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Fine-computable Functions on the Unit Square and their Integral

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Abstract: We discuss the integral and Fubini's Theorem for a Fine-computable function $F(x, y)$ on the upper-right open unit square $[0, 1) \times [0, 1)$. The core objective is Fine-computability of $f(x) = \int_{[0,1)} F(x, y)dy$ as a function of $x \in [0, 1)$.

Key Words: Fine-computable function, Fubini's Theorem, integral operator.
Category: F.0, F.m

1 Introduction

Notions of Fine-continuity and of Fine-computabilities on $[0, 1)$ are defined with respect to the Fine-topology, which is equivalent to the one defined by the Fine-metric (cf. Section 2, [Fine 1949], [Mori 2002a], [Mori et al. 2007]). We note that a Fine-computable function may be discontinuous at dyadic rationals and may be unbounded (cf. [Mori 2002a] Example 4.3). We have defined effective integrability for Fine-computable functions on $[0, 1)$ and effectivized some fundamental theorems of integral theory [Mori et al. 2007], [Mori et al. 2008b].

We then studied some notions of Fine-computability of functions on the upper-right open unit square $[0, 1)^2$ as well as some properties of their integrals [Mori et al. 2008c]. This article has come out ot it, extended and revized.

In classical analysis, the integral operator with a kernel $F(x, y)$, which maps a function $g(x)$ on X to $(Tg)(x) = \int_X g(y)F(x, y)dy$, is a central subject. Measurability and integrability of Tg are fundamental properties to be proved and Fubini's Theorem is a fundamental tool to deal with investigations of such an operator.

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Theorem 1. (Fubini's Theorem) *Let $F(x, y) \geqslant 0$ be a measurable and integrable function on the upper-right open unit square $[0,1)^2$. Then the following holds.*

(i) *For almost all x, $F(x,\cdot)$ and $F(\cdot,x)$ are measurable and integrable.*
(ii) $\int_{[0,1)} F(x,y)dy$ *and* $\int_{[0,1)} F(x,y)dx$ *are measurable.*
(iii) $\iint_{[0,1)^2} F(x,y)dxdy$
$$= \int_{[0,1)} \left(\int_{[0,1)} F(x,y)dy \right) dx = \int_{[0,1)} \left(\int_{[0,1)} F(x,y)dx \right) dy.$$

In this article, we discuss an effectivization of Fubini's Theorem for a uniformly Fine-computable function on $[0,1)^2$ (Definition 23) and for bounded Fine-computable functions (Definition 28). We also make some observations on the transformation T. In effectivization, *Fine-computability* and *effective integrability* correspond to *measurability* and *integrability* respectively.

From the standpoint of computable analysis, it is plausible that $f(x) = \int_{[0,1)} F(x,y)dy$ is defined everywhere on $[0,1)$ and $f(x)$ is Fine-computable for a Fine-computable function $F(x,y)$ on $[0,1)^2$. So, we assume that $F(x,y)$ is integrable with respect to y for all $x \in [0,1)$.

Since Fine-computable functions are continuous at all dyadically irrational points with respect to the Euclidean topology, they are measurable, and Fubini's Theorem holds classically for integrable Fine-computable functions. Therefore, effectivization of Fubini's Theorem boils down to the proof of Fine-computability of $f(x)$, and hence this property is the main objective of this paper.

Roughly speaking, continuity of Tg is deduced from that of $F(x,y)$. Hence, by modifying the proof of Fine-computability of $f(x)$, we can easily prove Fine-computability of Tg under some suitable conditions.

Our main assertions are that Fine-computability of $f(x)$ holds for a "uniformly Fine-computable" function $F(x,y)$ and for a "bounded Fine-computable" function $F(x,y)$.

In Section 2, we review Fine-computability and effective integrability for a function on $[0,1)$.

In Section 3, we define the two-dimensional Fine-space and notions of Fine-computability and prove prove some elementary properties.

In Section 4, we prove that $f(x) = \int_{[0,1)} F(x,y)dy$ is uniformly Fine-computable if $F(x,y)$ is uniformly Fine-computable (Theorem 26).

In Section 5, we prove that $f(x)$ is Fine-computable for a bounded Fine-computable $F(x,y)$ (Theorem 32).

In Section 6, we give such examples that Fine-computability of $f(x)$ does not hold in general and give an sufficient condition for $F(x,y)$ to assure that $f(x)$ is Fine-computable.

Consult [Fine 1949] as to Fine-continuous functions on $[0,1)$.

2 Preliminaries

We summarize Fine-computability properties on $[0,1)$ and effective integrability of such functions (cf. [Mori et al. 2007], [Mori et al. 2008a], [Mori et al. 2008b]). We assume basic knowledge of computability on the Euclidean space (cf. [Pour-El and Richards 1989]).

A left-closed right-open interval with dyadic end points is called a *dyadic interval.* We call $I(n,k) = [k2^{-n},(k+1)2^{-n})$ a *fundamental dyadic interval* (of level n) and $J(x,n)$, the unique fundamental dyadic interval $I(n,k)$ which contains x, the *fundamental dyadic neighborhood* of x (of level n).

Lemma 2. [Mori et al. 2007] (1) *The following three properties are equivalent for any $x, y \in [0,1)$ and any nonnegative integer n.*

(i) $y \in J(x,n)$. (ii) $x \in J(y,n)$. (iii) $J(x,n) = J(y,n)$.

(2) *If $\{x_m\}$ is Fine-computable, then we can decide effectively whether $x_m \in I(n,k)$ or not for all m, n and $k, 0 \leq k \leq 2^n - 1$.*

$\{J(x,n)\}$ satisfies the axioms of the effective uniformity [Tsujii et al. 2001]. We call the topology generated by $\{I(n,k)\}$ the *Fine topology* and put prefix *Fine-* to such notions. We put no prefix to the notions which are defined by means of Euclidean topology.

A double sequence of dyadic rationals $\{r_{n,m}\}$ is said to be *recursive* if there exist recursive functions $\alpha(n,m), \beta(n,m)$ such that $r_{n,m} = \beta(n,m)2^{-\alpha(n,m)}$.

Definition 3. (1) (Effective Fine-convergence of reals) A double sequence $\{x_{n,m}\}$ is said to *Fine-converge effectively* to $\{x_n\}$ if there exists a recursive function $\alpha(n,k)$ which satisfies that $m \geqslant \alpha(n,k)$ implies $x_{n,m} \in J(x_n,k)$.

(2) (Fine-computable sequence of reals) A sequence of real numbers $\{x_m\}$ in $[0,1)$ is said to be *Fine-computable* if there exists a recursive double sequence of dyadic rationals $\{r_{m,n}\}$ which Fine-converges effective ly to $\{x_m\}$.

