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# On the Effective Existence of Schauder Bases

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**Abstract:** We construct a computable Banach space which possesses a Schauder basis, but does not possess any computable Schauder basis.
**Key Words:** computatable functional analysis, Schauder basis
**Category:** F.1, F.m, G.m

## 1 Introduction

Let $X$ be an infinite-dimensional Banach space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. A sequence $(x_i)_i \in X^\omega$ is called a *Schauder basis* (or simply a *basis*) of $X$ if for every $x \in X$ there is a unique sequence $(\alpha_i)_i \in \mathbb{F}^\omega$ such that $x$ is the limit of the norm convergent series $\sum_i \alpha_i x_i$. If $X$ is a finite-dimensional vector space then a finite sequence $(x_1, \dots, x_n) \in \mathbb{F}^{<\omega}$ is a (Schauder) basis of $X$ if for every $x \in X$ there are unique $\alpha_1, \dots, \alpha_n$ such that $x = \sum_{i=1}^n \alpha_i x_i$. A finite or infinite sequence is called *basic* if it is a basis of the closure of its linear span. (Finite sequences are hence basic if, and only if, they are linearly independent.)

The theory of Schauder bases is a central area of research and also an important tool in functional analysis. Background information can be found in e.g. [Singer 1970, Singer 1981, Megginson 1998, Albiac and Kalton 2006].

In computable analysis[2], [Brattka and Dillhage 2007] have shown that computable versions of a number of classical theorems on compact operators on Banach spaces can be proved under the assumption that the computable Banach spaces under consideration possess computable bases (with certain additional properties). The restriction to spaces with computable bases does not seem to be too costly in terms of generality because virtually all of the separable Banach spaces that are important for applications are known to possess a computable basis.

Complete orthonormal sequences in Hilbert spaces are particularly well-behaved examples of Schauder bases. It is a fundamental fact that every separable Hilbert space contains such a complete orthonormal sequence. It is furthermore known (see [Brattka and Yoshikawa 2006, Lemma 3.1]) that every computable Hilbert space contains a computable complete orthonormal sequence and hence a computable basis. Can this be generalized to arbitrary computable

[1] This paper is an adaption of a chapter of the author's doctoral dissertation.
[2] See [Brattka et al. 2008, Weihrauch 2000] for introductions to computable analysis.

Banach spaces with bases? More precisely: If a computable Banach space possesses a basis, does it necessarily possess a computable basis? The aim of the present note is to show that the answer is “no” in general. Our example will be a subspace of the space of zero-convergent sequences in *Enflo’s space* – a famous example of a separable Banach space that lacks the *approximation property* (see below). The construction will proceed by direct diagonalization.

Some remarks on notation: As we will never consider more than one norm on the same linear space, we will denote every norm by $\|\cdot\|$; which norm is meant will be clear from what it is applied to. On spaces of continuous linear operators we always consider the usual operator norm. If $x_1, x_2, \ldots$ are elements of a Banach space, denote by $[x_1, x_2, \ldots]$ the closure of their linear span; analogously, let $[x_1, \ldots, x_n]$ denote the linear span of $x_1, \ldots, x_n$. Whenever we speak of the *rational span* of a set of vectors, we mean all their finite linear combinations with coefficients taken from $\mathbb{Q}$ (if $\mathbb{F} = \mathbb{R}$) or $\mathbb{Q}[i]$ (if $\mathbb{F} = \mathbb{C}$), respectively. If $X$ is a normed space, put $B_X := \{x \in X \ : \ \|x\| \le 1\}$. Denote by $\mathcal{K}(X)$ the hyperspace of compact subsets of $X$.

As far as *computable Banach spaces* are concerned, we refer the reader to the literature for the necessary definitions and basic results; see e.g. [Brattka 2001, Brattka and Presser 2003, Brattka and Dillhage 2007]. Computability of points in a computable Banach space shall be understood as computability with respect to the *Cauchy representation*. We use the following representations for open and compact sets: an open set is represented by a sequence of basic open balls that exhausts it (this corresponds to the representation $\delta_{\mathcal{O}(X)}$ in [Brattka 2001]); a compact set is represented by a list of all minimal finite covers[3] consisting of basic open balls (this corresponds to $\delta_{\mathrm{min-cover}}$ in [Brattka and Presser 2003]). Tuples and sequences of objects from represented spaces, as well as continuous functions on represented spaces shall always be represented by the derived standard representations. We recall that $f^{-1}(U)$ can be computed given continuous $f$ and open $U$. Similary, $f(K)$ can be computed given continuous $f$ and compact $K$. We can also compute minima and maxima of continuous real-valued functions on compact sets.

## 2 Computable Enflo’s space

The question whether every separable Banach space has a basis had been posed by Banach in 1932; fourty years later, it was answered in the negative by [Enflo 1973]. Enflo in fact constructed a Banach space that lacks the *approximation property* (AP): A Banach space is said to have AP if the identity operator can be approximated uniformly on every compact subset by finite rank operators

[3] A cover of a set is called *minimal* if every element of the cover has nonempty intersection with the set.

(see [Megginson 1998, Definition 3.4.26, Theorem 3.4.32]). A Banach space with a basis necessarily has AP (see [Megginson 1998, Theorem 4.1.33]).

Enflo's example was simplified by Davie [Davie 1973]. It is not surprising that the space defined by Davie is computable. For the reader's convenience, we shall give some details of the construction: For any $k \in \mathbb{N}$, let $G_k$ be the additive group $\mathbb{Z}/(3 \cdot 2^k)\mathbb{Z}$. For $j = 1, \cdots, 3 \cdot 2^k$, let $\gamma_j^{(k)}$ be the (unique) group homomorphism from $G_k$ into the multiplicative group $\mathbb{C} \setminus \{0\}$ with

$$\gamma_j^{(k)}(1) = \exp\left(2\pi \mathrm{i}\, \frac{j}{3 \cdot 2^k}\right).$$

