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Rearranging Series Constructively

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Abstract: Riemann's theorems on the rearrangement of absolutely convergent and conditionally convergent series of real numbers are analysed within Bishop-style constructive mathematics. The constructive proof that every rearrangement of an absolutely convergent series has the same sum is relatively straightforward; but the proof that a conditionally convergent series can be rearranged to converge to whatsoever we please is a good deal more delicate in the constructive framework. The work in the paper answers affirmatively a question posed many years ago by Beeson.
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1 Introduction

We are interested in the constructive content of Riemann's two famous theorems about the rearrangement of series of real numbers:

RST$_1$ *If a series* $\sum_{n=1}^{\infty} a_n$ *of real numbers is absolutely convergent, then for each permutation* σ *of* $\mathbf{N}^+$*, the series* $\sum_{n=1}^{\infty} a_{\sigma(n)}$ *converges to the same sum as* $\sum_{n=1}^{\infty} a_n$*.*

RST$_2$ *If a series* $\sum_{n=1}^{\infty} a_n$ *of real numbers is conditionally convergent, then for each real number* x *there exists a permutation* σ *of* $\mathbf{N}^+$ *such that* $\sum_{n=1}^{\infty} a_{\sigma(n)}$ *converges to* x*.*

When we speak of *constructive mathematics* (**BISH**), we mean mathematics in which *there exists* x *with property* P is interpreted strictly as *we have an algorithm for computing* x *and one for showing that it has the property* P. In practice, doing mathematics constructively, in our sense, means working only with intuitionistic logic, an appropriate set theory—one, such as the Aczel-Rathjen CZF [Aczel and Rathjen, 2001], that, taken with intuitionistic logic, does not allow us to derive such nonconstructive principles as the law of excluded middle—and dependent choice. Background material in constructive analysis can be found in [Bishop, 1967; Bridges and Vîţă, 2006]. Note that constructive mathematics is not characterised by the negative requirement that we exclude the law of excluded middle: it is characterised positively by its requirement that all proofs of existence must embody algorithms for constructing/computing the object whose existence is posited. Note also that intuitionistic logic does not allow proofs of many weaker, but essentially nonconstructive, forms of the law of excluded middle. One of these is Markov's principle,[1]

MP If $(a_n)_{n\geqslant 1}$ is binary sequence for which it is impossible that all terms equal 0, then there exists n such that $a_n = 1$.

We shall return to **MP** later.

The constructive proof of $\mathbf{RST}_1$ is more or less the standard classical one. What about $\mathbf{RST}_2$?[2] Before answering that question, we need to clarify the constructive interpretation of two notions: the series $\sum_{n=1}^{\infty} a_n$ of real numbers

- **diverges to** ∞, if for each $c > 0$ there exists ν such that $\sum_{n=1}^{N} a_n > c$ for all $N \geqslant \nu$;
- is **conditionally convergent** if it is convergent and $\sum_{n=1}^{\infty} |a_n|$ diverges to ∞.

For a constructive proof of the conditional convergence of a convergent series $\sum_{n=1}^{\infty} a_n$ it is not enough to allow the impossibility of convergence of $\sum_{n=1}^{\infty} |a_n|$. This distinction is tied up with Markov's principle, and will be discussed at the end of the paper.

¿From a constructive viewpoint, there are several problems with Riemann's proof of $\mathbf{RST}_2$ as presented in many texts. First, it requires us to be able to decide, for any given $x \in \mathbf{R}$, whether $x \leqslant 0$ or $x > 0$. In our context, that decision would require an algorithm which, applied to any $x \in \mathbf{R}$, would, for example, output -1 if $x \leqslant 0$ and 1 if $x > 0$. Such an algorithm could easily be adapted to give a constructive proof of the principle

LPO For each binary sequence $(a_n)_{n\geqslant 1}$, either $a_n = 0$ for all n or else there exists n such that $a_n = 1$,

which is known to be essentially nonconstructive. (To convince yourself of this, try interpreting **LPO** algorithmically.) Secondly, the standard proof of $\mathbf{RST}_2$ requires us to be able to identify, and then discard from our consideration, those

[1] Some practitioners of recursive constructive mathematics—that is, constructive mathematics (in our sense) supplemented by the Church-Markov-Turing thesis that all partial functions from $\mathbf{N}$ to $\mathbf{N}$ are recursive—admit the use of **MP**.

