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Non-Denumerable Infinitary Modal Logic

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Abstract: Segerberg established an analogue of the canonical model theorem in modal logic for infinitary modal logic. However, the logics studied by Segerberg and Goldblatt are based on denumerable sets of pairs $\langle \Gamma, \alpha \rangle$ of sets Γ of well-formed formulae and well-formed formulae α. In this paper I show how a generalisation of the infinite cut-rule used by Segerberg and Goldblatt enables the removal of the limitation to denumerable sets of sequents.
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1 Introduction

In (Segerberg 1994) Krister Segerberg discusses infinitary modal logic and establishes an analogue of the familiar canonical model theorem in ordinary finitary modal logic. Similar results are found in Chapters 8 and 9 of Goldblatt 1993. An example of an infinitary modal logic is provided on pp. 340-343 of Segerberg 1994 in the 'logic of common knowledge'. This is actually a multi-modal logic, but will serve to illustrate the utility of infinitary logic. Assume a family of knowledge operators $\{K_{\text{i}} : i \in \text{I}\}$ indexed by a (denumerable) set I, each with its own accessibility relation, together with a 'common knowledge' operator C, whose relation R* is the ancestral of all the R_is. (*op cit*, p. 340.) Of the rules which govern this operator the essentially infinite one is listed on p. 364, whose effect is to say that $\Gamma \vdash C\alpha$ provided Γ contains all combinations of iterated K_i operators applied to α. Segerberg points out on p. 341f that no finite subset of Γ will do the trick.

The philosophical importance of this kind of investigation in justifying the claim that modal operators in natural language are quantifiers over worlds is studied in Cresswell 2006, where the task is to embed any set of sentences which is jointly possible in some intuitive sense of possibility, into a maximally possible set. Such a set can be thought of as specifying a possible world. Given that intuitive possibility need not be finitary, stronger principles of construction are required. What the present paper does is describe an infinitary operation which corresponds exactly with truth-preservation in a modal model, and enables a completely general canonical model result. Of course the existence of a canonical model for every such system still leaves open all the usual questions about completeness in a class of frames.

The logics studied by Segerberg and Goldblatt are based on denumerable sets of pairs $\langle \Gamma,\alpha\rangle$ where Γ is a set of well-formed formulae (wff) and α is a well-formed formula (a wff) and $\langle \Gamma,\alpha\rangle$ is understood to mean that α is derivable from Γ[1] The purpose of the present note is to show how a generalisation of the infinite 'cut' rule used by Segerberg and Goldblatt enables the removal of the limitation to denumerable sets of sequents.

2 Preliminaries

Assume a language of propositional modal logic with a denumerable set of propositional variables, p, q, r, ... etc., the material implication operator $\supset$, the standard false proposition (the 'falsum') $\perp$ and the modal operator $\Box$[2]. I assume the standard formation rules and definitions. The model theory for such a language is the usual one. A *frame* is an ordered pair $\langle$W,R$\rangle$, where W is a non-empty set of objects (worlds), and R is a dyadic relation defined over the members of W. A *model* is an ordered triple $\langle$W,R,V$\rangle$ where $\langle$W,R$\rangle$ is a frame and V is a value-assignment satisfying the following conditions for any $w \in$ W:

[**V**p] For any propositional variable, p, and any $w \in$ W, either V(p,w) = 1 or V(p,w) = 0

[**V**$\perp$] V($\perp$,w) = 0

[**V**$\supset$] For any wff α and β, and for any $w \in$ W, V($\alpha \supset \beta$,w) = 1 if either V(α,w) = 0 or V(β,w) = 1; otherwise V($\alpha \supset \beta$,w) = 0.

[**V**$\Box$] For any wff α and for any $w \in$ W, V($\Box\alpha$,w) = 1 if for every $w\prime \in$ W such that wR$w\prime$, V(α,$w\prime$) = 1; otherwise V($\Box\alpha$,w) = 0.

With $\tilde{}\alpha$ defined as $\alpha \supset \perp$ we have that V($\tilde{}\alpha$,w) = 1 iff V(α,w) = 0.

On p. 343 Segerberg lists the following principles for a consequence relation $\vdash$ between a set of wff and a wff.