It is shown in [Davie 1973] (with a probabilistic argument) that there is a constant $A_2$ such that for every $k$, the set

$$\{\gamma_j^{(k)} \ : \ 1 \leq j \leq 3 \cdot 2^k\}$$

can be partitioned into two sets

$$\{\sigma_j^{(k)} \ : \ 1 \leq j \leq 2^k\} \qquad \text{and} \qquad \{\tau_j^{(k)} \ : \ 1 \leq j \leq 2 \cdot 2^k\}$$

with

$$(\forall\, g \in G_k) \ \left( \left| 2 \sum_{j=1}^{2^k} \sigma_j^{(k)}(g) - \sum_{j=1}^{2 \cdot 2^k} \tau_j^{(k)}(g) \right| < A_2 (k+1)^{1/2} 2^{k/2} \right).$$

Similarly, it is shown that there is a constant $A_3$ such that for every $k \geq 1$, there are $\varepsilon_j^{(k)} \in \{-1, 1\}$ $(j = 1, \ldots, 2^k)$ with

$$(\forall\, g \in G_k)(\forall\, h \in G_{k-1}) \ \left( \left| \sum_{j=1}^{2^k} \varepsilon_j^{(k)} \frac{\tau_j^{(k-1)}(h)}{\sigma_j^{(k)}(g)} \right| < A_3 (k+1)^{1/2} 2^{k/2} \right).$$

By exhaustive search, $\sigma_j^{(k)}$, $\tau_j^{(k)}$, $\varepsilon_j^{(k)}$ $(1 \leq j \leq 3 \cdot 2^k)$ such that the above two inequalities are fulfilled can be found effectively in $k$. Let $G$ be the disjoint union $\bigcup_{k \in \mathbb{N}} G_k$. Let $\nu$ be a computable bijection between the set

$$\{(k, j) \ : \ k, j \in \mathbb{N}, 1 \leq j \leq 2^k\}$$

and $\mathbb{N}$. We define a mapping $e$ from $\mathbb{N}$ into the linear space of bounded complex functions on $G$ by

$$e(\nu(k,j))(g) := \begin{cases} \tau_j^{(k-1)}(g) & \text{if } k \geq 1 \text{ and } g \in G_{k-1}, \\ \varepsilon_j^{(k)} \sigma_j^{(k)}(g) & \text{if } g \in G_k, \\ 0 & \text{otherwise.} \end{cases}$$

We equip the space of bounded complex functions on $G$ with the sup-norm. In this Banach space, we consider the subspace

$$Z := [e(0), e(1), \ldots].$$

Davie showed that $Z$ lacks AP. Furthermore, it is straightforward to check that $(Z, \|\cdot\|, e)$ is a computable Banach space. We hence have:

**Lemma 1.** *The computable Banach space $(Z, \|\cdot\|, e)$ constructed above lacks AP.* □

## 3 Basis constants and local basis structure

The following characterization of basic sequences (see [Megginson 1998, Corollary 4.1.25]) is due to Banach:

**Proposition 2.** *Let $X$ be an infinite-dimensional Banach space over $\mathbb{F}$. A sequence $(x_i)_i \in X^\omega$ is basic if, and only if,*

*(1) no $x_i$ is equal to zero, and*

*(2) there exists a constant $M \in \mathbb{R}$ such that*

$$\|\sum_{i=0}^{m} \alpha_i x_i\| \leq M \|\sum_{i=0}^{n} \alpha_i x_i\| \tag{1}$$

*for all $m, n \in \mathbb{N}$, $m \leq n$, and $\alpha_0, \ldots, \alpha_n \in \mathbb{F}$.*

□

If $(x_i)_i \in X^\omega$ is basic, then the *basis constant* $\mathrm{bc}((x_i)_i)$ of $(x_i)_i$ is defined as the minimum $M$ such that (1) holds for all $m, n \in \mathbb{N}$, $m \leq n$, and all $\alpha_0, \ldots, \alpha_n \in \mathbb{F}$. If $(x_1, \ldots, x_n) \in X^{<\omega}$ is basic, then the *basis constant* $\mathrm{bc}((x_1, \ldots, x_n))$ of $(x_1, \ldots, x_n)$ is defined as the minimum $M$ such that

$$\|\sum_{i=1}^{m} \alpha_i x_i\| \leq M \|\sum_{i=1}^{n} \alpha_i x_i\|$$

holds for all $m \in \mathbb{N}$, $m \leq n$, and all $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$. If $X$ is a Banach space with a basis, then the *basis constant* $\mathrm{bc}(X)$ of $X$ is defined as the infimum over the basis constants of all bases of $X$.

It is obvious that any basis constant must be at least 1. A basis $(x_i)_i$ of some infinite-dimensional Banach space $X$ is called *monotone* if $\mathrm{bc}((x_i)_i) = 1$. $(x_i)_i$ is called *shrinking* if

$$(\forall\, f \in X^*) \ \left(\lim_{n\to\infty} \sup\{|f(x)| \ : \ x \in B_{[x_n, x_{n+1}, \ldots]}\} = 0\right).$$

A Banach space $X$ is said to have *local basis structure* if there exists a constant $C$ such that for every finite-dimensional subspace $V$ of $X$, there is a finite-dimensional space $W$ with $V \subseteq W \subseteq X$ and $\mathrm{bc}(W) < C$. This notion was introduced by [Pujara 1975] (under a different name; cf. [Singer 1981, p. 820]).

For every $n \geq 1$, the Banach space $\ell_\infty^n$ is defined as the linear space $\mathbb{C}^n$ equipped with the sup-norm (cf. [Megginson 1998, Example 1.2.9]). The following criterion for local basis structure is found in [Szarek 1987, Proposition 1.3]:

**Proposition 3.** *Let $X$ be a Banach space such that there exists a constant $C$ such that for every $n \geq 1$, there is a subspace $V_n$ of $X$ and an isomorphism $F_n : V_n \to \ell_\infty^n$ with $\|F_n\|\|F_n^{-1}\| \leq C$. Then $X$ has local basis structure.* □

**Corollary 4.** *The space $Z$ from Lemma 1 has local basis structure.*

*Proof.* For every $n \geq 1$, there is even an isometric isomorphism from a subspace of $Z$ onto $\ell_\infty^n$: For any $k, j \in \mathbb{N}$, the function $e(\nu(k, j)) : G \to \mathbb{C}$ is supported on $G_{k-1} \cup G_k$. So we can choose $k_1, \ldots, k_n$ such that $e(k_1), \ldots, e(k_n)$ have pairwise disjoint supports. The norm of $Z$ (just like the norm of $\ell_\infty^n$) is the sup-norm. So it is obvious that the subspace $[e(k_1), \ldots, e(k_n)]$ of $Z$ is isometrically isomorphous to $\ell_\infty^n$ via $F_n$ with

$$F_n(e(k_i)) = (0, \ldots, 0, \underbrace{1}_{i\text{-th}}, 0, \ldots, 0), \qquad i = 1, \ldots, n.$$