[2] Our proof of $\mathbf{RST}_2$ within **BISH** answers, affirmatively, a question posed by Michael Beeson in a talk (heard by Bridges) at Oxford University in the mid-1970s.

terms of the series which equal 0. In order to do this, we have to be able to prove yet another statement equivalent to **LPO**: namely,

$$\forall_{x \in \mathbf{R}} \, (x = 0 \vee |x| > 0)\,.$$

Thirdly, we have to be careful about the meaning of the convergence/divergence of a series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms. Since the monotone convergence theorem for sequences is constructively equivalent to **LPO** (we leave that as an exercise), in order to establish the convergence of $\sum_{n=1}^{\infty} a_n$ it is not enough for us to prove that the partial sums of the series form a bounded sequence; we must show that those partial sums form a Cauchy sequence, so that we can invoke the completeness[3] of $\mathbf{R}$. Likewise, to establish the divergence to ∞ of a series of nonnegative terms, we need to prove that for each $c > 0$ there exists N such that $\sum_{n=1}^{N} a_n > c$ (and therefore $\sum_{n=1}^{\nu} a_n > c$ for all $\nu \geqslant N$).

2 Proofs of the main results

Now let us turn to details of constructive proofs of Riemann's series theorems. These proofs are quite elementary, but, unlike their classical counterparts, require careful handling of estimates. For completeness, we first remind ourselves of a natural (constructive) proof of $\mathbf{RST}_1$.

Proof. Supposing that $\sum_{n=1}^{\infty} a_n$ converges absolutely to the sum s, let σ be any permutation of $\mathbf{N}^+$. Given $\varepsilon > 0$, choose N such that $\sum_{n=N+1}^{\infty} |a_n| < \varepsilon$ and $\left| s - \sum_{n=1}^{N} a_n \right| < \varepsilon$. There exists K such that

$$\{1, \ldots, N\} \subset \{\sigma(1), \ldots, \sigma(K)\}\,.$$

For each $k \geqslant K$,

$$\sum_{i=1}^{k} a_{\sigma(i)} = \sum_{n=1}^{N} a_n + \sum \left\{ a_{\sigma(i)} : \sigma(i) > N \text{ and } i \leqslant k \right\},$$

so

$$\begin{aligned} \left| s - \sum_{i=1}^{k} a_{\sigma(i)} \right| &\leqslant \left| s - \sum_{n=1}^{N} a_n \right| + \sum \left\{ \left| a_{\sigma(i)} \right| : \sigma(i) > N \text{ and } i \leqslant k \right\} \\ &< \varepsilon + \sum_{n=N+1}^{\infty} |a_n| < 2\varepsilon. \end{aligned}$$

Hence $\sum_{n=1}^{\infty} a_{\sigma(n)}$ converges to s.

[3] Despite a common disbelief in the constructive completeness of $\mathbf{R}$, that completeness is relatively easy to establish; see [Bridges and Vîţă, 2006]. (Chapter 2).

For our first lemma we make two observations. First, playing with rational approximations to the numbers in question, we can prove that if $x_1, \ldots, x_n$ are real numbers such that $x_1 + \cdots + x_n > 0$, then there exists k such that $x_k > 0$ [Bridges and Vîţă, 2006] (Chapter 2). Second, we call a real number x **nonzero**, and write $x \neq 0$, if $|x| > 0$ (equivalently, if either $x > 0$ or $x < 0$). Note that this is a stronger property, constructively, than the impossibility that $x = 0$: the statement

$$\forall_{x \in \mathbf{R}} \left(\neg (x = 0) \Rightarrow x \neq 0 \right)$$

is equivalent to **MP**.

