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Computing the Solution Operators of Symmetric Hyperbolic Systems of PDE

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Abstract: We study the computability properties of symmetric hyperbolic systems of PDE $A\frac{\partial \mathbf{u}}{\partial t} + \sum\limits_{i=1}^{m} B_i \frac{\partial \mathbf{u}}{\partial x_i} = 0$, $A = A^* > 0$, $B_i = B_i^*$, with the initial condition $\mathbf{u}|_{t=0} = \varphi(x_1, \ldots, x_m)$. Such systems first considered by K.O. Friedrichs can be used to describe a wide variety of physical processes. Using the difference equations approach, we prove computability of the operator that sends (for any fixed computable matrices $A, B_1, \ldots, B_m$ satisfying certain conditions) any initial function $\varphi \in C^{p+1}(Q, \mathbb{R}^n)$ (satisfying certain conditions), $p \geq 2$, to the unique solution $\mathbf{u} \in C^p(H, \mathbb{R}^n)$, where $Q = [0,1]^m$ and H is the nonempty domain of correctness of the system.
Key Words: hyperbolic system, PDE, computability, metric space, norm, matrix pencil, difference scheme, stability, finite-dimensional approximation.
Category: F.2, F.2.1

1 Introduction

In this paper we study the computability properties of symmetric hyperbolic systems

$$\begin{cases} A\frac{\partial \mathbf{u}}{\partial t} + \sum\limits_{i=1}^{m} B_i \frac{\partial \mathbf{u}}{\partial x_i} = 0, \\ \mathbf{u}|_{t=0} = \varphi(x_1, \ldots, x_m), \end{cases} \tag{1}$$

where $A = A^*$ and $B_i = B_i^*$ are constant symmetric $n \times n$-matrices, A is positively definite (which is denoted as $A > 0$), $t \geq 0$, $x = (x_1, \ldots, x_m) \in Q = [0,1]^m$, $\varphi : Q \to \mathbb{R}^n$ and $\mathbf{u} : Q \times [0, +\infty) \rightharpoonup \mathbb{R}^n$ is a partial function. In particular, we prove computability of the operator that sends (for any fixed computable $A, B_1, \ldots, B_m$ satisfying some natural conditions) any initial function

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$\varphi \in C^{p+1}(Q, \mathbb{R}^n)$, $p \geq 2$, such that for some M we have $||\frac{\partial \varphi}{\partial x_i}||_{L_2}, ||\frac{\partial^2 \varphi}{\partial x_i \partial x_j}||_{L_2} \leq M$ for all $i, j \in \{1, \ldots, m\}$, to the unique solution $\mathbf{u} \in C^p(H, \mathbb{R}^n)$ of (1), where $H \neq \emptyset$ is the largest set where the unique solution of the initial value problem (1) exists. Note that $\mathbf{u}$ and φ may be considered as n-tuples of functions; sometimes it will be easier not to use the boldface font to denote such vector-functions.

Such systems first considered by K.O. Friedrichs [Fr54] can be used to describe a wide variety of physical processes like those considered in the theories of elasticity, acoustics, electromagnetism, etc. (see Section 6 below for additional details). Some of these processes can also be described via the wave equation, but in many cases it is more convenient to solve the equivalent first-order systems (1) which actually form a more general class of PDE. Some motivation for doing so was given in [Fr54] where the well-posedness of (1) (i.e., the existence, uniqueness and continuous dependence of the solution on initial data) was established. The notion of a hyperbolic system (applicable also to broader classes of systems) is due to I.G. Petrovskii [Pe37].

The Friedrichs method (described in more modern terminology in [Go71]) to prove the existence theorem is based on finite difference approximations, in contrast with the Cauchy-Kovalevskaya method based on approximations by analytic functions and a careful study of infinite series (see e.g. [Sc55]). This feature of the Friedrichs method is interesting from the computational point of view because, under some additional assumptions of the first and second derivatives of the initial function, it yields (as we show here) algorithms for solving PDE in the exact sense of computable analysis [We00] which are based on methods really used in numerical analysis.

In this way we make a step to fill the large gap between the exact approach of computable analysis and heuristic algorithms (the correctness of which is not always clear) widely used in numerical analysis. The fact that algorithms based on difference schemes sometimes imply the computability of solution operators in the sense of computable analysis is nontrivial because in the theory of difference schemes people usually concentrate on the grid functions (i.e., elements of finite-dimensional spaces) while notions of computable analysis appeal to the elements of functional (i.e., infinite-dimensional) spaces. Accordingly, our proof relies on some observations concerning the approximability of infinite-dimensional spaces by finite-dimensional ones. More exactly, we rely on the well-known classical theorem of the theory of difference schemes (see e.g. [GR62]) stating that the approximation and stability properties of a difference scheme imply its convergence to the solution of the correspondent differential equation in the grid norm uniformly on steps (see Sections 2.3 and 4 for some additional details). We use this theorem and multilinear interpolations to prove a convergence result (Theorem 8) in suitable functional norms. Though the constant from that theorem seems to be principal for the study of computabilty properties, we have not found in the literature any information on its value. We show that the existence of this constant is sufficient for proving the computabilty of the solution operators in some cases but we were not able to derive useful convergence rates implying the existence of feasible algorithms.

For other methods of proving the computability of PDE solutions see e.g. [PER89, WZ02, WZ05]. In particular, the computability of the wave equation follows from the existence [Ev98] of explicit expression of the solution through the Fourier transform. Since the wave equation is equivalent to a system considered in this paper [Ev98], one could hope to use a similar method for our systems. But the explicit solutions seem to be known only under some strong smoothness assumptions or assumptions of invariance under rotations [Ev98, GM98] which are not assumed in our result below. Moreover, we hope that the methods of our paper are applicable in more general situations where the explicit solutions probably do not exist.

The well-posedness of the initial value problem (1) in the maximal domain H was established for a general case by I.G. Petrovskii. A proof for systems close to ours based on the difference equation method is presented in [Go71] where sufficient details are given for the one-dimensional case $m = 1$. We work here with a slightly modified difference scheme described in detail in [Go76] and widely used in applications (cf. e.g. [GS06, Se05] and references therein). Some reasons to use the modified scheme here are the fact that it is more convenient for $m > 1$ and the possibilities of generalizing our proofs to some other processes (including the shock waves) and of constructing analogous higher-order schemes in a hope to make an insight into complexity of the resulting algorithms.

The set H above (which is, under some restrictions on $A, B_1, \ldots, B_m$, a convex polyhedron being an intersection of $2m+1$ semispaces of $\mathbb{R}^{m+1}$, details are given below) is the nonempty domain of existence and uniqueness of the solution $\mathbf{u}$ of (1). It is computable from $A, B_1, \ldots, B_m$ using the eigenvalues of the so called regular matrix pencils $\lambda A - B_i$ [Ga67] for $i = 1, \ldots, m$. The computability of spectrum of a symmetric matrix follows from [ZB01]. Using this result we show that a canonical form for the one-dimensional case $m = 1$ of (1) known as "form in Riemann invariants" is computable from A, B_1 and cardinalities of the spectra of the matrix A and of the pencil $\lambda A - B_1$. This leads to a stable scheme and to an algorithm for computing the solution of (1).

Although we need here only *eigenvalues* of the mentioned matrix pencils, the procedure of finding them involves the search of *eigenvectors* of the matrix A and of another symmetric matrix as well, and these eigenvectors will also be used in an essential way in some proofs below. We prove the computability of eigenvectors of regular matrix pencils in Section 3, slightly generalizing the result in [ZB01] that the eigenvectors of a symmetric matrix are computable if and only if the cardinality of spectrum of the matrix is known in advance.

The rest of the paper is organized as follows. In Section 2 we recall some notation, notions and facts. In Section 3 we prove the computability of the canonical form and of the set H. In Section 4 we describe the difference scheme for (1) and some of its properties. In Section 5 we prove the computability of the solution operator for (1). In Section 6 we provide some illustrative examples. We conclude in Section 7 with a short discussion on possible future work.

This paper is an extended and corrected version of the conference paper [SS08]. Trying to satisfy some requests of the referees, we provided a more detailed analysis of some rates of convergence and added the additional assumption on the constant M which was missed in [SS08].

2 Notation, Notions and Known Facts

2.1 Matrix Pencils and Description of H

We need some facts about eigenvalues of the matrix pencils $\lambda A - B_i$, $i = 1, \ldots, m$. For the theory of matrix pencils see e.g. [Ga67]. Following [Ga67], by a *regular matrix pencil* we mean a matrix $\lambda A - B$ where λ is a real parameter, A and B are symmetric real $n \times n$-matrices, and $A > 0$ is positively definite (i.e., $\langle A\mathbf{z}, \mathbf{z}\rangle > 0$ for all nonzero vectors $\mathbf{z} \in \mathbb{R}^n$). The determinant $\det(\lambda A - B)$ is called the *characteristic polynomial* of $\lambda A - B$. The roots of the characteristic polynomial are called *eigenvalues* of the pencil. For any eigenvalue λ of $\lambda A - B$ there is a non-zero vector $\mathbf{z} = (z_1, \ldots, z_n)^T$ (called *eigenvector of* $\lambda A - B$ related to this eigenvalue) such that $(\lambda A - B)\mathbf{z} = 0$. The following fact is well-known (see e.g. [Ga67], p. 281):

Theorem 1. *For any regular matrix pencil $\lambda A - B$ there exist eigenvalues $\lambda_1, \ldots,$ λ_n and associated to them A-orthogonal eigenvectors $\mathbf{z}_1, \ldots, \mathbf{z}_n$ (this means that $\langle A\mathbf{z}_j, \mathbf{z}_k\rangle = \delta_{j,k}$ for all $j, k = 1, \ldots, n$ where $\delta_{j,k}$ is the Kronecker symbol, i.e., $\delta_{j,k} = 0$ for $j \neq k$ and $\delta_{j,k} = 1$ for $j = k$).*

Now let us recall the structure of the domain of correctness $H \subseteq \mathbb{R}^{m+1}$, i.e., the maximal set where, for any $p \geq 1$ and $\varphi \in C^{p+1}(Q, \mathbb{R}^n)$, there exists a unique solution $\mathbf{u} \in C^p(H, \mathbb{R}^n)$ of the initial value problem (1).