(Note that all functions in the range of $e$ have norm 1.) □

We need an effective version of the previous statement:

**Lemma 5.** *Let $(X, \|\cdot\|, e)$ be an infinite-dimensional computable Banach space such that $X$ has local basis structure, witnessed by a constant $C \in \mathbb{N}$. There exists a computable linearly independent sequence $(x_i)_i \in X^\omega$ and a strictly increasing computable function $\sigma : \mathbb{N} \to \mathbb{N}$ such that $[x_0, x_1, \ldots] = X$ and*

$$(\forall\, n \in \mathbb{N})\ (\mathrm{bc}([x_0, \ldots, x_{\sigma(n)}]) < C).$$

Before we prove this lemma, we have to provide a number of auxiliary propositions: For every linear space $X$ over $\mathbb{F}$, define

$$\mathrm{IND}_X := \{(x_1, \ldots, x_n) \in X^{<\omega} \ : \ n \geq 1,\ x_1, \ldots, x_n \text{ linearly independent}\}.$$

If $(x_1, \ldots, x_n) \in \mathrm{IND}_X$, there is a unique vector $(f_1, \ldots, f_n)$ of continuous *coordinate functionals*: The domain of each $f_i :\subseteq X \to \mathbb{F}$ is $[x_1, \ldots, x_n]$, and the $f_i$ are uniquely defined by the condition

$$(\forall\, x \in [x_1, \ldots, x_n])\ \left(x = \sum_{i=1}^{n} f_i(x) x_i\right).$$

**Lemma 6.** *Let $(X, \|\cdot\|, e)$ be a computable Banach space over $\mathbb{F}$.*

*(1) The mapping*

$$(x_1, \dots, x_n) \mapsto (f_1, \dots, f_n)$$

*that takes $(x_1, \dots, x_n) \in \mathrm{IND}_X$ to the corresponding vector $(f_1, \dots, f_n)$ of coordinate functionals is computable.*

*(2) The mapping*

$$(x_1, \dots, x_n) \mapsto B_{[x_1,\dots,x_n]}$$

*that takes $(x_1, \dots, x_n) \in \mathrm{IND}_X$ to the compact set $B_{[x_1,\dots,x_n]}$ is computable.*

*Proof.* Given linearly independent $x_1, \dots, x_n$. For $i = 1, \dots, n$, we can compute the mappings

$$(\alpha_1, \dots, \alpha_n) \mapsto x_i + \sum_{\substack{1 \le j \le n \\ j \ne i}} \alpha_j x_j, \qquad \alpha_1, \dots, \alpha_n \in \mathbb{F}. \tag{2}$$

It is furthermore easy to see that we can compute

$$\{(\alpha_1, \dots, \alpha_n) \in \mathbb{F}^n \ : \ (\forall\, 1 \le i \le n)\ (|\alpha_i| \le 1)\}$$

as a compact set. Applying the mappings (2) to this set, we can compute the compact sets

$$C_i := \Big\{x_i + \sum_{\substack{1 \le j \le n \\ j \ne i}} \alpha_j x_j \ : \ \alpha_j \in \mathbb{F},\, |\alpha_j| \le 1\Big\}, \qquad i = 1, \dots, n.$$

We can compute the minimum value $M$ that $\|\cdot\|$ obtains on the sets $C_1, \dots, C_n$. The linear independence of $x_1, \dots, x_n$ yields $M > 0$.

Let $x \in [x_1, \dots, x_n] \setminus \{0\}$ be arbitrary, say

$$x = \sum_{i=1}^{n} \alpha_i x_i.$$

Let $\ell \in \{1, \dots, n\}$ be such that

$$|\alpha_\ell| = \max_{1 \le i \le n} |\alpha_i|.$$

Then

$$\sum_{i=1}^{n} \frac{\alpha_i}{\alpha_\ell} x_i \in C_\ell,$$

which implies

$$M \le \left\| \sum_{i=1}^{n} \frac{\alpha_i}{\alpha_\ell} x_i \right\|.$$

For any $1 \leq j \leq n$ we have

$$|f_j(x)| = \left| f_j \left( \sum_{i=1}^{n} \alpha_i x_i \right) \right| = |\alpha_j| \leq |\alpha_\ell| \leq |\alpha_\ell| M^{-1} \left\| \sum_{i=1}^{n} \frac{\alpha_i}{\alpha_\ell} x_i \right\|$$
$$= M^{-1} \left\| \sum_{i=1}^{n} \alpha_i x_i \right\| = M^{-1} \|x\|.$$

As $x$ was arbitrary, we have $\|f_j\| \leq 1/M$.

It is sufficient to demonstrate how to compute $f_j$ on any given $x \in [x_1, \ldots, x_n]$ up to precision $2^{-k}$ for any given $k \in \mathbb{N}$. This can be done by using exhaustive search to find rational scalars $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ such that

$$\left\| x - \sum_{i=1}^{n} \alpha_i x_i \right\| < M 2^{-k}.$$

Then

$$\|f_j(x) - \alpha_j\| = \left\| f_j \left( x - \sum_{i=1}^{n} \alpha_i x_i \right) \right\| < 2^{-k}.$$

We have proved item (1).