If $x_{n,m} = x_m$ and $x_n = x$, we obtain the definition of effective Fine-convergence of $\{x_m\}$ to x.

Remark. (1) The original definition of a Fine-computable sequence of real numbers is that $\{r_{n,m}\}$ be a recursive sequence of rational numbers. The present definition is equivalent to the original one. (cf. [Yasugi et al. 2005])

(2) The set of computable numbers and that of Fine-computable numbers coincide.

(3) A Fine-computable sequence is (Euclidean) computable, but the converse fails [Yasugi et al. 2005].

(4) $\{e_i\}$ will denote an effective enumeration of all dyadic rationals in $[0,1)$. It is an effective separating set of the Fine-space $[0,1)$ (cf. [Mori et al. 1996]).

Lemma 4. (Monotone convergence, [Pour-El and Richards 1989]) *Let $\{x_{n,k}\}$ be a computable sequence of reals which converges monotonically to $\{x_n\}$ as k tends to infinity for each n. Then $\{x_n\}$ is computable if and only if the convergence is effective.*

We will subsequently use this lemma without mention.

Definition 5. (Uniformly Fine-computable sequence of functions, [Mori 2002a], [Mori et al. 2007]) A sequence of functions $\{f_n\}$ is said to be *uniformly Fine-computable* if (i) and (ii) below hold.

(i) (Sequential Fine-computability) The double sequence $\{f_n(x_m)\}$ is computable for any Fine-computable sequence $\{x_m\}$.

(ii) (Effectively uniform Fine-continuity) There exists a recursive function $\alpha(n,k)$ such that, for all n,k and all $x,y \in [0,1)$, $y \in J(x,\alpha(n,k))$ implies $|f_n(x) - f_n(y)| < 2^{-k}$.

Definition 6. (Effectively uniform convergence of functions, [Mori 2002a], [Mori et al. 2007]). A double sequence of functions $\{g_{m,n}\}$ is said to *converge effectively uniformly* to a sequence of functions $\{f_m\}$ if there exists a recursive function $\alpha(m,k)$ such that, for all m,n and k, $n \geqslant \alpha(m,k)$ implies $|g_{m,n}(x) - f_m(x)| < 2^{-k}$ for all $x \in [0,1)$.

Definition 7. (Fine-computable sequence of functions, [Mori et al. 2007]) A sequence of functions $\{f_n\}$ is said to be *Fine-computable* if it satisfies the following.

(i) $\{f_n\}$ is sequentially Fine-computable.

(ii) (Effective Fine-Continuity) There exists a recursive function $\alpha(n,k,i)$ such that

(ii-a) $x \in J(e_i,\alpha(n,k,i))$ implies $|f_n(x) - f_n(e_i)| < 2^{-k}$,

(ii-b) $\bigcup_{i=1}^{\infty} J(e_i,\alpha(n,k,i)) = [0,1)$ for each n,k.

Definition 8. (Effective Fine-convergence of functions, [Mori et al. 2007]) We say that a double sequence of functions $\{g_{m,n}\}$ *Fine-converges effectively* to a sequence of functions $\{f_m\}$ if there exist recursive functions $\alpha(m,k,i)$ and $\beta(m,k,i)$, which satisfy

(a) $x \in J(e_i,\alpha(m,k,i))$ and $n \geqslant \beta(m,k,i)$ imply $|g_{m,n}(x) - f_m(x)| < 2^{-k}$,

(b) $\bigcup_{i=1}^{\infty} J(e_i,\alpha(m,k,i)) = [0,1)$ for each m and k.

Definition 9. (Computable sequence of dyadic step functions, [Mori 2002a], [Mori et al. 2007]) A sequence of functions $\{\varphi_n\}$ is called a *computable sequence of dyadic step functions* if there exist a recursive function $\alpha(n)$ and a computable sequence of reals $\{c_{n,j}\}$ ($0 \leqslant j < 2^{\alpha(n)}$, $n = 1,2,\ldots$) such that

$$\varphi_n(x) = \sum_{j=0}^{2^{\alpha(n)}-1} c_{n,j}\chi_{I(\alpha(n),j)}(x),$$

where χ_A denotes the indicator (characteristic) function of A.

Proposition 10. [Mori et al. 2007] *Let f be a Fine-computable function. The computable sequence of dyadic step functions $\{\varphi_n\}$, which is defined by*

$$\varphi_n(x) = \sum_{j=0}^{2^n-1} f(j2^{-n})\chi_{I(n,j)}(x), \tag{1}$$

Fine-converges effectively to f.

Moreover, if f is uniformly Fine-computable, then $\{\varphi_n\}$ converges effectively uniformly to f.

We will briefly review effective integrability of functions on $[0, 1)$. For details, see [Mori et al. 2007], [Mori et al. 2008a], [Mori et al. 2008b].

Definition 11. (Effective integrability of a sequence of functions, [Mori et al. 2008a], [Mori et al. 2008b]) A sequence of Fine-computable functions $\{f_n\}$ is called *effectively integrable* if each f_n is integrable and both of $\{\int_{[0,1)} f_n^+(x)dx\}$ and $\{\int_{[0,1)} f_n^-(x)dx\}$ are computable sequences of real numbers.

A Fine-computable function is said to be *effectively integrable* if the sequence $f, f, \ldots$ is effectively integrable.

Integral on a finite union of fundamental dyadic intervals E is defined to be $\int_{[0,1)} f(x)\chi_E(x)dx$.

It is easy to prove that a computable sequence of dyadic step functions is effectively integrable.

Theorem 12. *Let $\{g_n\}$ be a uniformly bounded Fine-computable sequence of functions which is effectively integrable and Fine-converges effectively to f. Then, f is Fine-computable and $\{\int_{[0,1)} g_n(x)dx\}$ converges effectively to $\int_{[0,1)} f(x)dx$. As a consequence, f is effectively integrable.*

Theorem 13. [Mori et al. 2008a], [Mori et al. 2008b]
A bounded Fine-computable function is effectively integrable.

Theorem 14. [Mori et al. 2008a], [Mori et al. 2008b] *Let $\{f_n\}$ be Fine-computable and effectively bounded, that is, there exists a computable sequence of reals $\{M_n\}$ such that $|f_n(x)| \leqslant M_n$ for all x. Then $\{f_n\}$ is effectively integrable.*

Theorem 15. (Effective dominated convergence theorem, [Mori et al. 2008a], [Mori et al. 2008b]) *Let $\{g_n\}$ be an effectively integrable Fine-computable sequence which Fine-converges effectively to f. Suppose that there exists an effectively integrable Fine-computable function h such that $|g_n(x)| \leqslant h(x)$. Then, $\{\int_{[0,1)} g_n(x)dx\}$ converges effectively to $\int_{[0,1)} f(x)dx$.*

Proposition 3.10 in [Mori et al. 2008b] can be easily extended.