The set H is known to be (see e.g. [Go71]) a nonempty intersection of the semispaces

$$t \geq 0,\ x_i - \lambda^{(i)}_{\max} t \geq 0,\ x_i - 1 - \lambda^{(i)}_{\min} t \leq 0,\ (i = 1, \ldots, m)$$

of $\mathbb{R}^{m+1}$ where, for each $i = 1, \ldots, m$, $\lambda^{(i)}_{\max}$ is the maximal and $\lambda^{(i)}_{\min}$ is the minimal eigenvalue of the matrix pencil $\lambda A - B_i$. We are especially interested in the case when H is a compact subset of $Q \times [0, +\infty)$ (obviously, a sufficient condition for this to be true is $\lambda^{(i)}_{\min} < 0 < \lambda^{(i)}_{\max}$ for all $i = 1, \ldots, m$; this is often the case for natural physical systems, see Section 6).

2.2 Spaces under Consideration

Let $\mathbb{R}^m$ be the Euclidean space with the usual norm. As it is known, this space is separable, i.e., it has a countable dense subset (e.g., the set of vectors with rational coordinates). We will consider some subspaces of $\mathbb{R}^m$ (as a metric space) with the induced metric, in particular the m-dimensional unitary cube $Q = [0,1]^m$. Such subspaces are also separable. In particular, a countable dense subset of Q is formed by the binary-rational vectors $(x_1, \dots, x_m)$ where $x_i = \frac{y_i}{2^k}$ and $y_i \in \{0, 1, \dots, 2^k\}$ for some $k \geq 0$. Below we often use the uniform grids G_k on Q formed by such vectors.

We will work with several functional spaces most of which are subsets of the set $C(\mathbb{R}^m, \mathbb{R}^n) \simeq C(\mathbb{R}^m, \mathbb{R})^n$ of integrable continuous functions $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ equipped with the L_2-norm. In particular, we deal with the space $C(Q, \mathbb{R}^n) \simeq C(Q, \mathbb{R})^n$ (resp. $C^k(Q, \mathbb{R}^n)$) of continuous (resp. continuously k-time differentiable) functions $\varphi : Q \to \mathbb{R}^n$ equipped with the L_2-norm

$$||\varphi||_{L_2} = \left(\int_Q |\varphi(x)|^2 dx \right)^{\frac{1}{2}}, \ |\varphi(x)|^2 = \langle \varphi, \varphi \rangle = \sum_{i=1}^{n} \varphi_i^2(x).$$

We will also use the sup-norm $||\varphi||_s = \sup_{x \in Q} |\varphi(x)|$ on $C(Q, \mathbb{R}^n)$ and the sL_2-norm

$$||u||_{sL_2} = \sup_{0 \leq t_0 \leq T} \sqrt{\int_Q |u(x, t_0)|^2 dx}$$

on $C(Q \times [0, T], \mathbb{R}^n)$ where $T > 0$. Whenever we want to emphasize the norm we use notation like $C_{L_2}(Q, \mathbb{R}^n)$, $C_s(Q, \mathbb{R}^n)$ or $C_{sL_2}(Q \times [0, T], \mathbb{R}^n)$. In case when the domain of correctness H is compact and $H \subseteq Q \times [0, T]$, we consider also the space $C_{sL_2}(H, \mathbb{R}^n)$ defined in the same way as the last of the mentioned spaces.

Note that the space $C(Q, \mathbb{R}^n)$ is separable w.r.t. any of the norms. A countable dense set in $C(Q, \mathbb{R}^n)$ frequently used in dealing with difference equations is formed as follows. Take a rectangular grid on Q with rational coordinates (in fact, the uniform grids G_k with step $\frac{1}{2^k}$ on each coordinate suffice). Associate to any function $f_k : G_k \to \mathbb{Q}$ on the finite set G_k of grid nodes the continuous extension $\tilde{f}_k : Q \to \mathbb{R}$ of f obtained by piecewise-linear interpolation on each coordinate. Such interpolations known also as multilinear interpolations are the simplest class of splines (see e.g. [ZKM80]). Note that the restriction of $\tilde{f}_k$ to any grid cell is a polynomial of degree m. The extensions $\tilde{f}_k$ induce a countable dense set in $C(Q, \mathbb{R}^n)$ with any of the three norms. Additional information on the multilinear interpolations is given in Subsection 4.2 below.

For all $\tau > 0$ and integer $k \geq 0, L \geq 1$, let G_k^τ be the grid in $Q \times [0, T]$, $T = L\tau$, with step $h = \frac{1}{2^k}$ on the space coordinates x_i and step τ on the time coordinate t. Just as above, such grids induce a countable dense set in $C(Q \times [0, T], \mathbb{R}^n)$ with any of the three norms.

In the study of difference equations the interaction between the infinite-dimensional space $C(Q, \mathbb{R}^n)$ (with a given norm) and the corresponding finite-dimensional spaces $(\mathbb{R}^n)^G$ of grid functions $f : G \to \mathbb{R}^n$ (with the discrete analog of the given norm) plays a crucial role. The discrete analogs of the norms in $C_s(Q, \mathbb{R}^n)$, $C_{L_2}(Q, \mathbb{R}^n)$ and $C_{sL_2}(Q \times [0, T], \mathbb{R}^n)$ are defined in the natural way. For example, in the last case the norm of a grid function $f : G_k^\tau \to \mathbb{R}$ is defined by

$$||f||_{sL_2} = \max_{0 \leq l\tau \leq T} \left(h^m \sqrt{\sum_{x \in G_k} f^2(x, l\tau)} \right).$$

We also need the Cantor space Σ^ω of infinite words over a finite alphabet Σ containing at least two symbols. This space plays a crucial technical role in the exact definition of computability over metric (and even more general) spaces. It is well-known that Σ^ω is a complete separable metric space.

2.3 Preliminaries on Difference Schemes

Here we briefly recall some relevant notions and facts about difference schemes (for more details see any book on the subject, e.g. [GR62]).

Let us consider a (system of) PDE with a boundary condition

$$L\mathbf{u} = \mathbf{f},\ \mathcal{L}\mathbf{u}|_{\partial\Omega} = \varphi \tag{2}$$

in an open bounded area $\Omega \subseteq \mathbb{R}^{m+1}$ where $\partial\Omega$ is the boundary of Ω, L and $\mathcal{L}$ are differential operators, f and φ are given functions (in our case, $f \equiv 0$ and $\mathcal{L}\mathbf{u}|_{\partial\Omega} = \mathbf{u}(\mathbf{x}, 0)$). Difference approximations to (2) are written in the form

$$L_h\mathbf{u}^{(h)} = \mathbf{f}^{(h)},\ \mathcal{L}_h\mathbf{u}^{(h)} = \varphi^{(h)} \tag{3}$$

where $L_h, \mathcal{L}_h$ are difference operators (which are often linear), and all functions are defined on some grids in Ω or $\partial\Omega$ (the grids are not always uniform, as in our simplest case). For simplicity we use the restriction notation $\mathbf{g}|_{G_k}$ to denote the projection of $\mathbf{g} : \Omega \to \mathbb{R}^n$ to the grid G_k in Ω though in general the projection operator may be more complicated. Both sides of (3) depend on the grid step h.

Note that in Section 8 we consider a little more complicated grids than the uniform grids discussed above, namely grids with the integer time steps $i\tau$, $i \geq 0$, (for some $\tau > 0$) and half-integer steps $(n + \frac{1}{2})$ for the space variables (this is illustrated by Figure 1). The theory for such slightly modified grids remains the same.

Let the space of grid functions defined on the same grid as $\mathbf{f}^{(h)}$ (resp. as $\mathbf{u}^{(h)}$, $\varphi^{(h)}$) carry some norm $||\cdot||_{F_h}$ (resp. some norms $||\cdot||_{U_h}$, $||\cdot||_{\Phi_h}$). In our case, $||\cdot||_{F_h}$ and $||\cdot||_{U_h}$ are $||\cdot||_{sL_2}$ while $||\cdot||_{\Phi_h}$ is $||\cdot||_{L_2}$.

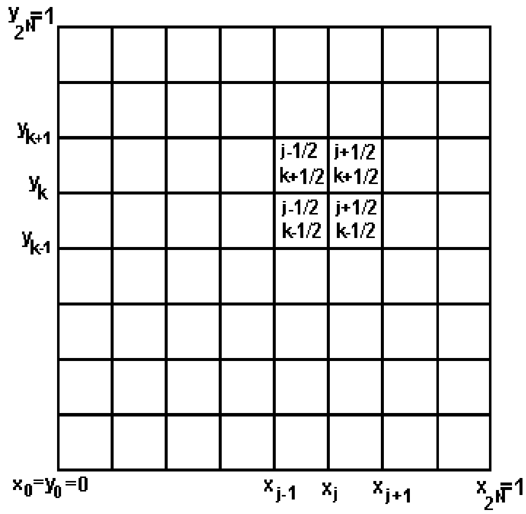

Figure 1. A grid on $Q = [0, 1]^2$.

Definition 2. Difference equations (3), also called difference schemes, *approximate the differential equation (2) with order of accuracy* h^l *on a solution* $\mathbf{u}(\mathbf{x}, t)$ *of (2)* if $||(L\mathbf{u})|_{G_k} - L_h\mathbf{u}^{(h)}||_{F_h} \leq M_1h^l$, $||f|_{G_k} - f^{(h)}||_{F_h} \leq M_2h^l$, $||(\mathcal{L}\mathbf{u})|_{G_k} - \mathcal{L}_h\mathbf{u}^{(h)}||_{\Phi_h} \leq M_3h^l$ and $||\varphi|_{G_k} - \varphi^{(h)}||_{\Phi_h} \leq M_4h^l$, for some constants M_1, M_2, M_3 and M_4 not depending on h.

The definition is usually checked by working with the Taylor series for the corresponding functions. As a result, the degrees of smoothness of the functions become essential when one is interested in the order of accuracy of a difference scheme. Note that the definition assumes the existence of a solution of (2). For our initial value problem (1) it is well-known (see e.g. [Fr54, Go71, Mi73]) that there is a unique solution.

The following notion identifies a property of difference schemes which is crucial for computing "good" approximations to the solutions of (2).

Definition 3. Difference scheme (3) is called *stable* if its solution $\mathbf{u}^{(h)}$ satisfies $||\mathbf{u}^{(h)}||_{U_h} \leq N_1||f^{(h)}||_{F_h} + N_2||\varphi^{(h)}||_{\Phi_h}$ for some constants N_1 and N_2 not depending on h, $f^{(h)}$ and $\varphi^{(h)}$.