(RX) $\{\alpha\} \vdash \alpha$

(MN) If $\Gamma \vdash \alpha$ then $\Gamma \cup \Delta \vdash \alpha$

(CT) If $\Gamma \vdash \gamma$ for every $\gamma \in \Omega$, and $\Omega \vdash \alpha$, then $\Gamma \vdash \alpha$

(SB) If $\Gamma \vdash \alpha$ then $s\Gamma \vdash s\alpha$ for every substitution function s

[1] Segerberg uses Γ and Ω, and I follow him in using upper case Greek letters in this way, with Θ for sets of sets of wff. But I use lower case Greek letters for wff, and I use $\dashv$ to mean 'not $\vdash$' and $\Dashv$ to mean 'not $\models$'.

[2] I am using $\Box$ rather than my usual L in Krister's honour! I only consider mono-modal languages, though the results can almost certainly be extended to multiply modal languages.

A substitution function s replaces every propositional variable by a wff uniformly in a wff or set of wff. For particular operators I shall only need the following (my labels):

(MP) $\{\alpha,\alpha \supset \beta\} \vdash \beta$

(CP) If $\Gamma \cup \{\alpha\} \vdash \beta$ then $\Gamma \vdash \alpha \supset \beta$

(DN) $\{\tilde{}\tilde{}\alpha\} \vdash \alpha$

For modality a normal propositional modal logic must satisfy what Segerberg (p. 344) calls 'Scott's Rule'. Scott's Rule is:

(SR) If $\Gamma \vdash \beta$ then $\{\Box\gamma : \gamma \in \Gamma\} \vdash \Box\beta$

Segerberg (pp. 343-345) defines a (normal modal propositional infinitary) logic L as set of $\langle\Gamma,\alpha\rangle$ sequents which includes a special *denumerable* set ρ of sequents, and for which $\Gamma \vdash \alpha$ satisfies all the conditions (RX)-(SR). In what follows $\Gamma \vdash_{\mathrm{L}} \alpha$ will indicate that $\langle\Gamma,\alpha\rangle \in$ L, and will be abbreviated to $\Gamma \vdash \alpha$ where a fixed but arbitrary logic is understood.

3 Extension of Segerberg's Result to Arbitrary Logics

The purpose of this note is to extend Segerberg's canonical model result to arbitrary logics that lack any denumerability constraint. For this purpose it is necessary to strengthen the cut rule CT. Like CT the new rule may be stated as a purely structural rule which makes no reference to any particular connective:

(CT$^+$) Where Δ is a set of wff and α a wff, suppose there is a set Θ of sets of wff such

(i) for every Λ in Θ, $\Lambda \vdash \alpha$, and

(ii) for every set Γ of wff, if

(ii^{a}) for every Λ in Θ, there is some β in Λ such that $\Gamma \cup \{\beta\} \vdash \alpha$, then

(ii^{b}) $\Delta \cup \Gamma \vdash \alpha$.

Then $\Delta \vdash \alpha$[3].

Theorem 1. *CT follows from CT$^+$.*

[3] A principle which may have a connection with CT$^+$ is described on p.189 of Dunn and Hardegree 2001 as 'global cut'. See also, p. 119 of Barwise and Seligman 1997.

Proof. Assuming that $\Delta \vdash \gamma$ for all $\gamma \in \Omega$, put $\Theta = \{\Omega\}$. Then if $\Omega \vdash \alpha$, condition (i) of CT$^+$ holds. Let Γ be any set for which $\Gamma \cup \{\gamma\} \vdash \alpha$ for some $\gamma \in \Omega$. Since $\Delta \vdash \gamma$ then $\Delta \cup \Gamma \vdash \alpha$, and so condition (ii) of CT$^+$ holds. So, by CT$^+$, $\Delta \vdash \alpha$. □

Given CT$^+$, a logic$^+$ L may be specified as any set of sequents which satisfies (RX)-(SR), but with CT$^+$ in place of CT. A logic$^+$ L is *consistent* unless $\emptyset \vdash_{\mathrm{L}} \bot$. Where ⟨W,R,V⟩ is a model I write $\Gamma \models_{\langle W,R,V\rangle} \alpha$ to indicate that for every $w \in$ W, if V(γ,w) = 1 for every $\gamma \in \Gamma$ then V(α,w) = 1. (When the same particular model is understood throughout I shall simply write $\Gamma \models \alpha$.) Let ⟨W,R,V⟩ be any model, and Δ any set of wff and α any wff:

Theorem 2. *Suppose there is a Θ such that*

(i) for every Λ in Θ, $\Lambda \models \alpha$, and

(ii) for every Γ, if

(ii^a) for every Λ in Θ, there is some β in Λ such that $\Gamma \cup \{\beta\} \models \alpha$, then

(ii^b) $\Delta \cup \Gamma \models \alpha$.