Lemma 1. *Let $\sum_{n=1}^{\infty} a_n$ be a convergent series of nonzero real numbers such that $\sum_{n=1}^{\infty} |a_n|$ diverges to ∞. Then for each n there exist $j, k \geqslant n$ such that $a_j > 0$ and $a_k < 0$.*

Proof. Replacing the original sequence by $(a_n, a_{n+1}, \ldots)$, we reduce the proof to the case $n = 1$. Without loss of generality, we may take $a_1 > 0$; so it suffices to find k with $a_k < 0$. Pick $N > 1$ such that $\left| \sum_{n=N}^{k} a_n \right| < 1$ for all $k \geqslant N$. Since $\sum_{n=N}^{\infty} |a_n|$ diverges to ∞, there exists $k > N$ such that $\sum_{n=N}^{k} |a_n| > 1$ and therefore

$$\sum_{n=N}^{k} \left(|a_n| - a_n \right) \geqslant \sum_{n=N}^{k} |a_n| - \left| \sum_{n=N}^{k} a_n \right| > 0.$$

It follows that there exists n such that $N \leqslant n \leqslant k$ and $|a_n| - a_n > 0$. For this n we have $a_n < 0$.

Note that the algorithm in the proof of Lemma 1 produces a bound for the number of terms beyond the nth we have to inspect in order to guarantee finding one with sign opposite to that of a_n.

Lemma 1 has a simple nonconstructive proof: if, for example, there were some n such that $a_k < 0$ for all $k \geqslant n$, then the series $\sum_{k=n}^{\infty} |a_k| = -\sum_{k=n}^{\infty} a_k$ would converge, which is absurd. But this proof does not show us how to compute the desired $k \geqslant n$ with $a_k > 0$.

Lemma 2. *Let $\sum_{n=1}^{\infty} a_n$ be a convergent series of nonzero real numbers such that $\sum_{n=1}^{\infty} |a_n|$ diverges to ∞. Then there exist an enumeration $f(1) < f(2) < \cdots$ of the indices of the positive terms of the series and an enumeration $g(1) < g(2) < \cdots$ of all the negative terms of the series. Moreover, the series $\sum_{n=1}^{\infty} a_{f(n)}$ and $\sum_{n=1}^{\infty} \left(-a_{g(n)} \right)$ both diverge to ∞.*

Proof. In view of Lemma 1, we readily construct the functions f, g inductively. For the last part of the lemma, pick N such that $\left| \sum_{n=N}^{k} a_n \right| < 1$ for each

$k \geqslant N$. Fix $c > 0$. Since $\sum_{n=N}^{\infty} |a_n|$ diverges to ∞, there exists $k > N$ such that $\sum_{n=N}^{k} |a_n| > 2c + 1$. Let

$$F \equiv \{n : N \leqslant f(n) \leqslant k\},$$
$$G \equiv \{n : N \leqslant g(n) \leqslant k\}.$$

Then

$$\left|\sum_{n\in F} a_{f(n)} - \sum_{n\in G} \left(-a_{g(n)}\right)\right| = \left|\sum_{n=N}^{k} a_n\right| < 1$$

and

$$\sum_{n\in F} a_{f(n)} + \sum_{n\in G} \left(-a_{g(n)}\right) = \sum_{n=N}^{k} |a_n| > 2c + 1,$$

from which it follows that both $\sum_{n\in F} a_{f(n)} > c$ and $\sum_{n\in G} \left(-a_{g(n)}\right) > c$. Since $c > 0$ is arbitrary, we conclude that the series $\sum_{n=1}^{\infty} a_{f(n)}$ and $\sum_{n=1}^{\infty}(-a_{g(n)})$ diverge to ∞.