Usually, for nonstationary processes (depending explicitly on the time variable t, as (1)), the difference equation (3) may be rewritten in the equivalent recurrent form $\mathbf{u}^{[i+1]} = R_h\mathbf{u}^{[i]} + \tau\rho^{[i]}$ where $\mathbf{u}^{[0]}$ is known, $\mathbf{u}^{[i]}$ is the restriction of the solution to the time level $t = i\tau$, $i \geq 0$, $\rho^{[i]}$ depends only on f and φ

, R_h is the difference operator obtained from L_h in a natural way. It is known (see e.g. [GR62]) that the stability of (3) on the interval $0 < t < T$ is equivalent to the uniform boundedness of the operators R_h and their powers: $||R_h^m|| < K$, $m = 1, 2, \ldots, \frac{T}{\tau}$, for some constant K not depending on h. In general, the investigation of the stability of difference schemes is a hard task; the most popular tool is the so called Fourier method (for the problem (1) and the scheme we use here, this is described in [Go76]).

Our main result on the computability of solutions of (1) makes an essential use of the following basic fact from the theory of difference schemes (see e.g. [GR62]):

Theorem 4. *Let the difference scheme (3) be stable and approximate (2) on the solution* $\mathbf{u}$ *with order* l. *Then the solution of (3) uniformly converges to the solution* $\mathbf{u}$ *in the sense that* $||\mathbf{u}|_{G_k^\tau} - \mathbf{u}^{(h)}||_{U_h} \le Nh^l$ *for some constant* N *not depending on* h *and* τ.

2.4 Computability Notions

We use the TTE-approach to computability over metric spaces developed in the K. Weihrauch's school (see e.g. [We00, WZ02, Br03] for more details).

Recall that a *computable metric space* is a triple (M, d, ν) where (M, d) is a metric space and $\nu : \omega \to M$ is a numbering of a dense subset $\mathrm{rng}(\nu)$ of M such that the set

$$\{(i, j, q, r) \mid i, j \in \omega,\ q, r \in \mathbb{Q},\ q < d(\nu_i, \nu_j) < r\}$$

is computably enumerable.

The *Cauchy representation* of a computable metric space (M, d, ν) is the partial surjection $\delta_M : \Sigma^\omega \rightharpoonup M$ defined exactly on the elements $p \in \Sigma^\omega$ which code (in a natural way) sequences $\{p_i\}$ of natural numbers such that $d(\nu(p_i), \nu(p_k)) \le 2^{-k}$ for $i \ge k$ and $\{\nu(p_i)\}_{i\in\omega}$ is convergent and sending any such code p to $\delta_M(p) = \lim_i \nu(p_i)$. (Note that the sequence $\{\nu(p_i)\}_{i\in\omega}$ is then fast convergent to $x = \delta_M(p)$ which means that $\forall i \ge 0 d(\nu(p_i), x) \le c2^{-k}$ for some constant c.) Following [We00], by *Cauchy sequences* we mean in this paper sequences $\{\nu(p_i)\}_{i\in\omega}$ satisfying the condition above. An element $x \in M$ of a computable metric space is called *computable* if $x = \delta_M(p)$ for a computable Cauchy sequence coded by $p \in \Sigma^\omega$

Metric spaces $\mathbb{R}^m$, Σ^ω, $C_s(Q, \mathbb{R}^n)$, $C_{L_2}(Q, \mathbb{R}^n)$ and $C_{sL_2}(Q \times [0, T], \mathbb{R}^n)$ discussed in Subsection 2.2 (w.r.t. the metrics induced by the norms and natural numberings of the dense subsets specified above) are computable. The same applies to spaces $C_s(H, \mathbb{R}^n)$, $C_{L_2}(H, \mathbb{R}^n)$ and $C_{sL_2}(H, \mathbb{R}^n)$ provided that matrices $A, B_1, \ldots, B_m$ are computable (as elements of $\mathbb{R}^{n\times n}$) and $\lambda_{\min}^{(i)} < 0 < \lambda_{\max}^{(i)}$ for all $i = 1, \ldots, m$.

A partial function $f : M \rightharpoonup M_1$ on the elements of computable metric spaces (M, d, ν) and (M_1, d_1, ν_1) is *computable* if there is a computable partial function $\hat{f} : \Sigma^\omega \rightharpoonup \Sigma^\omega$ which represents f w.r.t. the Cauchy representations of M and M_1, i.e., $\delta_{M_1}(\hat{f}(p)) = f(\delta_M(p))$ for each $p \in dom(\delta_M)$. Informally, f is computable if there is an algorithm (realized as a Turing machine sending infinite input words over Σ to infinite output words over Σ) which sends any convergent Cauchy sequence $\{a_i\}$ of elements of $\mathrm{rng}(\nu)$ with $\lim_i a_i \in \mathrm{dom} f$ to a convergent Cauchy sequence $\{b_i\}$ of elements of $\mathrm{rng}(\nu_1)$ such that $\lim_i b_i = f(\lim_i a_i)$.

Let G be the grid in Q with step $h = \frac{1}{2^k}$ on each coordinate. From well-known facts of computable analysis [We00] it follows that $\varphi \mapsto \varphi|_G$ is a computable operator from $C_s(Q, \mathbb{R}^n)$ to $(\mathbb{R}^n)^G$. From well-known properties of the multilinear interpolations (see e.g. [Go71, ZKM80]) it follows that $f \mapsto \tilde{f}$ is a computable operator from $((\mathbb{R}^n)^G)_s$ to $C_{L_2}(Q, \mathbb{R}^n)$ (see also the estimate (14) below).

3 Computing the Canonical Form and Domain

In this section we observe the computability of a canonical form for the one-dimensional system (1), and of the set of correctness H.

First we show that for any one-dimensional (i.e., for $m = 1$) system (1) of the form

$$A\frac{\partial \mathbf{u}}{\partial t} + B\frac{\partial \mathbf{u}}{\partial x} = 0, \ \mathbf{u} = (u_1, u_2, \dots, u_n)^T \tag{4}$$

we can compute an equivalent system in the canonical form

$$\frac{\partial \mathbf{v}}{\partial t} + M\frac{\partial \mathbf{v}}{\partial x} = 0, \quad M = \mathrm{diag}\{\mu_1, \mu_2, \dots, \mu_n\}. \tag{5}$$

The components of the vector $\mathbf{v}$ are called *Riemann invariants*. They are invariant along the lines $x = \mu_i t + const$, $1 \le i \le n$, called *characteristics* of the systems (4,5) (see e.g. [Go71, Ev98] for additional details).

Theorem 5. *Given as inputs symmetric real matrices A, B with $A > 0$ and the cardinalities of the spectrum of A and $\lambda A - B$, we can compute a system in the canonical form (5) equivalent to (4).*

Proof. We have to compute the matrix M and the non-degenerate linear transformation T of variables $\mathbf{u} = T\mathbf{v}$. Since matrices A and B are symmetric and $A > 0$, there is a non-degenerate matrix T such that $T^*AT = I_n$ (I_n is the identity n-dimensional matrix) and $T^*BT = \mathrm{diag}\{\mu_1, \mu_2, \dots \mu_n\}$. The matrix T may be constructed in the following way:

a) Applying the orthogonal matrix L, $L^*L = I_n$, formed by the coordinates of the normed eigenvectors of A (which are computable by [ZB01]), transform A to the diagonal form

$$L^*AL = \Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\}$$

where $\lambda_j > 0$ are the eigenvalues of A. The unknown functions are then transformed to $\mathbf{u} = L\mathbf{v_1}$ while the matrix B is transformed to $B_1 = L^*BL$.

b) Applying the transform $\mathbf{v_1} = D\mathbf{v_2}$, where $D = \Lambda^{-\frac{1}{2}}$, make the coefficient of $\dfrac{\partial}{\partial t}$ the identity matrix $D^*\Lambda D = I_n,\ D^*B_1D = B_2$.

c) Applying the transform $\mathbf{v_2} = K\mathbf{v}$, where K is the orthogonal matrix formed by the coordinates of normed eigenvectors of B_2, make the coefficient of $\dfrac{\partial}{\partial x}$ a diagonal matrix; note that the identity coefficient of $\dfrac{\partial}{\partial t}$ remains unchanged.

Finally, we set $T = LDK$ and $\mathbf{u} = T\mathbf{v}$. In this way, the linear substitution $\mathbf{v} = T^{-1}\mathbf{u}$ transforms the system (4) to the canonical form (5). From the main result of [ZB01] it follows that, given A and the cardinality of spectrum of A, one can compute the eigenvectors of A. Thus, matrices L, Λ, K, B_1, B_2 are computable from A, B and the cardinalities of spectra of A and $\lambda A - B$. □

From well-known facts of computable analysis we immediately obtain

Corollary 6. *If A, B are computable symmetric real matrices with $A > 0$ then there exist a computable diagonal matrix M and a computable linear transformation $T, \mathbf{u} = T\mathbf{v}$, such that* (5) *is equivalent to* (4)*.*

Next we observe that the domain H for the problem (1) is computable from $A, B_1, \ldots, B_m$ (more exactly, vector $(\lambda^{(1)}_{\max}, \ldots, \lambda^{(m)}_{\max}, \lambda^{(1)}_{\min}, \ldots, \lambda^{(m)}_{\min})$ from Subsection 2.1 is computable from $A, B_1, \ldots, B_m$; this implies the computability of H in the sense of computable analysis [We00]).

Since, for each $i = 1, \ldots, m$, $\lambda^{(i)}_{\max}$ is the maximal and $\lambda^{(i)}_{\min}$ is the minimal eigenvalue of the matrix pencil $\lambda A - B_i$, and maximum and minimum of a vector of reals are computable [We00], it suffices to show that the vector $(\lambda_1, \ldots, \lambda_n)$ of eigenvalues of $\lambda A - B$ is computable from A, B. But $(\lambda_1, \ldots, \lambda_n)$ is the vector of roots of the characteristic polynomial, hence it is computable [We00].