Then $\Delta \models \alpha$.

Proof. Suppose $\Delta \not\models \alpha$. Then there is a $w \in$ W such that, for every $\delta \in \Delta$, V(δ,w) = 1, but V(α,w) = 0. Consider any $\Lambda \in \Theta$. If $\models$ is to satisfy (i) at least one β in Λ must be false in w. Let Γ be the set of $\beta \supset \alpha$ for every such β. Then $\Gamma \cup \{\beta\} \models \alpha$, but V($\beta$,$w$) = 0, and so V($\beta \supset \alpha$,$w$) = 1, and therefore $\Delta \cup \Gamma \not\models \alpha$; but then $\models$ does not satisfy (ii). □

The analogue of theorem 2 holds for all the other rules (RX)-(SR) except SB. However the analogue holds for SB provided ⟨W,R,V⟩ is *generalisable*, i.e. that it satisfies the condition that if $\Gamma \models_{\langle W,R,V\rangle} \alpha$ then $s\Gamma \models_{\langle W,R,V\rangle} s\alpha$ for every substitution function s. Let $\mathrm{L}_{\langle W,R,V\rangle}$ be the set of all sequents ⟨Γ,α⟩ such that $\Gamma \models_{\langle W,R,V\rangle} \alpha$.

Theorem 3. *If ⟨W,R,V⟩ is generalisable $L_{\langle W,R,V\rangle}$ is a logic$^+$.*

Proof. From the fact that all generalisable models satisfy (RX)-(SR) in respect of $\models$. □

Γ will be said to be L-*consistent* iff $\Gamma \nvdash_{\mathrm{L}} \bot$. Γ is said to be *maximal* iff for any wff α either $\alpha \in \Gamma$ or ˜$\alpha \in \Gamma$. A key result in what follows is the *extension result*:

Theorem 4. *If Δ is L-consistent then there is a maximal L-consistent set Γ containing Δ.*

Proof. Suppose that $\Delta \dashv \bot$. Let Θ be the set of all sets of the form $\{\gamma, {\sim}\gamma\}$ for every wff γ.

Then $\Lambda \vdash \bot$ for every $\Lambda \in \Theta$. So for the case where α is $\bot$ condition (i) of CT^+ holds. So, given the falsity of $\Delta \vdash \bot$, and assuming that CT^+ holds, we therefore have the falsity of condition (ii) — that is to say there is some Γ^* such that

(a) for every Λ in Θ there is some β in Λ with $\Gamma^* \cup \{\beta\} \vdash \bot$, and

(b) $\Delta \cup \Gamma^* \dashv \bot$.

Let Γ be defined as follows. Each $\Lambda \in \Theta$ has the form $\{\gamma, {\sim}\gamma\}$. From (a) either $\Gamma^* \cup \{\gamma\} \vdash \bot$ or $\Gamma^* \cup \{{\sim}\gamma\} \vdash \bot$ If the former put ${\sim}\gamma \in \Gamma$; if the latter put $\gamma \in \Gamma$. Since, for every wff γ, γ or ${\sim}\gamma$ occurs in some $\Lambda \in \Theta$ then Γ is maximal, and so $\Delta \cup \Gamma$ is also maximal and contains Δ. So, to establish theorem 4 it is sufficient to shew that $\Delta \cup \Gamma \dashv \bot$. And to shew the latter it is sufficient to shew that if $\Delta \cup \Gamma \vdash \bot$ then $\Delta \cup \Gamma^* \vdash \bot$ (since by (b) $\Delta \cup \Gamma^* \dashv \bot$). Take any δ in $\Delta \cup \Gamma$. If $\delta \in \Delta$ then $\Delta \cup \Gamma^* \vdash \delta$; and if $\delta \in \Gamma$ then δ is ${\sim}\gamma$, where $\Gamma^* \cup \{\gamma\} \vdash \bot$, or δ is γ, where $\Gamma^* \cup \{{\sim}\gamma\} \vdash \bot$. In either case $\Gamma^* \vdash \delta$, and so here too $\Delta \cup \Gamma^* \vdash \delta$. So by CT, $\Delta \cup \Gamma^* \vdash \bot$, contradicting (b). □