Proposition 16. [Mori et al. 2008b] *Let f be a nonnegative integrable Fine-computable function. Then f is effectively integrable if and only if $\{\int_{[0,1)} g_n(x)dx\}$ converges effectively to $\int_{[0,1)} f(x)dx$ for an effectively integrable Fine-computable sequence $\{g_n\}$ which Fine-converges effectively to f and satisfies $|g_n(x)| \leqslant f(x)$ for every n and x.*

Proposition 17. [Mori et al. 2008b] *Let f be an effectively integrable Fine-computable function and let I_n be a computable sequence of dyadic intervals such that $\bigcup_{n=1}^{\infty} I_n = [0,1)$. Put $E_n = \bigcup_{i=1}^{n} I_i$. Then, $\int_{E_n} f(x)dx$ converges effectively to $\int_{[0,1)} f(x)dx$, or equivalently, $\int_{E_n^C} f(x)dx$ converges effectively to zero.*

Definition 18. A sequence of sets $\{E_k\}$ from $[0,1)$ is said to be a *computable sequence of elementary sets* if there exist a recursive function $N(k)$ and recursive sequences of dyadic rationals $\xi(k,\ell)$ and $\eta(k,\ell)$ $(\ell \leqslant N(k))$ such that $E_k = \bigcup_{\ell=1}^{N(k)} I(\xi(k,\ell),\eta(k,\ell))$. We say that a set E is a *computable elementary set* if $\{E, E, \ldots\}$ is a computable sequence of elementary sets.

We can also prove the following proposition.

Proposition 19. *Let f be an integrable positive Fine-computable function. Then, f is effectively integrable if and only if there exists a computable sequence of elementary sets $\{E_n\}$ such that $\{\int_{E_n} f(x)dx\}$ is a computable sequence and converges effectively to $\int_{[0,1)} f(x)dx$. The latter condition is equivalent to effective convergence to zero of $\{\int_{E_n^C} f(x)dx\}$.*

Definition 20. Let $\{f_n\}$ be a computable sequence of Fine-computable functions, where each f_n is integrable, and $\{E_m\}$ be a computable sequence of elementary sets. Then, $\{f_n\}$ is said to be effectively integrable on $\{E_m\}$ if $\{\int_{E_m} f_n(x)dx\}$ is a computable sequence.

3 Uniformly Fine-computable functions on $[0,1)^2$

The main objective of this section is to prove uniform Fine-computability of $f(x) = \int_{[0,1)} F(x,y)dy$ for a uniformly Fine-computable function $F(x,y)$ on the upper-right open unit square $[0,1)^2$.

We denote $[k2^{-n},(k+1)2^{-n}) \times [\ell 2^{-m},(\ell+1)2^{-m})$ with $I_2(n,m;k,\ell)$ and call it a *fundamental dyadic rectangle*. We also denote $J(x,n) \times J(y,m)$ by $J_2(x,y;n,m)$ and call it a *fundamental dyadic neighborhood* of (x,y). We call the topology generated by the set $\{J_2(e_i,e_j;n,m)\}_{i,j,n,m}$ the *Fine-topology* on $[0,1)^2$ and the space $[0,1)^2$ with this topology the *two-dimensional Fine-space*. Notions of computability on $[0,1)^2$ are defined with respect to the Fine-topology.

Note that $\{J_2(x,y;n,n)\}$ satisfies the axioms of the effective uniformity (cf. [Tsujii et al. 2001]).

Definition 21. (1) A double sequence $\{(x_{p,q}, y_{p,q})\}$ from $[0,1)^2$ is said to *Fine-converge effectively* to $\{(x_p, y_p)\}$ if there exists a recursive function $\alpha(p,n,m)$ such that $q \geqslant \alpha(p,n,m)$ implies $(x_{p,q}, y_{p,q}) \in J_2(x_p, y_p; n, m)$.

(2) A sequence $\{(x_p, y_p)\}$ is said to be *Fine-computable* if there exist recursive sequences of dyadic rationals $\{s_{p,q}\}$ and $\{t_{p,q}\}$ such that $\{s_{p,q}\}$ and $\{t_{p,q}\}$ Fine-converge effectively to $\{x_p\}$ and $\{y_p\}$ respectively.

Lemma 22. (cf. Lemma 2) (1) *The following three properties are equivalent for any $(x,y),(z,w) \in [0,1)^2$ and any positive integers n,m.*

(i) $(z,w) \in J_2(x,y;n,m)$. (ii) $(x,y) \in J_2(z,w;n,m)$. (iii) $J(x,y;n,m) = J(z,w;n,m)$.

(2) *If $\{(x_p, y_p)\}$ is Fine-computable, then we can decide effectively whether $(x_p, y_p) \in I_2(n,m;k,\ell)$ or not.*

In the following, we use the notation $F(x,\cdot)$ to designate the function $F(x,y)$ regarded as a function of y (for each fixed x).

Definition 23. (Uniform Fine-computability) A function $F(x,y)$ on $[0,1)^2$ is said to be *uniformly Fine-computable* if it satisfies the following two conditions.

(i) (Sequential computability) $\{F(x_n, y_m)\}$ is a computable double sequence of reals for every Fine-computable sequence $\{(x_n, y_m)\}$.

(ii) (Effective uniform Fine-continuity) There exist recursive functions $\alpha_1(k)$ and $\alpha_2(k)$ such that $(x,y) \in J_2(z,w;\alpha_1(k),\alpha_2(k))$ implies $|F(x,y) - F(z,w)| < 2^{-k}$.

Proposition 24. *Let $F(x,y)$ be uniformly Fine-computable as a function of (x,y). Then the following hold.*

(1) *If $\{x_n\}$ is a Fine-computable sequence, then $\{f_n(y)\} = \{F(x_n, y)\}$ is a uniformly Fine-computable sequence of functions on* $[0,1)$ *(Definition 5).*

(2) *If a Fine-computable sequence $\{x_{m,n}\}$ Fine-converges effectively to $\{x_m\}$, then $\{F(x_{m,n},\cdot)\}$ converges effectively uniformly to $\{F(x_m,\cdot)\}$ (Definition 6).*

Proof. Let $\alpha_1(k)$ and $\alpha_2(k)$ be as in Definition 23.