Again we immediately obtain

Corollary 7. *If $A, B_1, \ldots, B_m$ are computable symmetric real matrices with $A > 0$ then the eigenvalues $\lambda^{(1)}_{\max}, \ldots, \lambda^{(m)}_{\max}, \lambda^{(1)}_{\min}, \ldots, \lambda^{(m)}_{\min}$ are computable reals.*

In the case when, along with matrices $A, B_1, \ldots, B_m$, the cardinality of the spectrum of A is given we may use the algorithm from the proof of Theorem 5 to compute the eigenvalues $\lambda^{(1)}_{\max}, \ldots, \lambda^{(m)}_{\max}, \lambda^{(1)}_{\min}, \ldots, \lambda^{(m)}_{\min}$ in a way different from finding the roots of the characteristic polynomials and computing maxima and minima. Indeed, again it suffices to show that the vector $(\lambda_1, \ldots, \lambda_n)$ of eigenvalues of $\lambda A - B$ is computable from A, B and the cardinality of spectrum of A.

Let L be the orthogonal matrix formed by coordinates of the normed eigenvectors of A. By [ZB01], L is computable. Then $L^*L = I_n$, and the matrix $L^*AL = \Lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$ is diagonal, where $\lambda_j > 0$ are the eigenvalues of

A. By the proof of Theorem 1 in [Ga67], the eigenvalues of $\lambda A - B$ coincide with those of the symmetric matrix $B_2 = D^* B_1 D$ where $B_1 = L^* B L$ and $D = \Lambda^{-\frac{1}{2}}$. Therefore, they may be computed by standard algorithms of linear algebra used in numerical methods (see e.g. [Go97]). It seems interesting to investigate the complexity of finding those eigenvalues.

4 Finite-Dimensional Approximation

In this section we shortly describe the difference scheme for the initial value problem (1) and formulate some of its properties most of which are known [Go76, GR62].

4.1 Description of the Difference Scheme

The difference scheme may be chosen in various ways. Our scheme taken from [Go76] is a little more complicated than the scheme used in [Go71] but it has some useful feature mentioned in the introduction. We describe it in few stages.

1. First we describe some discretization details. To simplify notation, we stick to the 2-dimensional case $x_1 = x$, $x_2 = y$, $B_1 = B$, $B_2 = C$, i.e., $m = 2$ and $n \geq 2$. For $m \geq 3$ the difference scheme is obtained in the same way as for $m = 2$ but the step from $m = 1$ to $m = 2$ is nontrivial.

Consider the uniform rectangular grid G on $Q = [0,1]^2$ defined by the family of lines $\{x = x_j\}, \{y = y_k\}$ where $1 \leq j, k \leq 2^N$ for some natural number N (because of the many indices, we slightly modified here the grid notation from Subsection 2.2, see Figure 1). Let $h = x_j - x_{j-1} = y_k - y_{k-1} = 1/2^N$ be the step of the grid. Associate to any function $g \in \{u_1, \ldots, u_n\}$ and any fixed time point $t = l\tau$, $l \in \mathbb{N}$, the vector of dimension 2^{2N} with the components

$$g_{j-\frac{1}{2},k-\frac{1}{2}} = g\left(\frac{j-\frac{1}{2}}{2^N}, \frac{k-\frac{1}{2}}{2^N}, t\right)$$

equal to the values of g in the centers of grid cells.

Note that, strictly speaking, we work with modifications of the grids G_k in Subsection 2.2 when the centers of grid cells are taken as nodes of the modified grids.

2. Consider the following two auxiliary one-dimensional systems with parameters obtained by fixing any of the variables x, y:

$$A\frac{\partial \mathbf{u}}{\partial t} + B\frac{\partial \mathbf{u}}{\partial x} = 0,\ A\frac{\partial \mathbf{u}}{\partial t} + C\frac{\partial \mathbf{u}}{\partial y} = 0$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_n)^T$. Transform the systems into the canonical forms

$$\frac{\partial \mathbf{v}_x}{\partial t} + M_x\frac{\partial \mathbf{v}_x}{\partial x} = 0, \quad \frac{\partial \mathbf{v}_y}{\partial t} + M_y\frac{\partial \mathbf{v}_y}{\partial y} = 0. \tag{6}$$

via the linear transformations $\mathbf{v}_x = T_x^{-1}\mathbf{u}$ and $\mathbf{v}_y = T_y^{-1}\mathbf{u}$, as described in Section 3.

3. Any of the systems (6) in the canonical form consists of n independent equations of the form

$$\frac{\partial w}{\partial t} + \mu\frac{\partial w}{\partial x} = 0 \tag{7}$$

where $w = w(x,t)$ is a scalar function and $\mu \in \mathbb{R}$. Consider for equation (7) the following difference scheme. The function $w(t_0, x)$, already computed at time level $t = t_0$ (initially $t = 0$), is substituted by the piecewise-constant function with the values $w_{j-\frac{1}{2}}$ within the corresponding grid cell $x_{j-1} < x \leq x_j$. The "large values" defined at the boundaries of the grid cells are computed as follows:

$$\mathcal{W}_j = \begin{cases} w_{j-\frac{1}{2}}, \text{ if } \mu \geq 0, \\ w_{j+\frac{1}{2}}, \text{ if } \mu < 0. \end{cases} \tag{8}$$

Values on the next time level $t = t_0 + \tau$ (τ is a time step depending on h as specified below) are then computed as

$$w^{j-\frac{1}{2}} = w_{j-\frac{1}{2}} - \mu\frac{\tau}{h}(\mathcal{W}_j - \mathcal{W}_{j-1}).$$

or, in an equivalent form,

$$w^{j-\frac{1}{2}} = \begin{cases} w_{j-\frac{1}{2}} - \mu\frac{\tau}{h}(w_{j-\frac{1}{2}} - w_{j-\frac{3}{2}}), \text{ if } \mu \geq 0, \\ \\ w_{j-\frac{1}{2}} - \mu\frac{\tau}{h}(w_{j+\frac{1}{2}} - w_{j-\frac{1}{2}}), \text{ if } \mu < 0. \end{cases} \tag{9}$$

Observe that the formulas above are in fact written for the unbounded grid and, strictly speaking, they do not define the value $\mathcal{W}_0$ (resp. $\mathcal{W}_{2^N}$) for $\mu \geq 0$ (resp. $\mu < 0$) for these "boundary" points. We define for these points the values $\mathcal{W}_0 = w_{-1}$ (resp. $\mathcal{W}_{2^N} = w_{2^N+\frac{1}{2}}$) as $w_{2^N-\frac{1}{2}}$ (resp. $w_{\frac{1}{2}}$) in case $\mu \geq 0$ (resp. $\mu < 0$). The same trick is used to define $\mathbf{u}_{0,k-\frac{1}{2}}$, $\mathbf{u}_{2^N,k-\frac{1}{2}}$, $\mathbf{u}_{j-\frac{1}{2},0}$, $\mathbf{u}_{j-\frac{1}{2},2^N}$ and similarly for $\mathbf{v}$.

The grid solutions computed in this way approximate solutions of the differential equation (7) with the first order of accuracy h (see Subsection 2.3).

Recall from Subsection 2.3 that a difference scheme is *stable* if the corresponding difference operators R_h (that send the grid function $\{w_{j-\frac{1}{2}}\}_{j=1}^{2^N}$ to the grid function $\{w^{j-\frac{1}{2}}\}_{j=1}^{2^N}$) are bounded uniformly on h, together with their powers. The investigation of stability of difference schemes usually relies on Fourier analysis (here we only mention without proofs some known related facts [GR62, Go76] applied to our scheme, see some details below). The necessary and sufficient condition for our scheme to be stable is $|\mu|\frac{\tau}{h} \leq 1$.

Let us consider only the case $\mu > 0$ (in case $\mu < 0$ the argument is similar). We denote by $\nu = |\mu|\dfrac{\tau}{h}$ the Courant number and check the scheme stability by the Fourier method. Substituting in (9) the values

$$w_{j-\frac{1}{2}} = w^* e^{ij\phi}, \quad w^{j-\frac{1}{2}} = \lambda w_{j-\frac{1}{2}},$$

we obtain the characteristic equation

$$\lambda(\phi) = 1 - \nu(1 - e^{-i\phi}).$$

The necessary and sufficient condition for the stability is the condition $|\lambda(\phi)| \le 1$ for all $\phi \in [0, 2\pi)$, which is equivalent to the condition $\nu \le 1$ for the Courant number (see e.g. [GR62]).

4. Taking the scheme above for each equation of the systems (6), we obtain for them schemes of the following form:

$$\frac{\mathbf{v_x}^{j-\frac{1}{2}} - \mathbf{v_x}_{j-\frac{1}{2}}}{\tau} + M_x \frac{(\mathcal{V}_x)_j - (\mathcal{V}_x)_{j-1}}{h} = 0$$

where lower indices mean step $l\tau$, upper indices mean step $(l+1)\tau$ and $\mathcal{V}_j$ is the vector consisting of the "large values" given by one-dimensional formulas (8).

As shown in [Go76], by making the inverse transformation we can write the following scheme that approximates the system (1) with the first order of accuracy:

$$A\frac{\mathbf{u}^{j-\frac{1}{2},k-\frac{1}{2}} - \mathbf{u}_{j-\frac{1}{2},k-\frac{1}{2}}}{\tau} + B\frac{\mathcal{U}_{j+1,k-\frac{1}{2}} - \mathcal{U}_{j,k-\frac{1}{2}}}{h} + C\frac{\mathcal{U}_{j-\frac{1}{2},k+1} - \mathcal{U}_{j-\frac{1}{2},k}}{h} = 0 \tag{10}$$

where $\mathcal{U}_{j,k-\frac{1}{2}} = T_x(\mathcal{V}_x)_j$ and $\mathcal{U}_{j-\frac{1}{2},k} = T_y(\mathcal{V}_y)_k$.

The stability condition looks now as

$$\tau\left(\frac{1}{\tau_x} + \frac{1}{\tau_y}\right) \le 1, \tag{11}$$

where $\tau_x = \max\limits_i\{\mu_i(\mu A - B)\}h$ and $\tau_y = \max\limits_i\{\mu_i(\mu A - C)\}h$ are the maximal time steps guaranteeing the stability of the corresponding one-dimensional schemes [Go76].

It is easy to check that $\left\{\mathbf{u}_{j-\frac{1}{2},k-\frac{1}{2}}\right\} \mapsto \left\{\mathbf{u}^{j-\frac{1}{2},k-\frac{1}{2}}\right\}$ is a linear operator on the corresponding spaces of grid functions.