We now proceed to the constructive proof of $\mathbf{RST}_2$. That proof is based on the standard classical one, but requires additional refinements. One of these arises from the undecidability of equality on $\mathbf{R}$, which forces us to consider only rational numbers to begin with; some trickery is then needed to remove the requirement of rationality. Here, then, is our proof of $\mathbf{RST}_2$.

Proof. Let $\sum_{n=1}^{\infty} a_n$ converge to the sum s, but $\sum_{n=1}^{\infty} |a_n|$ diverge to ∞. Fixing $x \in \mathbf{R}$, we seek a permutation σ of $\mathbf{N}^+$ such that $\sum_{n=1}^{\infty} a_{\sigma(n)}$ converges to x. We first assume that each a_n is nonzero and rational, and that x is rational. Without loss of generality, we may also assume that $f(1) = 1$. Let the mappings f, g be as in Lemma 2. In the following, we repeatedly use the fact that the equality on the set $\mathbf{Q}$ of rational numbers is decidable. Since, by Lemma 2, the series $\sum_{n=1}^{\infty} a_{f(n)}$ diverges to ∞, we can find, in turn,

$$p_1 = \min\left\{p : \sum_{n=1}^{p} a_{f(n)} > x\right\},$$
$$q_1 = \min\left\{q : \sum_{n=1}^{q} \left(-a_{g(n)}\right) > \sum_{n=1}^{p_1} a_{f(n)} - x\right\}.$$

We begin the inductive construction of the permutation σ of $\mathbf{N}^+$ by setting

$$\sigma(n) \equiv \begin{cases} f(n) & \text{if } 1 \leqslant n \leqslant p_1 \\ \\ g(n - p_1) & \text{if } p_1 < n \leqslant p_1 + q_1. \end{cases}$$

Note that $\sigma(i) \neq \sigma(j)$ if $1 \leqslant i < j \leqslant p_1 + q_1$; also, $\sigma(1) = f(1) = 1$, and $\sigma(p_1 + 1) = g(1)$. Next, having found certain positive integers p_k and q_k, we set

$$p_{k+1} = \min \left\{ p : \sum_{n=p_k+1}^{p} a_{f(n)} > x - \left(\sum_{n=1}^{p_k} a_{f(n)} + \sum_{n=1}^{q_k} a_{g(n)} \right) \right\},$$

$$q_{k+1} = \min \left\{ q : \sum_{n=q_k}^{q} \left(-a_{g(n)}\right) > \left(\sum_{n=1}^{p_{k+1}} a_{f(n)} + \sum_{n=1}^{q_k} a_{g(n)} \right) - x \right\}.$$

Suppose also that we have defined $\sigma(n)$ for $1 \leqslant n \leqslant p_k + q_k$, and that for each $j \leqslant k$ there exists n with $1 \leqslant n \leqslant p_k + q_k$ and $j = \sigma(n)$. We extend the definition of σ by setting

$$\sigma(n) \equiv \begin{cases} f(n - q_k) & \text{if } p_k + q_k < n \leqslant p_{k+1} + q_k \\ \\ g(n - p_{k+1}) & \text{if } p_{k+1} + q_k < n \leqslant p_{k+1} + q_{k+1}. \end{cases}$$

Then $\sigma(i) \neq \sigma(j)$ whenever $1 \leqslant i < j \leqslant p_{k+1} + q_{k+1}$. Moreover, since $p_{k+1} > p_k$ and $q_{k+1} > q_k$, either $k+1 = \sigma(n)$ for some $n \leqslant p_k + q_k$, or else there exists n such that $p_k + q_k < n \leqslant p_{k+1} + q_{k+1}$ and $k + 1 = \sigma(n)$. This completes the inductive construction of p_{k+1}, q_{k+1}, and the values $\sigma(n)$ for $p_k + q_k < n \leqslant p_{k+1} + q_{k+1}$.