(1) Let $\{y_m\}$ be a Fine-computable sequence of reals. Then $\{f_n(y_m)\} = \{F(x_n, y_m)\}$ is a computable sequence of reals due to the sequential computability of $F(x,y)$. $|f_n(y) - f_n(z)| = |F(x_n,y) - F(x_n,z)| < 2^{-k}$ if $y \in J(z,\alpha_2(k))$, and hence follows effective uniform Fine-continuity of $\{f_n\}$.

(2) From the effective Fine-convergence of $\{x_{m,n}\}$ to $\{x_m\}$, there exists a recursive function $\beta(m,\ell)$ such that $n \geq \beta(m,\ell)$ implies $x_{m,n} \in J(x_m,\ell)$.

If we take $\delta(m,k) = \beta(m,\alpha_1(k))$, then $|F(x_{m,n},y) - F(x_m,y)| < 2^{-k}$ for $n \geq \delta(m,k)$ and all $y \in [0,1)$. □

It is pointed out in [Mori 2002b] that a uniformly Fine-computable function $g(y)$ on $[0,1)$ is bounded and has a computable supremum. The latter property holds for a uniformly Fine-computable sequence of functions. These properties are easily deduced from Theorem 2 in [Mori 2002a]. We denote the supremum of $|g|$ by $||g||$.

Similarly, we can prove that a uniformly Fine-computable function $F(x,y)$ takes a computable supremum.

Regarding uniform Fine-computability of $F(x,y)$, we obtain the following theorem.

Theorem 25. *For a function $F(x,y)$, the following* (i) *and* (ii) *are equivalent.*

(i) *$F(x,y)$ is uniformly Fine-computable.*

(ii) (ii-a) *$\{F(x_n,\cdot)\}$ is a uniformly Fine-computable sequence of functions on* $[0,1)$ *for any Fine-computable sequence $\{x_n\}$.*

(ii-b) *There exists a recursive function $\alpha(k)$ such that, $y \in J(x,\alpha(k))$ implies $||F(x,\cdot) - F(y,\cdot)|| < 2^{-k}$ for all k.*

Proof. (i)⇒(ii): (ii-a) follows immediately from Proposition 24 (1).

To prove (ii-b), let us take $\alpha_1(k)$ and $\alpha_2(k)$ in Definition 23. If $x \in J(y,\alpha_1(k+1))$, then $(x,z) \in J_2(y,z;\alpha_1(k+1),\alpha_2(k+1))$ for all $z \in [0,1)$. So, $|F(x,z) - F(y,z)| < 2^{-(k+1)}$ and $||F(x,\cdot) - F(y,\cdot)|| < 2^{-k}$.

(ii)⇒(i): Let $\alpha(k)$ be the recursive function in (ii-b). Then, $z \in J(x,\alpha(k))$ implies $||F(x,\cdot) - F(z,\cdot)|| < 2^{-k}$. Put $r_{k,j} = j2^{-\alpha(k)}$ for $j = 0,1,\dots,2^{\alpha(k)}-1$. By (ii-a), the sequence $\{F(r_{k,j},\cdot)\}$ is a uniform Fine-computable sequence of functions on $[0,1)$. So, there exists a recursive function $\beta(k,j)$ such that $y \in J(w,\beta(k,j))$ implies $|F(r_{k,j},y) - F(r_{k,j},w)| < 2^{-k}$.

Define $\gamma(k) = \max\{\alpha(k+2),\beta(k+2,0),\beta(k+2,1),\dots,\beta(k+2,2^{\alpha(k+2)}-1)\}$ and suppose that $(x,y) \in J_2(z,w;\gamma(k),\gamma(k))$. Since $z \in J(x,\alpha(k+2))$, there exists a j, such that $[j2^{-\alpha(k+2)},(j+1)2^{-\alpha(k+2)})$ contains both x and z. Therefore, we obtain

$$
\begin{aligned}
&|F(x,y) - F(z,w)| \\
&\leqslant |F(x,y) - F(r_{k+2,j},y)| + |F(r_{k+2,j},y) - F(r_{k+2,j},w)| \\
&\qquad + |F(r_{k+2,j},w) - F(z,w)| \\
&< 3 \cdot 2^{-(k+2)} < 2^{-k}.
\end{aligned}
$$

This shows effective uniform Fine-continuity of $F(x,y)$.

Let $\{x_n\}$ and $\{y_m\}$ be Fine-computable sequences. Then $\{F(x_n,\cdot)\}$ is a uniformly Fine-computable sequence of functions. This implies that $\{F(x_n,y_m)\}$ is a computable sequence of reals. □

It is easy to check that a uniformly Fine-computable function on $[0,1)^2$ is Lebesgue integrable and that its integral is a computable number, similarly to the case of uniformly Fine-computable functions on $[0,1)$ [Mori et al. 2008a].

Theorem 26. (Effective Fubini's Theorem for uniform Fine-computable functions) *Let $F(x,y)$ be a uniformly Fine-computable function. Then the following hold.*

(i) *If $\{x_n\}$ is Fine-computable, then $\{F(x_n,\cdot)\}$ and $\{F(\cdot,x_n)\}$ are uniformly Fine-computable sequences of functions on $[0,1)$.*

(ii) *$\int_{[0,1)} F(x,y)dy$ and $\int_{[0,1)} F(x,y)dx$ are uniformly Fine-computable functions.*

(iii) *$\iint_{[0,1)^2} F(x,y)dxdy = \int_{[0,1)} dx \int_{[0,1)} F(x,y)dy = \int_{[0,1)} dy \int_{[0,1)} F(x,y)dx$ holds and the value is computable.*

Proof. (i) is Proposition 24 (1).

(ii) To prove sequential computability, let $\{x_n\}$ be a Fine-computable sequence. Then $\{F(x_n,\cdot)\}$ is a uniformly bounded uniformly Fine-computable sequence of functions. Hence, $\{\int_{[0,1)} F(x_n,y)dy\}$ is a computable sequence of reals by Theorem 14.

Effective uniform Fine-continuity follows from the inequality

$|\int_{[0,1)} F(x,y)dy - \int_{[0,1)} F(z,y)dy| \leqslant ||F(x,\cdot) - F(y,\cdot)||$

and Theorem 25 (ii-b).

(iii) follows from Theorem 13 and the comment before Theorem 25. □

We can easily extend (ii) above as follows.

Theorem 27. *Let $F(x,y)$ be a uniformly Fine-computable function on $[0,1)^2$ and let g be an effectively integrable Fine-computable function on $[0,1)$. Then $(Tg)(x) = \int_{[0,1)} g(y)F(x,y)dy$ is uniformly Fine-computable.*

Especially, the operator T maps any uniformly Fine-computable function to a uniformly Fine-computable function.