4.2 Some Estimates

Here we assume that the domain H is compact and $H \subseteq Q \times [0, T]$ for some $T > 0$. Given (11), it can be derived [Go76] that

$$||\{\mathbf{u}^{j-\frac{1}{2},k-\frac{1}{2}}\}||_A \le ||\{\mathbf{u}_{j-\frac{1}{2},k-\frac{1}{2}}\}||_A, \tag{12}$$

where

$$||\{\mathbf{u}_{j-\frac{1}{2},k-\frac{1}{2}}\}||_A = \sqrt{h^2 \sum_{j,k} \left\langle A\mathbf{u}_{j-\frac{1}{2},k-\frac{1}{2}}, \mathbf{u}_{k-\frac{1}{2},j-\frac{1}{2}} \right\rangle}.$$

Fixing the initial values at the right hand side of (12), taking the maximal value of the left hand side w.r.t. t such that $\left(\frac{j-\frac{1}{2}}{2^N}, \frac{k-\frac{1}{2}}{2^N}, t\right) \in H$ for some $j, k \in \{1, 2, \ldots, 2^N\}$, and using an equivalent norm on $\mathbb{R}^n$, we obtain

$$\max_{0 \le l\tau \le T} \left(h^2 \sum_{j,k} \mathbf{u}^2_{j-\frac{1}{2},k-\frac{1}{2}} \right) \le c \cdot \left(h^2 \sum_{j,k} \varphi^2_{j-\frac{1}{2},k-\frac{1}{2}} \right) \tag{13}$$

where $\mathbf{u}^2 = \langle \mathbf{u}, \mathbf{u} \rangle$ and $c = \left(\frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}\right)$ (by Section 3, c is computable). Note that, since φ is continuous, the expression $\sum\limits_{j,k} \varphi^2_{j-\frac{1}{2},k-\frac{1}{2}}$ tends to $\int\limits_Q \varphi^2 dxdy$ as h tends to 0, and is therefore bounded.

Remark. Note that the proof of existence theorem for the initial value problem in [Go71] uses similar estimates also for the difference schemes obtained by formal differentiation of the scheme considered above. To assure this differentiation be correct, one needs natural smoothness assumptions, which is one of reasons for the presence of such assumptions in our main theorem and the result in the next section. Another reason is that for using a first order difference scheme we need the solution of (1) to be at least C^2 (see e.g. [GR62]).

Finally, we derive an estimate for the multilinear interpolation $\widetilde{\varphi|_G}$ of the initial function on the grid G and the multilinear interpolation $\tilde{u}$ of of the grid function u on the grid on $Q \times [0, T]$ obtained from G and the time step τ.

In one-dimensional case, the interpolating function $\tilde{\mathbf{u}}$ is defined inside the grid rectangles

$$\left(j - \frac{1}{2}\right) h \le x \le \left(j + \frac{1}{2}\right) h; \; l\tau \le t \le (l+1)\tau$$

in the standard way as follows (the interpolations are illustrated by Figure 2):

$$\begin{aligned} \tilde{\mathbf{u}}(x,t) = \mathbf{u}_{j-\frac{1}{2}} \left(l + 1 - \frac{t}{\tau}\right) \left(j + \frac{1}{2} - \frac{x}{h}\right) + \mathbf{u}_{j+\frac{1}{2}} \left(l + 1 - \frac{t}{\tau}\right) \left(\frac{x}{h} - \left(j - \frac{1}{2}\right)\right) \\ + \mathbf{u}^{j-\frac{1}{2}} \left(\frac{t}{\tau} - l\right) \left(j + \frac{1}{2} - \frac{x}{h}\right) + \mathbf{u}^{j+\frac{1}{2}} \left(\frac{t}{\tau} - l\right) \left(\frac{x}{h} - \left(j - \frac{1}{2}\right)\right) \end{aligned}$$

where $\mathbf{u}_{j\pm\frac{1}{2}}$ and $\mathbf{u}^{j\pm\frac{1}{2}}$ are the grid functions on levels $t = l\tau$ and $t = (l+1)\tau$, respectively.

In two-dimensional case (and for higher dimensions) the interpolating function is defined in a similar way. Since the full expression is rather long we write down only two (of six) summands:

$$\begin{aligned}\tilde{\mathbf{u}}(x,y,t) &= \mathbf{u}_{j-\frac{1}{2},k-\frac{1}{2}}\left(l+1-\frac{t}{\tau}\right)\left(j+\frac{1}{2}-\frac{x}{h}\right)\left(k+\frac{1}{2}-\frac{y}{h}\right)\\ &+ \mathbf{u}_{j+\frac{1}{2},k-\frac{1}{2}}\left(l+1-\frac{t}{\tau}\right)\left(\frac{x}{h}-\left(j-\frac{1}{2}\right)\right)\left(k+\frac{1}{2}-\frac{y}{h}\right)\\ &+\cdots\end{aligned}$$

where $\left(k-\frac{1}{2}h\right) \leq y \leq \left(k+\frac{1}{2}\right)h$.

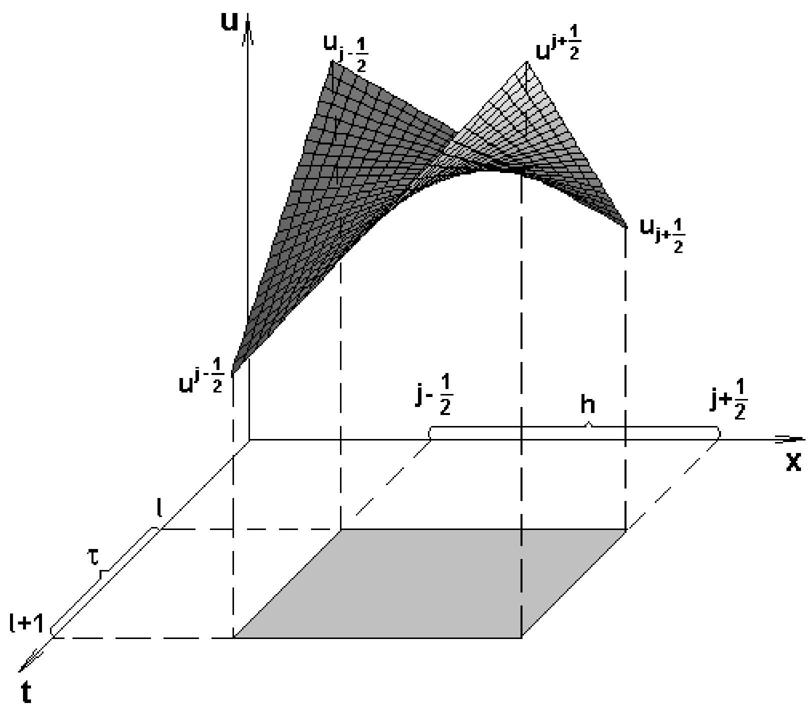

Figure 2. Bilinear interpolation in a grid cell.

From these formulas for multilinear interpolation, the linearity of the interpolation operator is obvious.

It is known [Go76] and easy to see by a direct computation that

$$\max_{0\leq l\tau\leq T} \int_{H\cap\{t=l\tau\}} |\tilde{u}(x,y,t)|^2 dxdy \leq \max_{0\leq l\tau\leq T}\left(h^2 \sum_{j,k} \mathbf{u}^2_{j-\frac{1}{2},k-\frac{1}{2}}\right). \tag{14}$$

Obviously,

$$h^2 \sum_{j,k} \varphi^2_{j-\frac{1}{2},k-\frac{1}{2}} \leq h^2 \frac{1}{h^2} \max_{j,k} \varphi^2_{j-\frac{1}{2},k-\frac{1}{2}} \leq \sup_{(x,y)\in Q} \widetilde{\varphi|_G}^2(x,y).$$

The last three estimates imply that for a constant c we have

$$||\tilde{u}|_H||_{sL_2} = \max_{0\leq l\tau\leq T} \sqrt{\int\limits_{H\cap\{t=l\tau\}} |\tilde{u}(x,y,t)|^2 dxdy} \leq$$
$$\leq c\cdot \sup_{(x,y)\in Q} \widetilde{\varphi|_G}(x,y) = c\cdot ||\widetilde{\varphi|_G}||_s \tag{15}$$

where $c = c(A) = \sqrt{\frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}}$.

4.3 Convergence of the Difference Scheme

In this section we show that the multilinear interpolations of grid solutions of the difference scheme (10) converge to the solution of the initial value problem (1).

Let $M > 0$ and let $\varphi \in C^{p+1}(Q,\mathbb{R}^n)$, $p \geq 2$, be a function satisfying $||\frac{\partial\varphi}{\partial x_i}||_{L_2}, ||\frac{\partial^2\varphi}{\partial x_i \partial x_j}||_{L_2} \leq M$ for all $i,j \in \{1,\ldots,m\}$. Let $A, B_1,\ldots,B_m$ be symmetric real matrices such that $A > 0$ and $\lambda^{(i)}_{\min} < 0 < \lambda^{(i)}_{\max}$ for all $i = 1,\ldots,m$. By Subsection 2.1, H is compact and there is a unique solution $\mathbf{u} \in C^p(H,\mathbb{R}^n)$ of (1). Choose $T > 0$ such that $H \subseteq Q\times[0,T]$. For any $k \geq 0$, let $\varphi|_{G_k}$ be the restriction of φ to the grid G_k in Q with step $\frac{1}{2^k}$ (recall that φ is actually an n-tuple of functions, hence the restrictions are componentwise). Note that G_k here denotes the modified grid in Subsection 4.1 rather than the grid in Subsection 2.2. For any $k \geq 0$, let τ_k be the time step such that the scheme (10) is stable (τ_k is any number satisfying (11) for $h = \frac{1}{2^k}$ and dividing T). For any $k \geq 0$, let u_k be the solution of the difference equation (10) on the grid G^τ_k in $Q\times[0,T]$ obtained from G_k and the time step $\tau = \tau_k$. Recall from Subsection 2.2 that $\tilde{f}$ denotes the multilinear interpolation of a grid function f.