Lemma 5. *If Γ is L-consistent and ${\sim}\Box\alpha \in \Gamma$ then $\{\beta{:}\Box\beta \in \Gamma\} \cup \{{\sim}\alpha\}$ is L-consistent.*

Proof. If $\{\beta{:}\Box\beta \in \Gamma\} \cup \{{\sim}\alpha\} \vdash \bot$ then $\{\beta{:}\Box\beta \in \Gamma\} \vdash \alpha$, and so by (SR) and (MN), $\Gamma \vdash \Box\alpha$. But ${\sim}\Box\alpha \in \Gamma$ and so $\Gamma \vdash \bot$. □

Lemma 6. *If Γ is a maximal L-consistent set of wff and ${\sim}\Box\alpha \in \Gamma$ then there is a maximal L-consistent Γ' such that ${\sim}\alpha \in \Gamma'$ and for any wff β if $\Box\beta \in \Gamma$ then $\beta \in \Gamma'$.*

Proof. By theorem 4 and lemma 5. □

Assume L is a consistent logic$^+$. The canonical model for L is a triple $\langle \mathrm{W_L}, \mathrm{R_L}, \mathrm{V_L} \rangle$, where $\mathrm{W_L}$ is the set of all sets of maximal L-consistent sets of wff, and $w\mathrm{R_L}w'$ iff $\alpha \in w'$ for $\Box\alpha \in w$. For any variable p, $\mathrm{V_L}(p,w) = 1$ iff $p \in w$.

Theorem 7. *$\mathrm{V_L}(\alpha,w) = 1$ iff $\alpha \in w$*

Proof. Theorem 7 is proved by induction in the usual way, relying crucially on lemma 6 for the inductive step for $\Box$. □

Theorem 8. *$\Gamma \vdash_{\mathrm{L}} \alpha$ iff $\Gamma \models_{\langle WL,RL,VL\rangle} \alpha$.*

Proof. Suppose $\Gamma \nvdash_{\mathrm{L}} \alpha$. Then $\Gamma \cup \{\tilde{}\alpha\}$ is L-consistent, and so $\Gamma \cup \{\tilde{}\alpha\} \subseteq w$ for some $w \in \mathrm{W_L}$. So by theorem 7, $\mathrm{V_L}(\beta,w) = 1$ for every $\beta \in \Gamma$ and $\mathrm{V_L}(\alpha,w) = 0$. So $\Gamma \nvDash_{\langle WL,RL,VL\rangle} \alpha$. Conversely if $\Gamma \nvDash_{\langle WL,RL,VL\rangle} \alpha$ then there is some $w \in \mathrm{W_L}$ with $\mathrm{V_L}(\beta,w) = 1$ for every $\beta \in \Gamma$ and $\mathrm{V_L}(\alpha,w) = 0$. By theorem 7, $\Gamma \cup \{\tilde{}\alpha\} \subseteq w$. So $\Gamma \cup \{\tilde{}\alpha\} \nvdash_{\mathrm{L}} \perp$, and therefore $\Gamma \nvdash_{\mathrm{L}} \alpha$. □

Theorem 9. *L is a consistent logic$^+$ iff L is $L_{\langle W,R,V\rangle}$ for some generalisable model $\langle W,R,V\rangle$.*

Proof. From theorems 3 and 8, noting that for every substitution function s, $s\Gamma \models_{\langle WL,RL,VL\rangle} s\alpha$ for every $\langle \Gamma,\alpha\rangle \in \mathrm{L}$. □