Proof. First, we note that $M = \sup_{(x,y)\in[0,1)^2} |F(x,y)|$ is computable if $F(x,y)$ is uniformly Fine-computable on $[0,1)^2$.

Let $\{x_m\}$ be Fine-computable. Then $\{g(y)F(x_m,y)\}$ is a Fine-computable sequence of functions of y by Theorem 26 (1). We take the approximating computable sequence of dyadic step functions $\{\varphi_{m,n}(y)\}$ obtained by Proposition 10. It Fine-converges effectively to $\{g(y)F(x_m,y)\}$, and it is an effectively integrable Fine-computable sequence satisfying $|\varphi_{m,n}(y)| \leqslant M|g(y)|$. Hence, $\{\int_{[0,1)} \varphi_{m,n}(y)dy\}$ converges effectively to $\{\int_{[0,1)} g(y)F(x_m,y)dy\}$ by Theorem 15. Therefore, $\{\int_{[0,1)} g(y)F(x_m,y)dy\}$ is a computable sequence.

Effective uniform continuity follows from the following inequality;

$|\int_{[0,1)} g(y)F(x,y)dy - \int_{[0,1)} g(y)F(z,y)dy| \leqslant ||F(x,\cdot) - F(z,\cdot)|| \int_{[0,1)} |g(z)|dz.$

□

4 Fine-computable functions on $[0,1)^2$

In the following, we treat Fine-computability of $f(x) = \int_{[0,1)} F(x,y)dy$ for a Fine-computable function $F(x,y)$. First we define Fine-computability of functions on $[0,1)^2$, which is weaker than uniform Fine-computability (Definition 23), as follows.

Definition 28. (Fine-computable functions on $[0,1)^2$) Let $F(x,y)$ be a function on $[0,1)^2$. F is said to be *Fine-computable* if it satisfies the following (i) and (ii).

(i) F is sequentially computable.

(ii) (Effective Fine-continuity) There exist recursive functions $\alpha_1(k,i,j)$ and $\alpha_2(k,i,j)$ which satisfy

(ii-a) $(x,y) \in J_2(e_i, e_j; \alpha_1(k,i,j), \alpha_2(k,i,j))$ implies $|F(x,y) - F(e_i,e_j)| < 2^{-k}$,

(ii-b) $\bigcup_{i,j=1}^{\infty} J_2(e_i, e_j; \alpha_1(k,i,j), \alpha_2(k,i,j)) = [0,1)^2$ for each k.

We state Proposition 3.1 in [Mori et al. 2007] for the case $\{r_i\} = \{e_i\}$.

Proposition 29. *A function g on $[0,1)$ is effectively Fine-continuous if and only if there exist a recursive sequence of dyadic rationals $\{r_{k,q}\}$ and a recursive function $\delta(k,q)$ which satisfy the following.*

(a) *$x \in J(r_{k,q}, \delta(k,q))$ implies $|g(x) - g(r_{k,q})| < 2^{-k}$.*

(b) *$\bigcup_{q=1}^{\infty} J(r_{k,q}, \delta(k,q)) = [0,1)$ for each k.*

(c) *The intervals in $\{J(r_{k,q}, \delta(k,q))\}$ are mutually disjoint with respect to q for each k.*

In the proof of Proposition 3.1 in [Mori et al. 2007], the crucial properties are those of Lemma 2, whose two-dimensional version is Lemma 22, and the fact that the complement of a finite (disjoint) union of fundamental dyadic intervals can be represented as a finite disjoint union of fundamental dyadic intervals. A similar fact also holds for fundamental dyadic rectangles. So, we can prove the following proposition.

Proposition 30. *Effective Fine-continuity of a function F on $[0,1)^2$ is equivalent to the following: There exist a recursive sequence of pairs of dyadic rationals $\{(s_{k,p}, t_{k,p})\}$ and recursive functions $\beta_1(k,p)$, $\beta_2(k,p)$ which satisfy the following three conditions.*

(a) *$(x,y) \in J_2(s_{k,p}, t_{k,p}; \beta_1(k,p), \beta_2(k,p))$ implies $|F(x,y) - F(s_{k,p}, t_{k,p})| < 2^{-k}$.*

(b) *$\bigcup_{p=1}^{\infty} J_2(s_{k,p}, t_{k,p}; \beta_1(k,p), \beta_2(k,p)) = [0,1)^2$ for each k.*

(c) *The fundamental dyadic neighborhoods in $\{J_2(s_{k,p}, t_{k,p}; \beta_1(k,p), \beta_2(k,p))\}$ are mutually disjoint with respect to p for each k.*

Remark. The conditions (b) and (c) in Proposition 30 signify that the unit square $[0,1)^2$ is partitioned into (infinitely many) disjoint rectangles $\{J_2(s_{k,p}, t_{k,p}; \beta_1(k,p), \beta_2(k,p))\}$ for each k. Hence, the following holds:

(a) There is the unique number $p(k,x,y)$ such that (x,y) is contained in $J_2(s_{k,p(k,x,y)}, t_{k,p(k,x,y)}; \beta_1(k, p(k,x,y)), \beta_2(k, p(k,x,y)))$, for any k and any $(x,y) \in [0,1)^2$.

Moreover, $(z,w) \in J_2(s_{k,p(k,x,y)}, t_{k,p(k,x,y)}; \beta_1(k, p(k,x,y)), \beta_2(k, p(k,x,y)))$ implies $p(k,x,y) = p(k,z,w)$.

(b) If $\{(x_n, y_n)\}$ is Fine-computable, then $i(k,n) = p(k, x_n, y_n)$ is a recursive function.

Proposition 31. *Let $F(x,y)$ be Fine-computable. Then the following hold.*

(1) *If $\{x_m\}$ is a Fine-computable sequence of reals, then $\{F(x_m, \cdot)\}$ is a Fine-computable sequence of functions.*

(2) *If $\{x_{m,n}\}$ is a Fine-computable sequence of reals and Fine-converges effectively to $\{x_m\}$, then $\{F(x_{m,n}, \cdot)\}$ Fine-converges effectively to $\{F(x_m, \cdot)\}$.*

Proof. Let us take $\{(s_{k,p}, t_{k,p})\}$ and $\beta_1(k,p)$, $\beta_2(k,p)$ in Proposition 30.

Proof of (1): We prove (i) and (ii) in Definition 7 for $\{F(x_m, \cdot)\}$.

(i): Sequential computability of $\{F(x_m, \cdot)\}$ is an easy consequence of sequential computability of $F(x,y)$.