Theorem 8. *Using the notation and assumptions of the previous paragraph, there are constants c_{diff}, c depending only on matrices $A, B_1,\ldots,B_m$ and on M such that for all $k \geq 0$ we have $||u_k - u|_{G^\tau_k}||_{sL_2} \leq c_{\text{diff}}\cdot\frac{1}{2^k}$ and $||\tilde{u}_k - u||_{sL_2} \leq c\cdot\frac{1}{2^k}$, where u is the solution of (1).*

Proof (sketch). The first estimate follows from Theorem 4. The fact that c_{diff} depends only on matrices $A, B_1,\ldots,B_m$ and on M follows from the proof of Theorem 4 in [GR62] according to which we can take $c_{\text{diff}} = c_1\cdot c_2$ where c_2 comes from the stability condition (in our case, $c_2 = \sqrt{\frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}}$) and c_1 from approximation ($||L_h u_h - (Lu)|_{G_k^\tau}||_{sL_2} \leq c_1 h$). Since we consider a first-order difference scheme, it follows from the Taylor decomposition of u that c_1 depends only on $A, B_1, \ldots, B_m$ and $||\frac{\partial^2 u}{\partial x_i \partial x_j}||_{sL_2}$, $||\frac{\partial^2 u}{\partial x_i \partial t}||_{sL_2}$. By the proof of the uniqueness theorem for (1) (details may be found in [Go71]), the norm of the derivatives above are bounded by

$$c(A, B_1, \ldots, B_m) \cdot \max_{i,j=\overline{1,n}} ||\frac{\partial^2 \varphi}{\partial x_i \partial x_j}||_{L_2} \leq \tilde{c}(A, B_1, \ldots, B_m, M)$$

For the second estimate, we have

$$||\tilde{u}_k - u||_{sL_2} \leq ||\tilde{u}_k - \widetilde{u|_{G_k^\tau}}||_{sL_2} + ||\widetilde{u|_{G_k^\tau}} - u||_{sL_2}.$$

By well-known properties of the multilinear interpolations (see e.g. [ZKM80]), $||\widetilde{u|_{G_k^\tau}} - u||_s \leq c_{\text{int}} \cdot \frac{1}{2^k}$ where c_{int} depends only on sL_2-norms of the first and second derivatives of u, hence only on $A, B_1, \ldots, B_m$ and L_2-norms of the first and second derivatives of φ. Therefore, $c_{\text{int}} = c_{\text{int}}(A, B_1, \ldots, B_m, M)$. Since the operator of multilinear interpolation is linear, from (14) we obtain

$$||\tilde{u}_k - \widetilde{u|_{G_k^\tau}}||_{sL_2} \leq ||u_k - u|_{G_k^\tau}||_{sL_2}.$$

Taking into account the first estimate, we obtain

$$||\tilde{u}_k - \widetilde{u|_{G_k^\tau}}||_{sL_2} \leq (c_{\text{diff}} + c_{\text{int}}) \cdot \frac{1}{2^k}.$$

This implies the second estimate. □

5 Computing the Solution Operator

We are ready to prove the main result of this paper:

Theorem 9. *Let $M > 0, p \geq 2$, let $A, B_1, \ldots, B_m$ be computable symmetric real matrices with $A > 0$, and let $\lambda_{\text{min}}^{(i)} < 0 < \lambda_{\text{max}}^{(i)}$ for all $i = 1, \ldots, m$. Then the operator $\varphi \mapsto \mathbf{u}$ sending any function $\varphi \in C^{p+1}(Q, \mathbb{R}^n)$ such that $||\frac{\partial \varphi}{\partial x_i}||_{L_2}, ||\frac{\partial^2 \varphi}{\partial x_i \partial x_j}||_{L_2} \leq M$ for all $i, j \in \{1, \ldots, m\}$ to the unique solution $\mathbf{u} \in C^p(H, \mathbb{R}^n)$ of (1) is a computable partial function from $C_s(Q, \mathbb{R}^n)$ to $C_{sL_2}(H, \mathbb{R}^n)$.*

Proof (sketch). We use the notation of Theorem 8. First let us observe that one can compute a rational number $T > 0$ such that $H \subseteq Q \times [0, T]$ (see Figure 3 that shows a projection of H). Indeed, since $\lambda_{\text{min}}^1 < 0$ is a computable real, we can compute a rational q with $\lambda_{\text{min}}^1 < q < 0$. Symmetrically, we can compute a rational r such that $0 < r < \lambda_{\text{max}}^1$. Let T be the cross-point of the lines $x_1 = rt$ and $x_1 = 1 + qt$ in the plane (x_1, t), i.e., $T = \frac{1}{r-q}$ (see Figure 3). By the description of H in Subsection 2.1, T has the desired property.

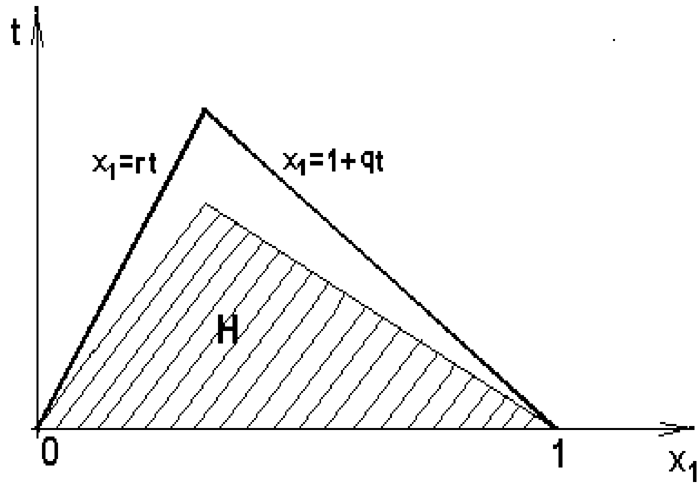

Figure 3. A computable bound for H.

Choose a computable sequence $\{\tau_k\}$ of rational numbers for which the estimates of Theorem 8 hold. According to Subsections 2.4 and 2.2, to prove the computability we have to find an algorithm that, given a sequence $\{f_k\}$ of grid functions $f_k : G_{i_k} \to \mathbb{Q}^n$ such that $\{\tilde{f}_k\}$ is a Cauchy sequence converging in $C_s(Q, \mathbb{R}^n)$ to φ, computes a sequence $\{v_k\}$ of grid functions $v_k : G^{\tau}_{i_k} \to \mathbb{Q}^n$ (for some sequence $\{i_k\}$ of natural numbers) such that $\{\tilde{v}_k|_H\}$ is a Cauchy sequence converging in $C_{sL_2}(H, \mathbb{R}^n)$ to u. W.l.o.g. we may assume that the sequence $\{i_k\}$ is increasing (otherwise, choose a suitable subsequence of $\{f_k\}$).

Let $A^{(k)}, B_1^{(k)}, \cdots, B_m^{(k)}$ be computable sequences of rational matrices fast converging (in the usual Eucleadean norm $||\cdot||_2$) to $A, B_1, \ldots, B_m$, respectively. Let v_k be constructed from f_k by the algorithm of the difference equation in Subsection 4.1 from approximations $A^{(k)}, B_1^{(k)}, \cdots, B_m^{(k)}$. Let $\hat{v}_k$ be constructed from f_k by the algorithm of the difference equation in Subsection 4.1 from the "exact" matrices $A, B_1, \ldots, B_m$.

It suffices to show that for some constant c (depending only on matrices $A, B_1, \ldots, B_m$, which are in fact fixed in our theorem, and on M) we have $||\tilde{v}_k|_H - u||_{sL_2} \leq c \cdot \frac{1}{2^k}$ for all k. We have

$$||\tilde{v}_k|_H - u||_{sL_2} \leq ||\tilde{v}_k|_H - \tilde{\hat{v}}_k|_H||_{sL_2} + ||\tilde{\hat{v}}_k|_H - \tilde{u}_k|_H||_{sL_2} + ||\tilde{u}_k|_H - u||_{sL_2}. \quad (16)$$

By Theorem 8, $||\tilde{u}_k|_H - u||_{sL_2} \leq c \cdot \frac{1}{2^{i_k}}$ for a constant $c = c(A, B_1, \ldots, B_m, M)$, hence it remains to get similar estimates for the first and second summands in

(16). For the second summand, from (15) we derive

$$||\tilde{\hat{v}}_k|_H - \tilde{u}_k|_H||_{sL_2} \leq \sqrt{\frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}} ||\tilde{f}_k - \widetilde{\varphi|_{G_k}}||_s.$$

Similarly to a fact in the proof of Theorem 8, $||\varphi - \widetilde{\varphi|_{G_k}}||_{L_2} \leq c_{\text{int}} \cdot \frac{1}{2^k}$ for some constant c_{int}. This implies the desired estimate.

Turning to the first summand, recall that v_k and $\hat{v}_k$ satisfy respectively the following difference schemes (see (10)) in which the index k in v_k, $\hat{v}_k$ and w_k is omitted for simplicity

$$A\frac{\mathbf{v}^{j-\frac{1}{2},l-\frac{1}{2}} - \mathbf{v}_{j-\frac{1}{2},l-\frac{1}{2}}}{\tau} + B\frac{\mathcal{V}_{j+1,l-\frac{1}{2}} - \mathcal{V}_{j,l-\frac{1}{2}}}{h} + C\frac{\mathcal{V}_{j-\frac{1}{2},l+1} - \mathcal{V}_{j-\frac{1}{2},l}}{h} = 0,$$

$$A^{(k)}\frac{\hat{\mathbf{v}}^{j-\frac{1}{2},l-\frac{1}{2}} - \hat{\mathbf{v}}_{j-\frac{1}{2},l-\frac{1}{2}}}{\tau} + B^{(k)}\frac{\hat{\mathcal{V}}_{j+1,l-\frac{1}{2}} - \hat{\mathcal{V}}_{j,l-\frac{1}{2}}}{h} + C^{(k)}\frac{\hat{\mathcal{V}}_{j-\frac{1}{2},l+1} - \hat{\mathcal{V}}_{j-\frac{1}{2},l}}{h} = 0$$

with the initial conditions $v|_{t=0} = f_k$ and $\hat{v}|_{t=0} = f_k$, respectively. Deducing the second equation from the first one and taking into account the linearity of the difference scheme, we obtain the scheme

$$A\frac{\mathbf{w}^{j-\frac{1}{2},l-\frac{1}{2}} - \mathbf{w}_{j-\frac{1}{2},l-\frac{1}{2}}}{\tau} + B\frac{\mathcal{W}_{j+1,l-\frac{1}{2}} - \mathcal{W}_{j,l-\frac{1}{2}}}{h} + C\frac{\mathcal{W}_{j-\frac{1}{2},l+1} - \mathcal{W}_{j-\frac{1}{2},l}}{h} = f,$$

with the initial condition $w|_{t=0} = 0$ where $w = v - \hat{v}$ and

$$f = (A^{(k)} - A)\frac{\hat{\mathbf{v}}^{j-\frac{1}{2},l-\frac{1}{2}} - \hat{\mathbf{v}}_{j-\frac{1}{2},l-\frac{1}{2}}}{\tau} + (B^{(k)} - B)\frac{\hat{\mathcal{V}}_{j+1,l-\frac{1}{2}} - \hat{\mathcal{V}}_{j,l-\frac{1}{2}}}{h}$$
$$+(C^{(k)} - C)\frac{\hat{\mathcal{V}}_{j-\frac{1}{2},l+1} - \hat{\mathcal{V}}_{j-\frac{1}{2},l}}{h}.$$