Now let Q be any set of sequents, denumerable or non-denumerable, and let Σ be the set of all logic$^+$s L such that $\mathrm{Q} \subseteq \mathrm{L}$. Say that Q is consistent iff $\mathrm{Q} \subseteq \mathrm{L}$ for some consistent logic$^+$ L, and assume for the next two theorems that Q is consistent. By theorem 9 any consistent $\mathrm{L} \in \Sigma$ is $\mathrm{L}_{\langle W,R,V\rangle}$ for some generalisable model $\langle$W,R,V$\rangle$. Let M_Σ be the set of all such models, one for each consistent $\mathrm{L} \in \Sigma$, and let $\langle \mathrm{W}_\Sigma,\mathrm{R}_\Sigma,\mathrm{V}_\Sigma\rangle$ be the disjoint union of every $\langle$W,R,V$\rangle \in \mathrm{M}_\Sigma$. $\langle \mathrm{W}_\Sigma,\mathrm{R}_\Sigma,\mathrm{V}_\Sigma\rangle$ will be a model, and will be divided into a number of separate 'cells' in such a way that each cell corresponds to some $\langle$W,R,V$\rangle$ in M_Σ and no world in any cell can see any world in any other cell. Further, since each $\langle$W,R,V$\rangle \in \mathrm{M}_\Sigma$ is generalisable so is $\langle \mathrm{W}_\Sigma,\mathrm{R}_\Sigma,\mathrm{V}_\Sigma\rangle$. Therefore

Lemma 10. *For any sequent $\langle \Gamma,\alpha\rangle$*

(i) If $\langle \Gamma,\alpha\rangle \in Q$ then $\Gamma \models_{\langle W\Sigma,R\Sigma,V\Sigma\rangle} \alpha$.

(ii) $\Gamma \models_{\langle W\Sigma,R\Sigma,V\Sigma\rangle} \alpha$ iff $\Gamma \models_{\langle W,R,V\rangle} \alpha$ for every $\langle W,R,V\rangle \in M_\Sigma$.

Theorem 11. *Let L_{Q} be $L_{\langle W\Sigma,R\Sigma,V\Sigma\rangle}$.*

(i) L_{Q} is a logic$^+$

(ii) L_{Q} is the intersection of all logic$^+$s containing Q

Proof. Since $\langle \mathrm{W}_\Sigma,\mathrm{R}_\Sigma,\mathrm{V}_\Sigma\rangle$ is generalisable then, by theorem 9, $\mathrm{L_Q}$ is a logic$^+$. By (i) of lemma 10, $\mathrm{Q} \subseteq \mathrm{L_Q}$, and, by (ii) of lemma 10, $\Gamma \vdash_{\mathrm{LQ}} \alpha$ iff $\Gamma \models_{\langle W,R,V\rangle} \alpha$ for every $\langle$W,R,V$\rangle \in \mathrm{M}_\Sigma$. But every $\mathrm{L} \in \Sigma$ is $\mathrm{L}_{\langle W,R,V\rangle}$ for some $\langle$W,R,V$\rangle \in \mathrm{M}_\Sigma$, and so $\Gamma \vdash_{\mathrm{LQ}} \alpha$ iff $\Gamma \vdash_{\mathrm{L}} \alpha$ for every $\mathrm{L} \in \Sigma$, i.e. for every logic$^+$ L such that $\mathrm{Q} \subseteq \mathrm{L}$. □

Say that $\Gamma \vdash_{\mathrm{Q}} \alpha$ iff $\Gamma \vdash_{\mathrm{L}} \alpha$ for every logic$^+$ L such that $\mathrm{Q} \subseteq \mathrm{L}$, and that $\Gamma \models_{\mathrm{Q}} \alpha$ iff for every generalisable model $\langle$W,R,V$\rangle$, if $\Delta \models_{\langle W,R,V\rangle} \beta$ for every $\langle \Delta,\beta\rangle \in \mathrm{Q}$ then $\Gamma \models_{\langle W,R,V\rangle} \alpha$.

Theorem 12. *For any set Q of sequents* $\Gamma \vdash_{\mathrm{Q}} \alpha$ *iff* $\Gamma \vDash_{\mathrm{Q}} \alpha$.