(ii-a): For each m, k and j, we can find effectively and uniquely such $p = p(m,k,j)$ that (x_m, e_j) is contained in $J_2(s_{k+1,p}, t_{k+1,p}; \beta_1(k+1,p), \beta_2(k+1,p))$ by Remark 4. Define $\alpha(m,k,j) = \beta_2(k+1, p(m,k+1,j))$ and suppose that $y \in J(e_j, \alpha(m,k,j))$.

Then (x_m, y) is also contained in $J_2(s_{k+1,p}, t_{k+1,p}; \beta_1(k+1,p), \beta_2(k+1,p))$. So, we obtain

$$\begin{aligned} & |F(x_m, y) - F(x_m, e_j)| \\ & \leqslant |F(x_m, y) - F(s_{k+1,p}, t_{k+1,p})| + |F(s_{k+1,p}, t_{k+1,p}) - F(x_m, e_j)| < 2^{-k}. \end{aligned}$$

(ii-b): Let us take $p = p(k, x_m, y)$ for arbitrary $y \in [0,1)$, as in Remark 4. Then, $J_2(s_{k+1,p}, t_{k+1,p}; \beta_1(k+1,p), \beta_2(k+1,p))$ contains (x_m, e_j) for some dyadic rational e_j. By Remark 4 (a), we obtain $p(k, x_m, y) = p(k, x_m, e_j)$ and $J_2(s_{k+1,p}, t_{k+1,p}; \beta_1(k+1,p), \beta_2(k+1,p)) = J_2(x_m, e_j; \beta_1(k+1,p), \beta_2(k+1,p))$. Hence, $\bigcup_{j=1}^{\infty} J(e_j, \alpha(m,k,j)) = [0,1)$ holds.

Proof of (2): We note first that $\{x_m\}$ is a Fine-computable sequence. Let $\gamma(m,\ell)$ be a recursive modulus of Fine-convergence. That is, it satisfies that $n \geqslant \gamma(m,\ell)$ implies $x_{m,n} \in J(x_m, \ell)$.

For any k, m and e_j, we can find effectively and uniquely such $p = p(k+1, m, j)$ that $J_2(s_{k+1,p}, t_{k+1,p}; \beta_1(k+1,p), \beta_2(k+1,p))$ contains (x_m, e_j). We note that $J(s_{k+1,p}, \beta_1(k+1,p)) = J(x_m, \beta_1(k+1,p))$ by Lemma 2.

If $n \geqslant \gamma(m, \beta_1(k+1,p))$ and $y \in J(t_{k+1,p}, \beta_2(k+1,p)) = J(e_j, \beta_2(k+1,p))$, then

$$|F(x_{m,n}, y) - F(x_m, y)|$$
$$\leqslant |F(x_{m,n}, y) - F(s_{k+1,p}, t_{k+1,p})| + |F(s_{k+1,p}, t_{k+1,p}) - F(x_m, y)|$$
$$< 2 \cdot 2^{-(k+1)} = 2^{-k}.$$

By Proposition 30 (b), $\bigcup_{J(s_{k+1,p}, \beta_1(k+1,p)) \ni x} J(t_{k+1,p}, \beta_2(k+1,p)) = [0,1)$ and hence, $\bigcup_j J(e_j, \beta_2(k+1,p)) = [0,1)$.

This proves the effective Fine-convergence of $\{F(x_{m,n}, \cdot)\}$ to $\{F(x_m, \cdot)\}$ with respect to $\alpha(k,j) = \beta_2(k+1, p(k+1,m,j))$ and $\delta(k,i) = \gamma(m, \beta_1(k+1, p(k+1,m,j)))$ (cf. Definition 8). □

5 Bounded Fine-computable functions on $[0,1)^2$

In this section, we first investigate Fine-computability of the function $f(x) = \int_{[0,1)} F(x,y)dy$ for a bounded Fine-computable function $F(x,y)$.

Theorem 32. *If $F(x,y)$ is bounded and Fine-computable on $[0,1)^2$, then $f(x) = \int_{[0,1)} F(x,y)dy$ is Fine-computable on $[0,1)$.*

Proof. Sequential computability: Let $\{x_n\}$ be Fine-computable. By Proposition 31, $\{F(x_n, \cdot)\}$ is a bounded Fine-computable sequence of functions on $[0,1)$. So $\{f(x_n)\}$ is computable by Theorem 14.

Effective Fine-continuity: Let us take $\{(s_{k,p}, t_{k,p})\}$ and $\beta_1(k,p)$, $\beta_2(k,p)$ in Proposition 30.

First, we define a function $N(k,x)$, on $\mathbb{N}^+ \times [0,1)$, functions $h(k,x,\ell)$, $\alpha_1(k,x,\ell)$, $\alpha_2(k,x,\ell)$ on $\mathbb{N}^+ \times [0,1) \times \{1,2,\ldots,N(k,x)\}$ and sequences of dyadic rationals $u_{k,x,\ell}$, $v_{k,x,\ell}$ for each k, x and ℓ, $1 \leqslant \ell \leqslant N(k,x)$, by means of the following procedure $\mathrm{PB}_{k,x}$.

Procedures $\mathrm{PB}_{k,x}$:

First Step: Take $(s_{k,1}, t_{k,1})$ and examine the following test $\mathrm{TB1}_{k,x}$.

Test $\mathrm{TB1}_{k,x}$: $J_2(s_{k,1}, t_{k,1}; \beta_1(k,1), \beta_2(k,1))$ intersects $\{x\} \times [0,1)$.

Test $\mathrm{TB1}_{k,x}$ is equivalent to checking the containment $x \in J(s_{k,1}, \beta_1(k,1))$.

If the answer of $\mathrm{TB1}_{k,x}$ is "No", then set $h(k,x,1) = 0$ and go to the next step.

If the answer of $\mathrm{TB1}_{k,x}$ is "Yes", then define $(u_{k,x,1}, v_{k,x,1})$ to be the left lower endpoint of the fundamental dyadic neighborhood $J_2(s_{k,1}, t_{k,1}; \beta_1(k,1), \beta_2(k,1))$. Define also $\alpha_1(k,x,1) = \beta_1(k,1)$ and $\alpha_2(k,x,1) = \beta_2(k,1)$.

Set $h(k,x,1) = 1$ and examine also the following $\mathrm{TB2}_{k,x}$.

$\mathrm{TB2}_{k,x}$: $2^{-\alpha_2(k,x,1)} > 1 - 2^{-k}$.

If the answer of $\mathrm{TB2}_{k,x}$ is "Yes", then terminate $\mathrm{PB}_{k,x}$.