Next, observe that for $p_k + 1 \leqslant j < p_{k+1}$ we have

$$\begin{aligned} x + a_{g(q_k)} &< \sum_{n=1}^{p_k} a_{f(n)} + \sum_{n=1}^{q_k - 1} a_{g(n)} + a_{g(q_k)} \\ &\leqslant \sum_{n=1}^{j} a_{f(n)} + \sum_{n=1}^{q_k} a_{g(n)} \\ &\leqslant \sum_{n=1}^{p_{k+1}-1} a_{f(n)} + \sum_{n=1}^{q_k} a_{g(n)} \leqslant x \end{aligned}$$

and therefore

$$\left| x - \left(\sum_{n=1}^{j} a_{f(n)} + \sum_{n=1}^{q_k} a_{g(n)} \right) \right| < \left| a_{g(q_k)} \right|.$$

It follows from this and the last part of Lemma 2 that for $p_k < j \leqslant p_{k+1}$,

$$\left| x - \left(\sum_{n=1}^{j} a_{f(n)} + \sum_{n=1}^{q_k} a_{g(n)} \right) \right| < \max \left\{ a_{f(p_{k+1})}, \left| a_{g(q_k)} \right| \right\} . \qquad (1)$$

Likewise, for $q_k + 1 \leqslant j < q_{k+1}$, since $a_{g(j)} < 0$, we have

$$\begin{aligned} x &\leqslant \sum_{n=1}^{p_{k+1}} a_{f(n)} + \sum_{n=1}^{q_{k+1}-1} a_{g(n)} \\ &\leqslant \sum_{n=1}^{p_{k+1}} a_{f(n)} + \sum_{n=1}^{j} a_{g(n)} \\ &< \sum_{n=1}^{p_{k+1}} a_{f(n)} + \sum_{n=1}^{q_k} a_{g(n)} \\ &< x + a_{f(p_{k+1})} \end{aligned}$$

and therefore

$$\left| x - \sum_{n=1}^{p_{k+1}} a_{f(n)} + \sum_{n=1}^{j} a_{g(n)} \right| < a_{f(p_{k+1})}$$

It follows from this and Lemma 2 that for $q_k < j \leqslant q_{k+1}$,

$$\left| x - \left(\sum_{n=1}^{p_{k+1}} a_{f(n)} + \sum_{n=1}^{j} a_{g(n)} \right) \right| < \max \left\{ a_{f(p_{k+1})}, \left| a_{g(q_{k+1})} \right| \right\}. \tag{2}$$

Now, since the functions f, g are strictly increasing and the series $\sum_{n=1}^{\infty} a_n$ converges, $\max \left\{ a_{f(n)}, \left| a_{g(n)} \right| \right\} \to 0$ as $n \to \infty$. Thus, given $\varepsilon > 0$, we can compute K such that $\max \left\{ a_{f(k)}, \left| a_{g(k)} \right| \right\} < \varepsilon$ for all $k \geqslant K$. Given $j > p_K + q_K$, pick $k \geqslant K$ such that $p_k + q_k < j \leqslant p_{k+1} + q_{k+1}$. In view of (1), (2), and our construction of the function σ, we see that

$$\left| x - \sum_{n=1}^{j} a_{\sigma(n)} \right| < \max \left\{ a_{f(p_{k+1})}, \left| a_{g(q_k)} \right|, \left| a_{g(q_{k+1})} \right| \right\} < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that the series $\sum_{n=1}^{\infty} a_{\sigma(n)}$ converges to x.