In order to show item (2), we need to demonstrate how to semidecide whether a given tuple $(U_1, \ldots, U_k)$ of basic open balls is a minimal cover of $B_{[x_1,\ldots,x_n]}$. We first semidecide whether every $U_m$ intersects $B_{[x_1,\ldots,x_n]}$. This can be done by exhaustively searching for rational $q_1, \ldots q_n \in \mathbb{F}$ with

$$\left\| \sum_{i=1}^{n} q_i x_i \right\| < 1, \quad \text{and} \quad \sum_{i=1}^{n} q_i x_i \in U_m.$$

It remains to semidecide whether $(U_1, \ldots, U_k)$ is a cover of $B_{[x_1,\ldots,x_n]}$. Note that $B_{[x_1,\ldots,x_n]} = B_X \cap K$, where

$$K := \Big\{ \sum_{i=1}^{n} \alpha_i x_i \ : \ \alpha_i \in \mathbb{F}, \ |\alpha_i| \leq 1/M \Big\}.$$

By similar arguments as for the $C_i$ above, we can compute $K$ as a compact set. $X \setminus K$ can be computed as an open set by [Brattka and Presser 2003, Corollary 4.11.1]; the same is true for $X \setminus B_X$ (as it can be written as the preimage of the computably open set $(1, \infty)$ over the computable function $x \mapsto \|x\|$). So

$$X \setminus B_{[x_1,\ldots,x_n]} = (X \setminus B_X) \cup (X \setminus K)$$

can be computed as an open set. We hence have an enumeration $(V_j)_j$ of basic open balls with

$$\bigcup_j V_j = X \setminus B_{[x_1,\ldots,x_n]}.$$

As $B_{[x_1,\dots,x_n]} \subseteq K$ and $K$ is compact, we have that $(U_1,\dots,U_k)$ is a cover of $B_{[x_1,\dots,x_n]}$ if, and only if, there exists an $\ell$ such that

$$(U_1,\dots,U_k,V_0,\dots V_\ell)$$

is a minimal cover of $K$. As we have a list of all minimal covers of $K$, this condition can be semidecided effectively. □

**Lemma 7.** *Let* $(X,\|\cdot\|,e)$ *be a computable Banach space over* $\mathbb{F}$*. The set* $\mathrm{IND}_X$ *is computably enumerable.*

*Proof.* Given a vector $(x_1,\dots,x_n) \in X^{<\omega}$, we need to semidecide whether the vector is in $\mathrm{IND}_X$. If $n=0$, the vector is not in $\mathrm{IND}_X$. If $n=1$, we simply semidecide $\|x_1\|>0$. If $n>1$ the procedure can be reduced to the procedure for $n-1$: First semidecide whether $x_1,\dots,x_{n-1}$ are linearly independent. In case this is detected, use Lemma 6.2 to compute $B_{[x_1,\dots,x_{n-1}]}$ as a compact set. We can now compute the distance between $x_n/\|x_n\|$ and $B_{[x_1,\dots,x_{n-1}]}$. It remains to semidecide whether this distance is positive. □

**Lemma 8.** *Let* $(X,\|\cdot\|,e)$ *be a computable Banach space.*

*(1) The mapping*

$$(x_1,\dots,x_n) \mapsto \mathrm{bc}((x_1,\dots,x_n))$$

*for* $(x_1,\dots,x_n) \in \mathrm{IND}_X$ *is computable.*

*(2) The mapping*

$$(x_1,\dots,x_n) \mapsto \mathrm{bc}([x_1,\dots,x_n])$$

*for* $(x_1,\dots,x_n) \in \mathrm{IND}_X$ *is upper semi-computable*[4].

*Proof.* Let $(x_1,\dots,x_n) \in \mathrm{IND}_X$ be given.

For item (1): By Lemma 6, we can compute the compact set $B_{[x_1,\dots,x_n]}$ as well as the coordinate functionals $(f_1,\dots,f_n)$. By effective maximization, we can hence compute the numbers

$$C_\ell := \sup\big\{\big\|\sum_{i=1}^{\ell} x_i f_i(x)\big\| \ : \ x \in B_{[x_1,\dots,x_n]}\big\}, \qquad \ell = 1\dots,n.$$

We can now compute $\mathrm{bc}((x_1,\dots,x_n))$ as the maximum of $C_1,\dots,C_n$.

For item (2): As a consequence of item (1), the basis constant of a finite basis depends continuously on the basis' elements. This implies that $\mathrm{bc}([x_1,\dots,x_n])$ is the infimum of the set

$$\{\mathrm{bc}(a_1,\dots,a_n) \ : a_1,\dots,a_n \text{ linearly independent elements of the rational span of } x_1,\dots,x_n\}.$$

[4] This means: Given any linearly independent $x_1,\dots,x_n$, we can effectively enumerate all rational numbers greater than $\mathrm{bc}([x_1,\dots,x_n])$.

By item (1) and Lemma 7, we can compute a sequence that exhausts this set. We can hence compute its infimum from above. □

*Proof of Lemma 5.* The function $\sigma$ and the sequence $(x_i)_i$ are computed recursively as follows: Search for an arbitrary $\ell$ with $e(\ell) \neq 0$. Put $\sigma(0) = 0$, $x_0 = e(\ell)$. Now suppose that $\sigma(n)$ and $x_0, \ldots, x_{\sigma(n)}$ have already been computed. By Lemma 6.2, we can compute $B_{[x_0,\ldots,x_{\sigma(n)}]}$, and so we can compute the sequence $(d_m^{(n)})_m$, where $d_m^{(n)}$ is the distance between $e(m)/\|e(m)\|$ and $B_{[x_0,\ldots,x_{\sigma(n)}]}$. For all $m, j \in \mathbb{N}$, we can semidecide both $d_m^{(n)} > 2^{-(n+j+1)}$ and $d_m^{(n)} < 2^{-(n+j)}$. We can hence compute a binary double sequence $(t_{m,j}^{(n)})_{m,j}$ such that

$$d_m^{(n)} \leq 2^{-(n+j+1)} \implies t_{m,j}^{(n)} = 0$$

and

$$d_m^{(n)} \geq 2^{-(n+j)} \implies t_{m,j}^{(n)} = 1$$

for all $m, j \in \mathbb{N}$. We finally search the triangular scheme

$$\begin{array}{l} t_{0,0}^{(n)}, \\ t_{0,1}^{(n)}, t_{1,1}^{(n)} \\ t_{0,2}^{(n)}, t_{1,2}^{(n)}, t_{2,2}^{(n)} \\ t_{0,3}^{(n)}, t_{1,3}^{(n)}, t_{2,3}^{(n)}, t_{3,3}^{(n)} \\ \vdots \end{array} \tag{3}$$

row by row and from left to right for the first occurrence of 1. There must be an occurrence of 1 in the scheme, because else one would have

$$(\forall\, m \in \mathbb{N})\ (d_m^{(n)} = 0),$$

which would imply

$$(\forall\, m \in \mathbb{N})\ (e(m) \in [x_0, \ldots, x_{\sigma(n)}]),$$

in contradiction to $X$ being infinite dimensional. If the first 1 occurs at index, say, $(m_0, j_0)$, put $x_{\sigma(n)+1} := e(m_0)$.