If the answer of $\mathrm{TB2}_{k,x}$ is "No", then go to the next step.

n-th Step ($n \geqslant 2$): Take $(s_{k,n}, t_{k,n})$ and suppose that we have obtained $h(k,x,\ell)$ $(1 \leqslant \ell \leqslant n-1)$, $u_{k,x,\ell}$, $v_{k,x,\ell}$ and $\alpha_1(k,x,\ell)$, $\alpha_2(k,x,\ell)$, $1 \leqslant \ell \leqslant h(k,x,n-1)$.

Apply $\mathrm{TB1}_{k,x}$ to $(s_{k,n}, t_{k,n})$ instead of $(s_{k,1}, t_{k,1})$.

If the answers of $\mathrm{TB1}_{k,x}$ is "No", then define $h(k,x,n) = h(k,x,n-1)$ and go to the next step.

If the answer of $\mathrm{TB1}_{k,x}$ is "Yes", define $h(k,x,n) = h(k,x,n-1)+1$. Define $(u_{k,x,h(k,x,n)}, v_{k,x,h(k,x,n)})$ to be the left lower endpoint of the fundamental dyadic neighborhood $J_2(s_{k,n}, t_{k,n}; \beta_1(k,n), \beta_2(k,n))$, and put $\alpha_1(k,x,h(k,x,n)) = \beta_1(k,n)$, $\alpha_2(k,x,h(k,x,n)) = \beta_2(k,n)$.

Examine also $\mathrm{TB2}_{k,x}$: $\sum_{\ell=1}^{h(k,x,n)} 2^{-\alpha_2(k,x,\ell)} > 1-2^{-k}$.

If the answer of $\mathrm{TB2}_{k,x}$ is "Yes", then terminate $\mathrm{PB}_{k,x}$.

If the answer of $\mathrm{TB2}_{k,x}$ is "No", then go to the next step.

By (b) in Proposition 30, $\bigcup_{p: J(s_{k+1,p}, \beta_1(k+1,p)) \ni x} J(t_{k+1,p}, \beta_2(k+1,p)) = [0,1)$. If $p \neq q$, $J(s_{k+1,p}, \beta_1(k+1,p)) \ni x$ and $J(s_{k+1,q}, \beta_1(k+1,q)) \ni x$ hold, then $J(t_{k+1,p}, \beta_2(k+1,p)) \cap J(t_{k+1,q}, \beta_2(k+1,q)) = \phi$ by (c) in Proposition 30. Therefore, $\sum_{p: J(s_{k+1,p}, \beta_1(k+1,p)) \ni x} |J(t_{k+1,p}, \beta_2(k+1,p))| = \sum_{p: J(s_{k+1,p}, \beta_1(k+1,p)) \ni x} 2^{-\beta_2(k+1,p)} = 1$, where $|J|$ denotes the length of the interval J. Hence, Procedure $\mathrm{PB}_{k,x}$ terminates within finite steps.

When Procedure $\mathrm{PB}_{k,x}$ terminates at Step m, we have $h(k,x,m)$ and $u_{k,x,\ell}$, $v_{k,x,\ell}$, $\alpha_1(k,x,\ell)$, $\alpha_2(k,x,\ell)$ for $1 \leqslant \ell \leqslant h(k,x,m)$. Define $N(k,x) = h(k,x,m)$. Then, the following properties hold.

(a) Dyadic intervals $\{J(v_{k,x,\ell}, \alpha_2(k,x,\ell))\}_{1 \leqslant \ell \leqslant N(k,x)}$ are mutually disjoint.

(b) $\sum_{\ell=1}^{N(k,x)} 2^{-\alpha_2(k,x,\ell)} > 1-2^{-k}$.

(c) $y \in J(v_{k,x,\ell}, \alpha_2(k,x,\ell))$ and $z \in J(u_{k,x,\ell}, \alpha_1(k,x,\ell))$ imply $|F(x,y) - F(z,y)| < 2^{-(k-1)}$ due to Proposition 30 (a) for $1 \leqslant \ell \leqslant N(k,x)$.

(d) $u_{k,x,\ell} \leqslant x < u_{k,x,\ell} + 2^{-\alpha_1(k,x,\ell)}$ for $1 \leqslant \ell \leqslant N(k,x)$.

Define $\xi(k,x) = \max_{1 \leqslant \ell \leqslant N(k,x)} u_{k,x,\ell}$ and $\eta(k,x) = \min_{1 \leqslant \ell \leqslant N(k,x)} u_{k,x,\ell} + 2^{-\alpha_1(k,x,\ell)}$.

Then, $[\xi(k,x), \eta(k,x))$ is a dyadic interval and contains x. So, we can define $\gamma(k,x)$ as $\min\{\ell \,|\, J(x,\ell) \subset [\xi(k,x), \eta(k,x))\}$.

The following properties of $\gamma(k,x)$ and $N(k,x)$ follow from (a) to (d) above:

(i) If $z \in J(x, \gamma(k,x))$, then $N(k,z) = N(k,x)$. Moreover, $u_{k,x,\ell} = u_{k,z,\ell}$, $v_{k,x,\ell} = v_{k,z,\ell}$ and $\alpha_i(k,x,\ell) = \alpha_i(k,z,\ell)$ for $1 \leqslant \ell \leqslant N(k,x)$, and hence $\gamma(k,z) = \gamma(k,x)$.

(ii) If $y \in \bigcup_{\ell=1}^{N(k,x)} J(v_{k,x,\ell}, \alpha_2(k,x,\ell))$ and $z \in J(x, \gamma(k,x))$, then $|F(x,y) - F(z,y)| < 2^{-(k-1)}$.

(iii) $\left|\bigcup_{n=1}^{N(k,x)} J(v_{k,x,\ell}, \alpha_2(k,x,\ell))\right| = \sum_{n=1}^{N(k,x)} 2^{-\alpha_2(k,x,\ell)} > 1-2^{-k}$.

Now, we prove the effective Fine-continuity. If x is a dyadic rational, then we can perform all the above procedure effectively, since we need only finite number of comparisons of dyadic rationals for Tests $\mathrm{TB1}_{k,x}$ and $\mathrm{TB2}_{k,x}$.