Then $||v - \hat{v}||_{sL_2} \leq c||f||_{L_2}$ for some constant $c = c(A, B_1, \ldots, B_m)$ [Go76]. By formal differentiation of the scheme in [Go71, Go76] it is checked that any of

$$||\frac{\hat{\mathbf{v}}^{j-\frac{1}{2},l-\frac{1}{2}} - \hat{\mathbf{v}}_{j-\frac{1}{2},l-\frac{1}{2}}}{\tau}||_{sL_2},\ ||\frac{\hat{\mathcal{V}}_{j+1,l-\frac{1}{2}} - \hat{\mathcal{V}}_{j,l-\frac{1}{2}}}{h}||_{sL_2},\ ||\frac{\hat{\mathcal{V}}_{j-\frac{1}{2},l+1} - \hat{\mathcal{V}}_{j-\frac{1}{2},l}}{h}||_{sL_2}$$

is below the difference derivatives of f_k, and so below a constant depending only on $A, B_1, \ldots, B_m, M$. Since $||A^{(k)} - A||_2$, $||B^{(k)} - B||_2$, and $||C^{(k)} - C||_2$ are below $\frac{1}{2^k}$, the desired estimate for the first summand follows. $\square$

As usual, we immediately obtain

Corollary 10. *Let $M > 0, p \geq 2$, let $A, B_1, \ldots, B_m$ be computable symmetric real matrices with $A > 0$, let $\lambda_{\min}^{(i)} < 0 < \lambda_{\max}^{(i)}$ for all $i = 1, \ldots, m$, and let $\varphi \in C^{p+1}(Q, \mathbb{R}^n)$ be a computable element of $C_s(Q, \mathbb{R}^n)$ such that for all $i, j \in \{1, \ldots, m\}$ it holds $||\frac{\partial \varphi}{\partial x_i}||_{L_2}, ||\frac{\partial^2 \varphi}{\partial x_i \partial x_j}||_{L_2} \leq M$. Then the unique solution $\mathbf{u} \in C^p(H, \mathbb{R}^n)$ of (1) is a computable element of $C_{sL_2}(H, \mathbb{R}^n)$.*

6 Examples

In this section we illustrate some notions and results above by some practically important examples of PDE.

6.1 Linear Acoustics

The equations of linear 1-dimensional acoustics ($m = 1, n = 2$) are derived from the isentropic compressible flow model under the assumptions that fluid motions are small. Here u, p are infinitesimal variations of velocity and pressure, respectively; $\rho_0 > 0$ is the density, and $c_0 > 0$ is the speed of sound.

$$\begin{cases} \rho_0 \dfrac{\partial u}{\partial t} + \dfrac{\partial p}{\partial x} = 0, \\ \dfrac{\partial p}{\partial t} + \rho_0 c_0^2 \dfrac{\partial u}{\partial x} = 0. \end{cases}$$

Multiplying the first equation by c_0 and dividing the second one by $\rho_0 c_0$, we obtain an equivalent symmetric hyperbolic system $A\frac{\partial \mathbf{u}}{\partial t} + B\frac{\partial \mathbf{u}}{\partial x} = 0$ where

$$\mathbf{u} = \begin{pmatrix} u \\ p \end{pmatrix}, \ A = \begin{pmatrix} \rho_0 c_0 & 0 \\ 0 & \frac{1}{\rho_0 c_0} \end{pmatrix} = A^* > 0, \ B^* = B = \begin{pmatrix} 0 & c_0 \\ c_0 & 0 \end{pmatrix}.$$

It is easy to find the eigenvalues $\pm c_0$ and the corresponding eigenvectors of B and to see that, in notation of Section 3, $M = \begin{pmatrix} c_0 & 0 \\ 0 & -c_0 \end{pmatrix}$ and

$$\mathbf{v} = T^{-1}\mathbf{u} = \begin{pmatrix} \sqrt{\frac{\rho_0 c_0}{2}} & \frac{1}{\sqrt{2\rho_0 c_0}} \\ \sqrt{\frac{\rho_0 c_0}{2}} & -\frac{1}{\sqrt{2\rho_0 c_0}} \end{pmatrix} \mathbf{u} = \begin{pmatrix} \sqrt{\frac{\rho_0 c_0}{2}} u + \frac{1}{\sqrt{2\rho_0 c_0}} p \\ \sqrt{\frac{\rho_0 c_0}{2}} u - \frac{1}{\sqrt{2\rho_0 c_0}} p \end{pmatrix}.$$

The characteristics are the lines $x \pm c_0 t = \text{const}$, hence the domain of correctness H looks as shown at Figure 4.

The equations of linear 2-dimensional acoustics ($m = 2, n = 3$) are a natural generalization of those for the 1-dimensional case:

$$\begin{cases} \rho_0 \dfrac{\partial u}{\partial t} + \dfrac{\partial p}{\partial x} = 0, \\ \rho_0 \dfrac{\partial v}{\partial t} + \dfrac{\partial p}{\partial y} = 0, \\ \dfrac{\partial p}{\partial t} + \rho_0 c_0^2 \left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} \right) = 0. \end{cases}$$

The eigenvalues $0, \pm c_0$ and the corresponding eigenvectors are found easily, the characteristics are the planes $x = \text{const}, y = \text{const}, x \pm c_0 t = \text{const}, y \pm c_0 t = \text{const}$, and the domain of correctness H looks now as shown at Figure 5.

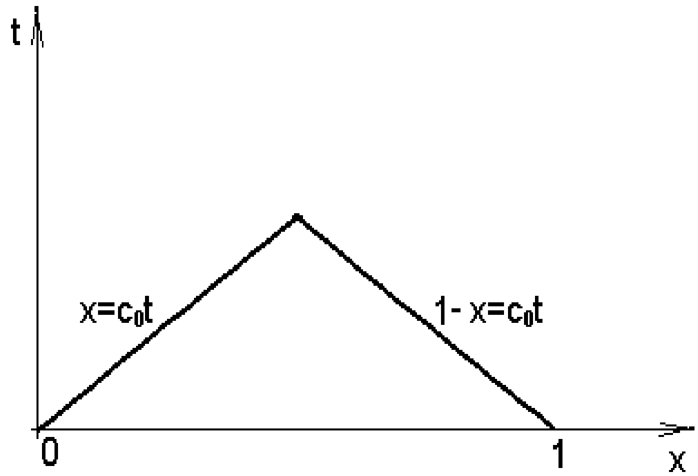

Figure 4. H for 1-dimensional acoustics.

Remark. Eliminating the variables u, v one may obtain the wave equation for the pressure variation:

$$\frac{\partial^2 p}{\partial t^2} = c_0^2 \Delta p$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplace operator. Considering the time-harmonic waves $p = W \times \exp(-i\omega t)$ one may reduce this wave equation to the Helmholtz equation:

$$\Delta W - k^2 W = 0, \quad k = \omega / c_0.$$

6.2 Maxwell Equations ($m = 3, n = 6$)

As is well known (see e.g. [LL04]), the Maxwell equations for electromagnetic waves look as follows

$$\begin{cases} \dfrac{\partial(\varepsilon E)}{\partial t} = \mathrm{rot} H, \\ \dfrac{\partial(\mu H)}{\partial t} = -\mathrm{rot} E, \end{cases}$$

where E, H are the electric and magnetic fields, respectively, and ε, μ are parameters characterizing the medium. The equations represent a symmetric hyperbolic system with $\mathbf{u} = (E_x, E_y, E_z, H_x, H_y, H_z)^T$.

Eigenvalues (in the canonical form) are $-c, -c, 0, 0, c, c$ where $c = 1/\sqrt{\varepsilon\mu}$ is the speed of propagation for the electromagnetic waves.

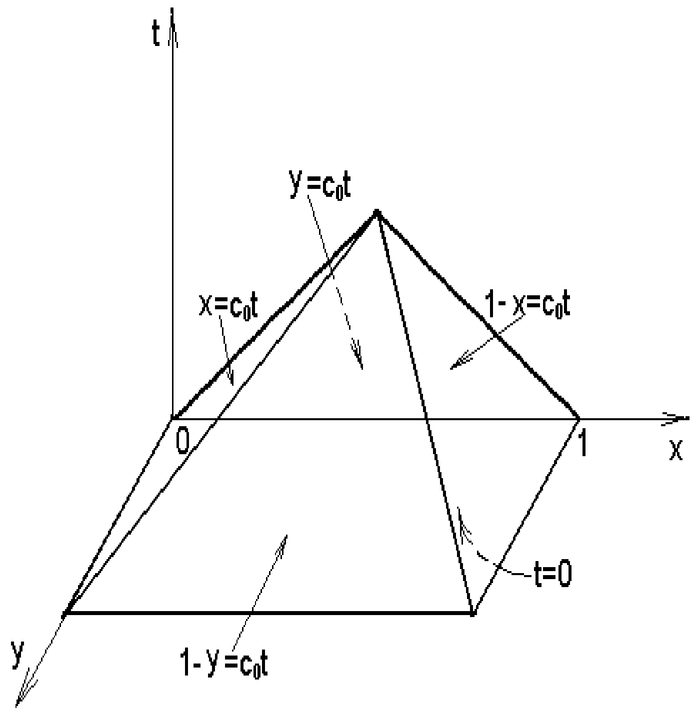

Figure 5. H for 2-dimensional acoustics.