Before we describe how to compute $\sigma(n+1)$ and $x_{\sigma(n)+2}, \ldots, x_{\sigma(n+1)}$, let us point out that our method for choosing $x_{\sigma(n)+1}$ already ensures that eventually $[x_0, x_1, \ldots] = X$: Assume the contrary. Then there is an $N \in \mathbb{N}$ such that

$$e(N) \notin [x_0, x_1, \ldots]. \tag{4}$$

Let $d > 0$ be the distance between $e(N)/\|e(N)\|$ and $B_{[x_0,x_1,\ldots]}$. Let $n$ be so large that

$$\{e(0), \ldots, e(N-1)\} \cap \{x_0, x_1, \ldots\} = \{e(0), \ldots, e(N-1)\} \cap \{x_0, x_1, \ldots, x_{\sigma(n)}\} \tag{5}$$

(such an $n$ exists because the sequence $(x_i)_i$ is constructed to be linearly independent and can hence not have dublicate elements) and $d \geq 2^{-n}$. Consider the construction of $x_{\sigma(n)+1}$. As

$$d_N^{(n)} \geq d \geq 2^{-n} \geq 2^{-(n+N)},$$

we have $t_{N,N}^{(n)} = 1$. Let $(m_0, j_0)$ be the first index in (3) with $t_{m_0,j_0} = 1$. If $(m_0, j_0) \neq (N, N)$, then necessarily $x_{\sigma(n)+1} = e(m_0)$ and $m_0 \leq N - 1$, in contradiction to (5). So necessarily $(m_0, j_0) = (N, N)$, and hence $x_{\sigma(n)+1} = e(N)$, in contradiction to (4).

We now resume the construction and describe how to compute $\sigma(n+1)$ and $x_{\sigma(n)+2}, \ldots, x_{\sigma(n+1)}$: As $X$ has local basis structure, there must be a number $k \in \mathbb{N}$ and points $a_1, \ldots, a_k \in X$ such that

$$(x_0, \ldots, x_{\sigma(n)+1}, a_1, \ldots, a_k) \in \mathrm{IND}_X$$

and

$$\mathrm{bc}([x_0, \ldots, x_{\sigma(n)+1}, a_1, \ldots, a_k]) < C.$$

Lemma 8.2 yields that suitable $a_1, \ldots, a_k$ can then be found in the rational span of $e$ (for reasons of continuity) and can furthermore be searched for effectively. So once such $k, a_1, \ldots, a_k$ are found, put $\sigma(n+1) = \sigma(n)+1+k$ and $x_{\sigma(n)+1+i} = a_i$ for $i = 1, \ldots, k$. $\square$

## 4 A class of computable spaces with bases

For any sequence $(X_n)_n$ of Banach spaces over $\mathbb{F}$, let $(X_0 \times X_1 \times \cdots)_{c_0}$ be the Banach space of all sequences $(x_n)_n$ with $x_n \in X_n$ for all $n$ and $\lim_{n\to\infty} \|x_n\| = 0$, equipped with the norm

$$\|(x_n)_n\| := \sup_{n\in\mathbb{N}} \|x_n\|.$$

The dual of a space of the form $(X_0 \times X_1 \times \cdots)_{c_0}$ has a simple description in terms of the duals of the spaces $X_i$:

**Lemma 9.** *Let $(X_n)_n$ be a sequence of Banach spaces over $\mathbb{F}$. Put*

$$X := (X_0 \times X_1 \times \cdots)_{c_0}.$$

*If $(f_n)_n$ is a sequence with $f_n \in X_n^*$ for every $n$ and $\sum_{n=0}^{\infty} \|f_n\| < \infty$, then $f$ with*

$$f((x_n)_n) = \sum_{n=0}^{\infty} f_n(x_n), \qquad (x_n)_n \in X, \tag{6}$$

*is a well-defined element of $X^*$ with $\|f\| = \sum_{n=0}^{\infty} \|f_n\|$. Furthermore, every element of $X^*$ is of this form.*

*Proof.* This is a straightforward generalization of [Megginson 1998, Example 1.10.4]. □

Consider the computable Banach space $(Z, \|\cdot\|, e)$ from Lemma 1. Define

$$Y := (Z \times Z \times \cdots)_{c_0}.$$

For every $n \in \mathbb{N}$, let $\mathrm{proj}^{(n)} : Y \to Z$ and $\mathrm{emb}^{(n)} : Z \to Y$ be given by

$$\mathrm{proj}^{(n)}((z_i)_i) := z_n, \qquad (z_i)_i \in Y$$

and

$$\mathrm{emb}^{(n)}(z) := (0, \ldots, 0, \underbrace{z}_{\text{index } n}, 0, 0, \ldots), \qquad z \in Z.$$

Apply Lemma 5 to $(Z, \|\cdot\|, e)$; let $\sigma$ and $(x_i)_i$ be as in the statement of that lemma. For every $n \in \mathbb{N}$, put

$$Z_n := [x_0, \ldots, x_{\sigma(n)}].$$

For every $\tau : \mathbb{N} \to \mathbb{N}$, define

$$Y_\tau := (Z_{\tau(0)} \times Z_{\tau(1)} \times \cdots)_{c_0}.$$

The fact that the $Z_n$ have uniformly bounded basis constants goes into the proof of the following proposition:

**Proposition 10.** *Let $\tau : \mathbb{N} \to \mathbb{N}$ be arbitrary. Then $Y_\tau$ has a shrinking basis.*

*Proof.* By Lemma 5, there is a constant $C$ such that every $Z_n$ has a basis $a_{n,0}, \ldots, a_{n,\sigma(n)}$ with basis constant less than $C$. For every $n \in \mathbb{N}$ and $0 \leq i \leq \sigma(\tau(n))$, put

$$b_{n,i} := \mathrm{emb}^{(n)}(a_{\tau(n),i}).$$

We will show that

$$b_{0,0}, \ldots, b_{0,\sigma(\tau(0))},\ b_{1,0}, \ldots, b_{1,\sigma(\tau(1))},\ \ldots,\ \ldots \tag{7}$$

is a shrinking basis of $Y_\tau$.