From boundedness of $F(x,y)$, there exists an integer K such that $|F(x,y)| < 2^K$ for all (x,y). Now, if we define $\delta(k,i) = \gamma(k+K+2, e_i)$, then δ is a recursive function. Suppose that $x \in J(e_i, \delta(k,i)) = J(e_i, \gamma(k+K+2, e_i))$, and put $E_{k,i} = \bigcup_{\ell=1}^{N(k,e_i)} J(v_{k,e_i,\ell}, \alpha_2(k,e_i,\ell))$. Then, $E_{k,i} = \bigcup_{\ell=1}^{N(k,x)} J(v_{k,x,\ell}, \alpha_2(k,x,\ell))$ by (i), and we obtain by (ii) and (iii)

$$\begin{aligned}|f(x) - f(e_i)| &\leqslant \int_{E_{k+K+2,i}} |F(x,y) - F(e_i,y)|dy + \int_{(E_{k+K+2,i})^C} |F(x,y)|dy \\ &\quad + \int_{(E_{k+K+2,i})^C} |F(e_i,y)|dy \\ &< 2^{-(k+K+2)} + 2 \cdot 2^K 2^{-(k+K+2)} < 2^{-k}\end{aligned}$$

For all $x \in [0,1)$, $J(x, \delta(k,x))$ contains a dyadic rational, say, e_i. By property (i), $J(x,\delta(k,i)) = J(e_i, \delta(k,i))$. So $x \in J(e_i, \delta(k,i))$ and we obtain $\bigcup_{i=1}^{\infty} J(e_i, \delta(k,i)) = [0,1)$. This proves effective Fine-continuity of $f(x)$. □

We now state the effective version of Theorem 1.

Theorem 33. (Effective Fubini's Theorem for bounded Fine-computable functions) *Let $F(x,y)$ be a positive bounded Fine-computable function. Then the following holds.*

(i) *If $\{x_n\}$ is Fine-computable, then $\{F(x_n,\cdot)\}$ and $\{F(\cdot,x_n)\}$ are uniformly bounded Fine-computable sequences of functions.*

(ii) *$\int_{[0,1)} F(x,y)dy$ and $\int_{[0,1)} F(x,y)dx$ are bounded Fine-computable functions.*

(iii) *$\int_{[0,1)^2} F(x,y)dxdy = \int_{[0,1)} dx \int_{[0,1)} F(x,y)dy = \int_{[0,1)} dy \int_{[0,1)} F(x,y)dx$ holds, and the value is computable.*

6 General Fine-computable functions on $[0,1)^2$

We give some examples of such a Fine-computable function $F(x,y)$ on $[0,1)^2$ that $f(x) = \int_0^1 F(x,y)dy$ is not a Fine-computable function on $[0,1)$.

Example 1. (Suggested by Yagishita) Let us define $F(x,y) = \frac{1}{1-y} e^{-(\frac{x}{1-y})^2}$. Then $F(x,y)$ is positive and continuous on $\mathbb{R} \times [0,1)$. It is easy to prove that the restriction of $F(x,y)$ to $[0,1)^2$ is Fine-computable.

It holds that $\int_0^1 F(x,y)dx = \int_0^1 \frac{1}{1-y} e^{-(\frac{x}{1-y})^2} dx = \int_0^{\frac{1}{1-y}} e^{-x^2} dx < \sqrt{\pi}$.

Hence $\int_{-1}^1 dy \int_0^1 F(x,y)dx < \infty$.

On the other hand, $F(0,y) = \frac{1}{1-y}$ and $\int_{[0,1)} F(0,y)dy$ diverges.

Example 1 shows that Fine-computability and integrability of $F(x,y)$ do not assure that $f(x)$ is a total function.

Example 2. Let $\alpha(k)$ be a recursive injection whose range is not recursive. Then

$$\varphi(y) = 2^k 2^{-\alpha(k)} \quad \text{if} \quad 1 - 2^{-(k-1)} \leqslant y < 1 - 2^{-k}, k = 1, 2, \dots$$

is Fine-computable and integrable but not effectively integrable [Brattka 2002].

Define $F(x, y) = \varphi(y)(1 - x)^{\varphi(y)-1}$ and $f(x) = \int_{[0,1)} F(x, y)dy$.

Then, $F(x, y)$ is Fine-computable and not bounded. It holds that $\int_{[0,1)} F(x, y)dx = 1$ and $\iint_{[0,1)^2} F(x, y)dxdy = 1$, and that $f(x)$ is total.

On the other hand, $f(0) = \int_{[0,1)} F(0, y)dy = \sum_{k=1}^{\infty} 2^{-\alpha(k)}$ is not a computable number, and hence sequential computability for $f(x)$ does not hold.

Example 2 shows that Fine-computability of $F(x, y)$ and computability of $\iint_{[0,1)^2} F(x, y)dxdy$ do not imply Fine-computability of $f(x)$ even if it is total.

We give a sufficient condition which assures the Fine-computability of $f(x)$ for a Fine-computable function $F(x, y)$.

Theorem 34. *If $F(x, y)$ is Fine-computable and there exists an effectively integrable Fine-computable function $g(y)$ which satisfies $|F(x, y)| \leqslant g(y)$ for all x, then $f(x) = \int_{[0,1)} F(x, y)dy$ is Fine-computable.*

Proof. Sequential computability can be proved in a similar way to the proof of Theorem 27.

To prove *effective Fine-continuity*, let $\alpha(k, i)$ be the effective modulus of effective Fine-continuity of $g(y)$. By effective integrability of $g(y)$ and Proposition 17, there exists a recursive function $M(k)$ such that $\int_{(E_k)^C} g(y)dy < 2^{-k}$, where $E_k = \bigcup_{i=1}^{M(k)} J(e_i, \alpha(k, i))$. This implies $\int_{(E_k)^C} |F(x, y)|dy < 2^{-k}$ for all $x \in [0, 1)$. On E_k, $g(y) \leqslant \max_{1 \leqslant i \leqslant M(k)} \{g(e_i) + 2^{-k}\}$. So, $F(x, y)$ is bounded on $[0, 1) \times E_k$.

We can apply the proof of Theorem 32 to the domain $[0, 1) \times E_k$ and obtain that $\tilde{F}_k(x) = \int_{E_k} F(x, y)dy$ is Fine-computable. Let $\theta(k, i)$ be a modulus of continuity, that is, it satisfies that $|\tilde{F}_k(x) - \tilde{F}_k(e_i)| < 2^{-k}$ if $x \in J(e_i, \theta(k, i))$ and $\bigcup_{i=1}^{\infty} J(e_i, \theta(k, i)) = [0, 1)$. From the proof of Theorem 32, $\theta(k, i)$ can be taken as recursive. If we define $\gamma(k, i) = \theta(k + 2, i)$, then $f(x)$ is effective Fine-continuous with respect to $\gamma(k, i)$, since $|f(x) - f(e_i)| \leqslant |\tilde{F}_{k+2}(x) - \tilde{F}_{k+2}(e_i)| + \int_{(E_{k+2})^C} |F(x, y)|dy + \int_{(E_{k+2})^C} |F(e_i, y)|dy < 2^{-k}$ if $x \in J(e_i, \gamma(k, i))$. □

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