It remains to remove the hypotheses that each a_n is nonzero and rational, and that x is rational. Construct a sequence $(b_n)_{n \geqslant 1}$ of positive rational numbers such that for each n, $a_n + b_n$ is nonzero and rational, and $\sum_{n=1}^{\infty} b_n$ converges. Then pick, in turn, a rational number $q > x + \sum_{n=1}^{\infty} b_n$ and a sequence $(c_n)_{n \geqslant 1}$ of positive rational numbers such that

$$\sum_{n=1}^{\infty} c_n = q - x - \sum_{n=1}^{\infty} b_n.$$

By the first part of the proof, there exists a permutation σ of $\mathbf{N}^+$ such that

$$\sum_{n=1}^{\infty} \left(a_{\sigma(n)} + b_{\sigma(n)} + c_{\sigma(n)} \right) = q.$$

But, by $\mathbf{RST}_1$, that the series $\sum_{n=1}^{\infty}\left(a_{\sigma(n)}+b_{\sigma(n)}\right), \sum_{n=1}^{\infty} c_{\sigma(n)}$ converge to $\sum_{n=1}^{\infty}\left(a_n+b_n\right), \sum_{n=1}^{\infty} c_n$ respectively. Elementary convergence theorems now yield

$$\begin{aligned} q &= \sum_{n=1}^{\infty}\left(a_{\sigma(n)}+b_{\sigma(n)}\right)+\sum_{n=1}^{\infty} c_{\sigma(n)} \\ &= \sum_{n=1}^{\infty} a_n+\sum_{n=1}^{\infty} b_n+\sum_{n=1}^{\infty} c_n \end{aligned}$$

and therefore

$$\sum_{n=1}^{\infty} a_n=\left(q-\sum_{n=1}^{\infty} b_n\right)-\sum_{n=1}^{\infty} c_n=x,$$

as we required.

We emphasise two aspects of our constructive proofs of $\mathbf{RST}_2$ and the lemmas used therein: first, those proofs yield *a priori* bounds for the searches involved in our choices of the numbers p_k, q_k; and second, the proof of $\mathbf{RST}_2$ is *a fortiori* one that the (implementable) algorithms it contains, for constructing the permutation σ and showing that $\sum_{n=1}^{\infty} a_{\sigma(n)}$ converges to x, are correct—that is (in computer science terms), meet their specifications.

3 A concluding Brouwerian example

We end by showing that $\mathbf{RST}_2$ cannot be proved constructively if we weaken the divergence of $\sum_{n=1}^{\infty}|a_n|$ to the impossibility that its partial sums be bounded. Let $(\lambda_n)_{n\geqslant 1}$ be an increasing binary sequence for which we can prove that it is impossible for all the terms to be 0. It is routine to show that the partial sums of the series $\sum_{n=1}^{\infty}(-1)^n \lambda_n/n$ form a Cauchy sequence. Suppose that the partial sums of the series $\sum_{n=1}^{\infty}\lambda_n/n$ are bounded above by some positive number. If there were N such that $\lambda_N=1-\lambda_{N-1}$, then the series $\sum_{n=1}^{\infty}\lambda_n/n$ would diverge to infinity, which is impossible. Hence $\lambda_n=0$ for all n, a contradiction from which we conclude that the partial sums of the series $\sum_{n=1}^{\infty}\lambda_n/n$ are not bounded above by any positive number. Now suppose that there exists a permutation σ of $\mathbf{N}$ such that

$$\sum_{n=1}^{\infty}(-1)^{\sigma(n)}\frac{\lambda_{\sigma(n)}}{\sigma(n)}=1.$$

Picking N such that

$$\sum_{n=1}^{N}(-1)^{\sigma(n)}\frac{\lambda_{\sigma(n)}}{\sigma(n)}>\frac{1}{2},$$

we see that $\lambda_{\sigma(n)}=1$ for some $n\leqslant N$. Thus the statement

If a series $\sum_{n=1}^{\infty} a_n$ *of real numbers converges, and the partial sums of* $\sum_{n=1}^{\infty} |a_n|$ *are not bounded above, then for each real number* x *there exists a permutation* σ *of* $\mathbf{N}^+$ *such that* $\sum_{n=1}^{\infty} a_{\sigma(n)}$ *converges to* x

implies Markov's principle.[4]

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[4] This Brouwerian counterexample is perhaps not surprising, since, as is easily proved, **MP** is constructively equivalent to the statement: *Every series of positive numbers whose partial sums are not bounded above diverges to* ∞.