Let $(z_n)_n \in Y_\tau$ be arbitrary. Suppose that there exists an expansion

$$\alpha_{0,0}b_{0,0}+\cdots+\alpha_{0,\sigma(\tau(0))}b_{0,\sigma(\tau(0))} + \alpha_{1,0}b_{1,0}+\cdots+\alpha_{1,\sigma(\tau(1))}b_{1,\sigma(\tau(1))} + \ldots + \ldots \tag{8}$$

of $(z_n)_n$ with respect to the sequence from (7). For every $n$, the continuity of $\mathrm{proj}^{(n)}$ yields

$$z_n = \sum_{i=0}^{\sigma(\tau(n))} \alpha_{n,i} a_{\tau(n),i},$$

so $\alpha_{n,0}, \dots, \alpha_{n,\sigma(\tau(n))}$ must be the unique coordinates of $z_n$ with respect to the basis

$$a_{\tau(n),0}, \dots, a_{\tau(n),\sigma(\tau(n))} \tag{9}$$

of $Z_{\tau(n)}$. Every element of $Y_\tau$ thus has at most one expansion with respect to the sequence from (7).

To show that (7) is a basis, it remains to show that (8) converges to $(z_n)_n \in Y_\tau$ if the $\alpha_{n,i}$ are chosen such that $\alpha_{n,0}, \dots, \alpha_{n,\sigma(\tau(n))}$ is the expansion of $z_n$ with respect to the basis (9) for every $n$. The partial sums of the series (8) have the form

$$\sum_{n=0}^{M-1} \sum_{i=0}^{\sigma(\tau(n))} \alpha_{n,i} b_{n,i} + \sum_{i=0}^{N} \alpha_{M,i} b_{M,i}$$

with $N, M \in \mathbb{N}$, $0 \leq N \leq \sigma(\tau(M))$. We have the following estimate for the distance to $(z_n)_n$:[5]

$$\begin{aligned}
&\Big\|(z_n)_n - \Big(\sum_{n=0}^{M-1} \sum_{i=0}^{\sigma(\tau(n))} \alpha_{n,i} b_{n,i} + \sum_{i=0}^{N} \alpha_{M,i} b_{M,i}\Big)\Big\| \\
&= \Big\| \sum_{n=0}^{\infty} \mathrm{e}^{(n)}(z_n) - \sum_{n=0}^{M-1} \sum_{i=0}^{\sigma(\tau(n))} \alpha_{n,i} \mathrm{e}^{(n)}(a_{\tau(n),i}) - \sum_{i=0}^{N} \alpha_{M,i} \mathrm{e}^{(M)}(a_{\tau(M),i}) \Big\| \\
&= \Big\| \sum_{n=0}^{\infty} \mathrm{e}^{(n)}(z_n) - \sum_{n=0}^{M-1} \mathrm{e}^{(n)}\Big( \underbrace{\sum_{i=0}^{\sigma(\tau(n))} \alpha_{n,i} a_{\tau(n),i}}_{=z_n} \Big) - \mathrm{e}^{(M)}\Big( \sum_{i=0}^{N} \alpha_{M,i} a_{\tau(M),i} \Big) \Big\| \\
&= \Big\| \sum_{n=M}^{\infty} \mathrm{e}^{(n)}(z_n) - \mathrm{e}^{(M)}\Big( \sum_{i=0}^{N} \alpha_{M,i} a_{\tau(M),i} \Big) \Big\| \\
&= \max\Big( \sup_{n>M} \|z_n\|,\ \Big\| z_M - \sum_{i=0}^{N} \alpha_{M,i} a_{\tau(M),i} \Big\| \Big) \\
&\leq \max\Big( \sup_{n>M} \|z_n\|,\ \|z_M\| + \Big\| \sum_{i=0}^{N} \alpha_{M,i} a_{\tau(M),i} \Big\| \Big) \\
&\leq \max\Big( \sup_{n>M} \|z_n\|,\ \|z_M\| + C\|z_M\| \Big).
\end{aligned}$$

In view of the fact that $\|z_M\| \to 0$ as $M \to \infty$, this estimate yields the convergence of (8) to $(z_n)_n$.

It remains to show that the basis (7) is shrinking. Let $f \in (Y_\tau)^*$ be arbitrary. $f$ has the form

$$f((z_n)_n) = \sum_{n=0}^{\infty} f_n(z_n)$$

[5] In this equation only, we abbreviate emb with e

with certain $f_n \in (Z_{\tau(n)})^*$ $(n \in \mathbb{N})$, and $\|f\| = \sum_{n=0}^{\infty} \|f_n\|$ (see Lemma 9). For every $\ell$, let $B_\ell$ be the closed unit ball in the subspace

$$[b_{\ell,0}, \ldots, b_{\ell,\sigma(\tau(\ell))},\ b_{\ell+1,0}, \ldots, b_{\ell+1,\sigma(\tau(\ell+1))},\ \ldots,\ \ldots] \\ = (\underbrace{\{0\} \times \cdots \times \{0\}}_{\ell\text{-times}} \times Z_{\tau(\ell)} \times Z_{\tau(\ell+1)} \times \cdots)_{c_0}$$

of $Y$. Then (again by Lemma 9)

$$\sup\{|f(y)|\ :\ y \in B_\ell\} = \sum_{n=\ell}^{\infty} \|f_n\|,$$

so

$$\lim_{\ell\to\infty} \sup\{|f(y)|\ :\ y \in B_\ell\} = 0.$$

This completes the proof. □

The following lemma and its corollary will be useful in the next section:

**Lemma 11.** *Let $X$ be a Banach space that has AP, and let $V$ be a closed subspace such that there is a linear bounded $F : X \to X$ with* range$(F) = V$ *and $F(v) = v$ for every $v \in V$. Then $V$ has AP.*