6.3 Linear Elasticity ($m = 3, n = 9$)

Consider the 3-dimensional linear elasticity equations in the following form (see e.g. [LL86, GR03]):

$$\begin{cases} \frac{1}{2\mu}\frac{\partial \sigma_{ij}}{\partial t} - \frac{\lambda}{2\mu(3\lambda+2\mu)}\delta_{ij}\frac{\partial(\sigma_{11}+\sigma_{22}+\sigma_{33})}{\partial t} - \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) = 0, \\ \rho\frac{\partial u_i}{\partial t} - \frac{\partial \sigma_{ij}}{\partial x_j} = 0, \end{cases} \tag{17}$$

where $i, j = 1, 2, 3$, u_i are the velocities, $\sigma_{ij} = \sigma_{ji}$ is the tensor of stresses, λ, μ are the Lame coefficients. The equations represent the symmetric hyperbolic system

$$A\frac{\partial \mathbf{u}}{\partial t} + B\frac{\partial \mathbf{u}}{\partial x} + C\frac{\partial \mathbf{u}}{\partial y} + D\frac{\partial \mathbf{u}}{\partial z} = 0,$$

where $\mathbf{u} = (\sigma_{11},\ \sigma_{22},\ \sigma_{33},\ \sigma_{12},\ \sigma_{13},\ \sigma_{23},\ u,\ v,\ w)^T$,

$$A = \begin{pmatrix} \frac{\lambda+\mu}{\mu(3\lambda+2\mu)} & -\frac{\lambda}{2\mu(3\lambda+2\mu)} & -\frac{\lambda}{2\mu(3\lambda+2\mu)} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{\lambda}{2\mu(3\lambda+2\mu)} & \frac{\lambda+\mu}{\mu(3\lambda+2\mu)} & -\frac{\lambda}{2\mu(3\lambda+2\mu)} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{\lambda}{2\mu(3\lambda+2\mu)} & -\frac{\lambda}{2\mu(3\lambda+2\mu)} & \frac{\lambda+\mu}{\mu(3\lambda+2\mu)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\mu} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\mu} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\mu} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix},\ D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$A = A^* > 0,\ B = B^*,\ C = C^*, D = D^*.$

The canonical form for the system $A\frac{\partial \mathbf{u}}{\partial t} + B\frac{\partial \mathbf{u}}{\partial x} = 0$ looks as follows:

$$\frac{\partial \mathbf{v_x}}{\partial t} + M\frac{\partial \mathbf{v_x}}{\partial x} = 0 \qquad (18)$$

where

$$M = \mathrm{diag}\left\{0, 0, 0, \sqrt{\frac{\mu}{\rho}}, \sqrt{\frac{\mu}{\rho}}, -\sqrt{\frac{\mu}{\rho}}, -\sqrt{\frac{\mu}{\rho}}, \sqrt{\frac{\lambda+2\mu}{\rho}}, -\sqrt{\frac{\lambda+2\mu}{\rho}}\right\}$$

and $\mathbf{v_x}$ is the vector of Riemann invariants with the components

$$\frac{\sigma_{23}}{\sqrt{\mu}},\ \frac{\sigma_{33}-\sigma_{22}}{2\sqrt{\mu}},\ \frac{1}{\sqrt{\mu(3\lambda+2\mu)(\lambda+2\mu)}}\left[\mu\left(\sigma_{22}+\sigma_{33}\right) + \frac{\lambda}{2}\left(\sigma_{22}+\sigma_{33}-2\sigma_{11}\right)\right]$$

(invariant along the vertical characteristics $x = const$);

$$\frac{1}{\sqrt{2}}\left(\frac{\sigma_{12}}{\sqrt{\mu}} - v\sqrt{\rho}\right),\ \frac{1}{\sqrt{2}}\left(\frac{\sigma_{13}}{\sqrt{\mu}} - w\sqrt{\rho}\right)$$

(invariant along the characteristics $x = \sqrt{\frac{\mu}{\rho}}t + const$);

$$\frac{1}{\sqrt{2}}\left(\frac{\sigma_{12}}{\sqrt{\mu}} + v\sqrt{\rho}\right), \ \frac{1}{\sqrt{2}}\left(\frac{\sigma_{13}}{\sqrt{\mu}} + w\sqrt{\rho}\right)$$

(invariant along the characteristics $x = -\sqrt{\frac{\mu}{\rho}}t + const$);

$$\frac{1}{\sqrt{2}}\left(-\frac{\sigma_{11}}{\sqrt{\lambda+2\mu}} + u\sqrt{\rho}\right)$$

(invariant along the characteristics $x = \sqrt{\frac{\lambda+2\mu}{\rho}}t + const$);

$$\frac{1}{\sqrt{2}}\left(\frac{\sigma_{11}}{\sqrt{\lambda+2\mu}} + u\sqrt{\rho}\right)$$

(invariant along the characteristics $x = -\sqrt{\frac{\lambda+2\mu}{\rho}}t + const$).

The canonical forms for analogous 1-dimensional systems with matrices C and D are obtained in the same way as (18), with the appropriate permutation of variables (see [Se05] for additional details). It is interesting to note that the computation of M (which was made "by hand") becomes much simpler with taking into account the invariance of (17) under rotations [GM98]. In fact, all systems considered in this section are invariant under rotations.

6.4 Nonlinear Elasticity

Here we briefly discuss an example of nonlinear equation (we hope that our proofs may be generalized to prove the computability for similar systems as well). The example is a nontrivial generalization of the previous one:

$$\begin{cases} \rho\dfrac{\partial u_i}{\partial t} - \dfrac{\partial s_{ij}}{\partial \xi_j} = 0, \\ \dfrac{\partial H_{s_{ij}}}{\partial t} - \dfrac{\partial u_i}{\partial \xi_j} = 0, \end{cases} \tag{19}$$

where s_{ij} are the components of the so called Piola–Kirchhof tensor which is a non-symmetric stress tensor in Lagrangian coordinates ξ_j; u_i are the velocities, $\rho = \rho(\xi_1, \xi_2, \xi_3)$ is the density of the medium; the convex function $H = H(u_i, s_{ij})$ is the so called generating potential.

This example includes, in particular, the equations of the form (1) for all crystal systems, i.e., the linear anisotropic elasticity theory. But it also models more complicated physical processes when the generating potential H (which is actually obtained as the Legendrian transform of the energy function) is nonlinear. Consequently, the matrix coefficient A by $\frac{\partial}{\partial t}$ in (1), consisting of the second derivatives of A may depend nonlinearly on (x_1, x_2, x_3) and even on the unknown variables $\mathbf{u}$ (the fact that A is positively definite follows from the convexity of H).

With the aid of representations of the rotation group $SO(3)$ this system can be written [Se08] in a certain invariant form that looks like

$$\begin{cases} \rho \dfrac{\partial}{\partial t}\mathbf{v}^{(\mathbf{1})} + \varDelta_- \varSigma^{(\mathbf{2})} + \varDelta_+ \varSigma^{(\mathbf{0})} + \varDelta_0 \varSigma^{(\mathbf{1})} = 0, \\ \hat{A}\dfrac{\partial}{\partial t}\begin{pmatrix} \varSigma^{(\mathbf{0})} \\ \varSigma^{(\mathbf{1})} \\ \varSigma^{(\mathbf{2})} \end{pmatrix} + \begin{pmatrix} \varDelta_- \mathbf{v}^{(\mathbf{1})} & 0 & 0 \\ 0 & \varDelta_0 \mathbf{v}^{(\mathbf{1})} & 0 \\ 0 & 0 & \varDelta_+ \mathbf{v}^{(\mathbf{1})} \end{pmatrix} = 0, \end{cases} \tag{20}$$

where

$$\varDelta_- \mathbf{u}^{(L)} = c_-(L) \sum_{i=-1}^{1} \frac{\partial}{\partial \xi_i} G^i_{1[L-1,L]} \mathbf{u}^{(L)},\ \varDelta_+ \mathbf{u}^{(L)} = c_+(L) \sum_{i=-1}^{1} \frac{\partial}{\partial \xi_i} G^i_{1[L+1,L]} \mathbf{u}^{(L)},$$

$$\varDelta_0 \mathbf{u}^{(L)} = c_0(L) \sum_{i=-1}^{1} \frac{\partial}{\partial \xi_i} G^i_{1[L,L]} \mathbf{u}^{(L)}$$

are invariant operators acting on vectors of dimension $2L+1$ (L is called the weight of the corresponding representation), and $G^i_{1[K,L]}$ $(K = L-1, L, L+1)$ are the so called Klebsh–Gordan matrices. They are obtained with the aid of products of representations of $SO(3)$ of vectors with the corresponding weights K, L. For the representation theory of $SO(3)$ see e.g. [GM98]. For more details on this subsection see [Se07, Se08].

7 Concluding Remarks

The main result of this paper was obtained under the rather strong (from a theoretical point of view) restriction on the derivatives of the function φ which was missed in the announcement of this result in [SS08]). We do not currently know whether the main result holds without the restriction. This restriction seems to correspond well to the experience of numerical analysts that using of initial functions with large derivatives may make the difference scheme nonstable. Moreover, in most of applications one can assume that the restriction holds because the existence and an estimate of M may be derived from physical reasons (see e.g. [KPS01, Se05]). So, the restriction is not too strong from the point of view of applications. It would be interesting to prove the computability of (1) by a different method (for example, with the aid of explicit formula for solution) and compare the proofs w.r.t. efficiency of the corresponding algorithms.

There are many other open questions and directions of possible future research related to this paper. For example, currently we do not see how to adjust our proof for the space $C_s(H,\mathbb{R}^n)$ instead of $C_{sL_2}(H,\mathbb{R}^n)$, and we do not know whether the operator sending arbitrary (not necessarily computable) matrices $A, B_1, \ldots, B_m$ to the corresponding solution of (1) is computable without the additional assumption that we are given also the cardinality of the spectrum of A and $\lambda A - B_i$, $i \in \{1,\ldots,m\}$. With this assumption, the computability probably holds but we did not try to give the exact formulation because it would lead to some technical complications.

We guess that some generalizations and strengthenings of our results hold (for example, for more complicated sets in place of Q, for Sobolev spaces (which are very popular in the study of equations similar to ours [Mi73]) in place of the spaces discussed above, for other types of differential and difference equations [KPS01], and so on). In this paper we concentrated on the technically simplest case, in order to show clearly a close relation of algorithms based on difference schemes to the exact notions of computable analysis. It would be also interesting to investigate schemes with higher order of accuracy and algorithms based on the popular finite element method.

The general idea is to work towards the study of the vast variety of algorithms used in numerical analysis (in particular, for solving differential equations satisfying some initial value and/or boundary conditions) in the context of computable analysis. This could be of interest for both fields: important practical algorithms of numerical analysis would get a solid mathematical foundation, while computable analysis would get a vast variety of interesting practical algorithms. The study of such algorithms may be of interest for the ongoing development of a sound complexity theory for computations in analysis and topology (see e.g. [Ko91, We00, TWW88]).

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