Computability, noncomputability and undecidability of maximal intervals of IVPs. Accepted and to appear in *Transactions of AMS*.

[Heuser, 1981] Heuser, H.: Lehrbuch der Analysis, Teil 2. B.G.Teubner, 1981.

[Ho, 1999] Ho, C. K.: Relatively recursive reals and real functions. *Theoretical Computer Science* 210 (1):99–120, 1999.

[Ko, 1983] Ko, Ker-I: On the computational complexity of ordinary differential equations. *Information and control*, 58:157–194, 1983.

[Ko, 1991] Ko, Ker-I: Computational Complexity of Real Functions. Birkhäuser, 1991.

[Odifreddi, 1989] Odifreddi, P.: Classical Recursion Theory. North-Holland, 1989.

[Perko, 1996] Perko, L.: Differential equations and dynamical systems. Springer-Verlag New York, 1996.

[Pour-El, Richards, 1979] Pour-El, M. B., Richards, J. I.: A computable ordinary differential equation which possesses no computable solution. *Annals of Mathematical Logic*, 17:61–90, 1979.

[Pour-El, Richards, 1989] Pour-El, M. B., Richards, J. I.: *Computability in Analysis and Physics*. Perspectives in Mathematical Logic, Springer, Berlin, 1989.

[Pour-El, Zhong, 1997] Pour-El, M. B., Zhong, N.: The wave equation with computable initial data whose unique solution is nowhere computable. *Mathematical Logic Quarterly*, 43 (4):499–509, 1997.

[Turing, 1936] Turing, A. M.: On computable numbers, with an application to the Entscheidungsproblem. *Proceedings of the London Mathematical Society*, 2 (42):230–265, 1936.

[Weihrauch, 1993] Weihrauch, K.: Computability on computable metric spaces. *Theoretical Computer Science*, 113:191–210, 1993.

[Weihrauch, 2000] Weihrauch, K.: *Computable Analysis*. Springer, Berlin, 2000.

[Weihrauch, 2005] Weihrauch, K.: Multi-functions on multi-represented sets are closed under flowchart programming. In Tanja Grubba, Peter Hertling, Hideki Tsuiki, and Klaus Weihrauch, editors, *Computability and Complexity in Analysis*, volume 326 of *Informatik Berichte*, pages 267–300. FernUniversität in Hagen, July 2005. Proccedings, Second International Conference, CCA 2005, Kyoto, Japan, August 25–29, 2005.

[Weihrauch, 2008] Weihrauch, K.: The computable multi-functions on multi-represented sets are closed under programming Submitted for publication.

[Weihrauch, Zhong, 2002] Weihrauch, K., Zhong, N.: Is wave propagation computable or can wave computers beat the Turing machine? *Proceedings of the London Mathematical Society*, 85 (2):312–332, 2002.

[Weihrauch, Zhong, 2003] Zhong, N., Weihrauch, K.: Computability theory of generalized functions. *Journal of the Association of Computer Machinery*, 50 (4):469–505, 2003.

[Weihrauch, Zhong, 2005a] Weihrauch, K., Zhong, N.: An algorithm for computing fundamental solutions. *SIAM Journal on Computing*, 35 (6):1283–1294, 2006.

[Weihrauch, Zhong, 2005b] Weihrauch, K., Zhong, N.: Computing the solution of the Korteweg-de Vries equation with arbitrary precision on Turing machines. *Theoretical Computer Science*, 332 (1–3):337–366, 2005.

[Zhou, 1996] Zhou, Q.: Computable real-valued functions on recursive open and closed subsets of Euclidean space. *Mathematical Logic Quarterly*, 42 (3):379–409, 1996.