*Proof.* The claim is trivial if $V = \{0\}$. So suppose otherwise. Then necessarily $\|F\| \geq 1$, in particular $\mathcal{F}\| \neq 0$. Let $K$ be compact in $V$ and $\varepsilon > 0$ arbitrary. $K$ is also compact in $X$. As $X$ has AP, there is a finite-rank linear $G : X \to X$ such that

$$\sup_{x\in K} \|G(x) - x\| \leq \varepsilon \|F\|^{-1}.$$

Put $G' := F \circ G|_V$. Then $G'$ has finite rank. Furthermore

$$\sup_{x\in K} \|G'(x) - x\| = \sup_{x\in K} \|F(G(x)) - F(x)\| \leq \|F\| \sup_{x\in K} \|G(x) - x\| \leq \varepsilon.$$

□

**Corollary 12.** *Let $(y_i)_i \in Y^\omega$ be a basic sequence. Then*

$$\mathrm{emb}^{(n)}(Z) \not\subseteq [y_0, y_1, \ldots].$$

*for every $n \in \mathbb{N}$.*

*Proof.* Let us assume that $\mathrm{emb}^{(n)}(Z) \subseteq [y_0, y_1, \ldots]$ for some $n$. The space $X := [y_0, y_1, \ldots]$ has a basis and hence has AP. The mapping

$$F := \mathrm{emb}^{(n)} \circ \mathrm{proj}^{(n)}|_X$$

is linear and bounded on $X$ with range$(F) = \mathrm{emb}^{(n)}(Z)$ and $F(v) = v$ for every $v \in \mathrm{emb}^{(n)}(Z)$. The previous lemma yields that $\mathrm{emb}^{(n)}(Z)$ has AP. $\mathrm{emb}^{(n)}(Z)$, however, is isometrically isomorphic to $Z$, which lacks AP. Contradiction! □

Let us finally equip $Y$ and the $Y_\tau$ with computability structures: It is straightforward to verify that $(Y, \|\cdot\|, h)$ with

$$h(\langle n, i\rangle) := \mathrm{emb}^{(n)}(e(i)), \qquad n, i \in \mathbb{N},$$

is a computable Banach space. Recall that a function $\tau : \mathbb{N} \to \mathbb{N}$ is *lower semicomputable* if there is a computably enumerable set $N \subseteq \mathbb{N}$ with

$$\tau(n) = \sup\{k \in \mathbb{N} \ : \ \langle n, k\rangle \in N\}, \qquad n \in \mathbb{N}.$$

If $\tau$ is lower semicomputable, it is easy to see that there is a computable enumeration $h_\tau : \mathbb{N} \to Y$ of the set

$$\{\mathrm{emb}^{(n)}(x_i) \ : \ n, i \in \mathbb{N}, \, 0 \leq i \leq \sigma(\tau(n))\};$$

the span of this set is dense in $Y_\tau$. We learn the following from [Brattka 2001, Proposition 3.10]:

**Lemma 13.** *Let $\tau : \mathbb{N} \to \mathbb{N}$ be lower semicomputable. Then $(Y_\tau, \|\cdot\|, h_\tau)$ is a computable Banach space. The injection $Y_\tau \hookrightarrow Y$ is computable.* $\square$

## 5 The diagonalization construction

In this section, we will prove our main result:

**Theorem 14.** *There exists a lower-semicomputable $\tau : \mathbb{N} \to \mathbb{N}$ such that the computable Banach space $(Y_\tau, \|\cdot\|, h_\tau)$ as defined above possesses a basis, but does not possess any computable basis.*

In view of the results of the previous section, it remains to construct a lower-semicomputable $\tau$ such that $(Y_\tau, \|\cdot\|, h_\tau)$ does not possess any computable basis. By Lemma 13, every computable sequence in $Y_\tau$ is computable in $Y$. So it is sufficient to compute $\tau$ such that every computable sequence $(y_i)_i \in Y^\omega$ has one of the following two properties:

- $(y_i)_i$ is not basic.
- $Y_\tau \not\subseteq [y_0, y_1, \ldots]$.

We will proceed by diagonalization over all computable sequences in $Y$. We first note the following fact which follows immediately from the definition of the Cauchy representation: Let $\alpha_h$ be a canonical notation of the rational span of $h$ (cf. [Brattka and Dillhage 2007]). For every computable sequence $(y_i)_i \in Y^\omega$, there is a total computable function $\psi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that $(\alpha_h(\psi(i,k)))_k$ converges rapidly to $y_i$ for every $i$. It is well-known that there exists a universal partial computable $\Psi :\subseteq \mathbb{N}\times\mathbb{N}\times\mathbb{N} \to \mathbb{N}$; that means, for every partial computable $\psi :\subseteq \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, there is an $n \in \mathbb{N}$ such that

$$(n, i, k) \in \mathrm{dom}(\Psi) \;\Leftrightarrow\; (i, k) \in \mathrm{dom}(\psi)$$

and

$$(i, k) \in \mathrm{dom}(\psi) \;\Rightarrow\; \Psi(n, i, k) = \psi(i, k)$$

for all $i, k \in \mathbb{N}$. In this case, $n$ is called a *Gödel number* of $\psi$. Let $\psi_n$ be the partial computable function with Gödel number $n$. Denote by TOT the set of all $n$ such that $\psi_n$ is total. Denote by SEQ the set of all $n$ such that $\psi_n$ corresponds to a computable sequence in $Y$ as described above.

**Lemma 15.** *There is a computably enumerable set $M \subseteq \mathbb{N}$ with*

$$\mathrm{TOT} \setminus \mathrm{SEQ} \subseteq M \subseteq \mathbb{N} \setminus \mathrm{SEQ}.$$

*Proof.* $M$ shall be defined as the set of all $n$ with

$$\begin{aligned}(\exists\, i \in \mathbb{N})(\exists\, k \in \mathbb{N})(\exists\, j \in \mathbb{N}) \,\big(j > k, \quad (i, k) \in \mathrm{dom}(\psi_n), \quad (i, j) \in \mathrm{dom}(\psi_n),\\ \text{and} \quad \|\alpha_h(\psi_n(i, k)) - \alpha_h(\psi_n(i, j))\| > 2^{-k}\big),\end{aligned}$$

which is easily seen to be computably enumerable. The claimed inclusions follow directly from the definition of rapid convergence. □

Let $(n, k) \mapsto \langle n, k\rangle$ be a canonical bijective tupling $\mathbb{N}\times\mathbb{N} \to \mathbb{N}$. We now define a lower-semicomputable $\tau : \mathbb{N} \to \mathbb{N}$ by giving an algorithm that enumerates a set $L\langle n, k\rangle \subseteq \mathbb{N}$ with

$$\tau\langle n, k\rangle = \max(L\langle n, k\rangle \cup \{0\}) \tag{10}$$

for any given $\langle n, k\rangle \in \mathbb{N}$.

We begin the description of the algorithm: Let $\langle n, k\rangle$ be given. The procedure consists of four parallel processes **A**, **B**, **C**, **D**. The set $L\langle n, k\rangle$ is defined to be the intersection of the sets put out by processes **A** and **D**. Each process can make a `terminate` call that causes itself and the other three processes to terminate immediately; this shall be the only way that a process can interfere with another process's execution.

**Process A** runs a loop over $\ell = 0, 1, 2, \ldots$. In the body of the loop, $\psi_n$ is called with input $(i, j)$ chosen such that $\ell = \langle i, j\rangle$. If this call returns, $\ell$ is put out and the loop continues.

**Process B** runs a semidecision procedure for "$n \in M$", where $M$ is the set from Lemma 15. As soon as "$n \in M$" (if ever), the process calls `terminate`.

We will only define the behaviour of processes **C**, **D** for $n \in \mathrm{SEQ}$. For $n \notin \mathrm{SEQ}$, the behaviour of **C** and **D** shall be undefined. So let $(y_i)_i \in Y^\omega$ be the computable sequence corresponding to $n$.

**Process C** performs an exhaustive search for $\ell, m \in \mathbb{N}$, $\ell \leq m$, $\alpha_0, \ldots, \alpha_m \in \mathbb{Q}[\mathrm{i}]$ with

$$\Big\| \sum_{i=0}^{\ell} \alpha_i y_i \Big\| > k \Big\| \sum_{i=0}^{m} \alpha_i y_i \Big\|.$$

Once such numbers are found, the process calls `terminate`.

**Process D** performs a loop over $\ell = 0, 1, \ldots$. In the body of the loop, first $\ell$ is put out, then an exhaustive search for elements $\widetilde{x}_0^{(\ell)}, \ldots, \widetilde{x}_{\sigma(\ell)}^{(\ell)}$ of the rational span of $\{y_i \ : \ i \in \mathbb{N}\}$ with

$$(\forall\, 0 \leq i \leq \sigma(\ell)) \ \big(\|\mathrm{emb}^{(\langle n,k\rangle)}(x_i) - \widetilde{x}_i^{(\ell)}\| < 2^{-\ell}\big)$$

is performed.[6] In case such elements are found, the loop continues.

This completes the description of the algorithm.

$\tau : \mathbb{N} \to \mathbb{N}$ is well-defined by (10) if, and only if, $L\langle n,k\rangle$ is finite for all $n, k$. So we have to make sure that the output of either **A** or **D** is finite:

**Case 1:** $n \notin$ TOT. Then **A** will sooner or later call the function $\psi_n$ with an argument from outside $\mathrm{dom}(\psi_n)$. This call will not return, so the process will "hang" and not produce any more output.

**Case 2:** $n \in$ TOT $\setminus$ SEQ. **B** will sooner or later detect that $n \in M$, so all processes are terminated after finite time.

**Case 3:** Otherwise. Then $n \in$ SEQ. Let $(y_i)_i$ be the corresponding sequence.

**Case 3a:** The nonzero elements of $(y_i)_i$ form a basic sequence. We show that the loop in process **D** will only be iterated a finite number of times: Suppose the contrary. Then all $\mathrm{emb}^{(\langle n,k\rangle)}(x_i)$, $i \in \mathbb{N}$, can be approximated arbitrarily well by elements from the rational span of $\{y_i \ : \ i \in \mathbb{N}\}$. This implies

$$\{\mathrm{emb}^{(\langle n,k\rangle)}(x_i) \ : \ i \in \mathbb{N}\} \subseteq [y_0, y_1, \ldots],$$

and thus

$$\begin{aligned} \mathrm{emb}^{(\langle n,k\rangle)}(Z) &= \mathrm{emb}^{(\langle n,k\rangle)}([x_0, x_1, \ldots]) \\ &= [\mathrm{emb}^{(\langle n,k\rangle)}(x_0), \mathrm{emb}^{(\langle n,k\rangle)}(x_1), \ldots] \\ &\subseteq [y_0, y_1, \ldots]. \end{aligned}$$

This contradicts Corollary 12.

**Case 3b:** Otherwise. The nonzero elements of $(y_i)_i$ do not form a basic sequence. Proposition 2 ensures that the exhaustive search performed by process **C** will succeed, so all processes will sooner or later be terminated.

It remains to show that $Y_\tau \not\subseteq [y_0, y_1, \ldots]$ for any computable basic sequence $(y_i)_i \in Y^\omega$: Let $n$ be a Gödel number of $(y_i)_i$. Choose $k \in \mathbb{N}$ greater then the basis constant of $(y_i)_i$. As

$$\mathrm{emb}^{(\langle n,k\rangle)}(Z_{\tau(\langle n,k\rangle)}) \subseteq Y_\tau,$$

[6] Recall that $\sigma$ and the $x_i$ were defined on page 11.

it is sufficient to show

$$\mathrm{emb}^{(\langle n,k\rangle)}(Z_{\tau(\langle n,k\rangle)}) \not\subseteq [y_0, y_1, \ldots].$$

This is fulfilled if, and only if,

$$\{\mathrm{emb}^{(\langle n,k\rangle)}(x_i) \ : \ 0 \le i \le \sigma(\tau(\langle n,k\rangle))\} \not\subseteq [y_0, y_1, \ldots]. \tag{11}$$

As $n \in \mathrm{SEQ}$, we have $n \notin M$, so process **B** will not terminate the other processes. As $k > \mathrm{bc}((y_i)_i)$, the exhaustive search performed by process **C** will not succeed, so **C** will not terminate the other processes, either. Process **A** will enumerate the entire set $\mathbb{N}$. Together, this implies that $L\langle n,k\rangle$ is equal to the output of process **D**. Consider the final iteration of the loop in process **D**, that means $\ell = \tau(\langle n,k\rangle)$. The exhaustive search in the body of the loop does not succeed (otherwise, this were not the final iteration). This directly implies (11). The proof is complete.

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