Proof. Suppose there is some generalisable model ⟨W,R,V⟩ such that $\Delta \vDash_{\langle W,R,V\rangle} \beta$ for every $\langle\Delta,\beta\rangle \in \mathrm{Q}$, but $\Gamma \nvDash_{\langle W,R,V\rangle} \alpha$. Then $\Gamma \nvdash_{\mathrm{L}\langle \mathrm{W,R,V}\rangle} \alpha$. But $\mathrm{Q} \subseteq \mathrm{L}_{\langle W,R,V\rangle}$, and so $\Gamma \nvdash_{\mathrm{Q}} \alpha$. Conversely, if $\Gamma \nvdash_{\mathrm{Q}} \alpha$ then, by theorem 11, $\Gamma \nvdash_{\mathrm{LQ}} \alpha$, and so $\Gamma \nvDash_{\langle W\Sigma,R\Sigma,V\Sigma\rangle} \alpha$. But by (i) of lemma 10, $\Delta \vDash_{\langle W\Sigma,R\Sigma,V\Sigma\rangle} \beta$, for every $\langle\Delta,\beta\rangle \in$ Q, and so, since $\langle \mathrm{W}_\Sigma,\mathrm{R}_\Sigma,\mathrm{V}_\Sigma\rangle$ is generalisable, $\Gamma \nvDash_{\mathrm{Q}} \alpha$. □

Theorem 12 holds even for inconsistent Q since the inconsistent logic$^+$ is characterised by the empty set of models[4].

4 On Uniform Substitution

In theorems 1-12 I have followed Segerberg in requiring a logic to satisfy (SB). This is in line with the practice in ordinary finitary modal logic of having a rule of uniform substitution for propositional variables and axiomatising systems using particular formulae. Thus T may be axiomatised by adding to K the single wff $\Box p \supset p$. But one can equally dispense with the uniform substitution rule, and axiomatise T by the schema $\Box\alpha \supset \alpha$. It is then easy to prove that if $\vdash_{\mathrm{T}} \beta$ then $\vdash_{\mathrm{T}} s\beta$ for any substitution function s. However the proof in this and analogous cases is by induction on the proof of β, and such inductive proofs are unavailable in the infinitary systems studied here. Nevertheless it turns out that the equivalent result applies. First note that the analogues of theorems 1-12 still hold when (SB) is omitted, *provided* that models are not required to be generalisable. Where Q is a set of sequents say that $\Gamma \vdash^{---}_{\mathrm{Q}} \alpha$ iff $\Gamma \vdash_{\mathrm{L}} \alpha$ for every logic$^+$ L containing Q, *where* L *is not required to satisfy* (SB), and say that $\Gamma \vDash^{---}_{\mathrm{Q}} \alpha$ iff in every model ⟨W,R,V⟩, *whether generalisable or not*, if $\Delta \vDash_{\langle W,R,V\rangle} \beta$ for every $\langle\Delta,\beta\rangle \in$ Q then $\Gamma \vDash_{\langle W,R,V\rangle} \alpha$. Then we have

Theorem 12a. $\Gamma \vdash^{---}_{\mathrm{Q}} \alpha$ *iff* $\Gamma \vDash^{---}_{\mathrm{Q}} \alpha$.

Where Q is any set of sequents, say that Q is *substitution-closed* provided that for any set Δ of wff, any wff β and any substitution function s, if $\langle\Delta,\beta\rangle \in$ Q then $\langle s\Delta,s\beta\rangle \in$ Q. What we must shew is that where Q is substitution-closed, $\Gamma \vdash^{---}_{\mathrm{Q}} \alpha$ iff $\Gamma \vdash_{\mathrm{Q}} \alpha$. Given theorems 12 and 12a it is sufficient to shew that $\Gamma \vDash_{\mathrm{Q}} \alpha$ iff $\Gamma \vDash^{---}_{\mathrm{Q}} \alpha$. Let ⟨W,R,V⟩ be any model. Define the *generalisation*

[4] This paper has appeared in a pre-published form in a collection of essays in honour of Krister Segerberg's 70th birthday. (pp. 111-116, of Lagerlund, Lindström and Sliwinski) The paper has been enormously improved as a result of discussions with Rob Goldblatt. His help and encouragement are especially welcome because I know that his regard for Krister is as great as my own. While we both accept that Krister had to return to his homeland we both regret New Zealand's loss. I am also grateful to Lloyd Humberstone for discussions on possible ways of differentiating CT from CT^+.

$\langle \mathrm{W}^+,\mathrm{R}^+,\mathrm{V}^+\rangle$ of $\langle \mathrm{W},\mathrm{R},\mathrm{V}\rangle$ as follows. For each substitution function s let $\langle \mathrm{W}_s,\mathrm{R}_s\rangle$ be a disjoint isomorphic copy of $\langle \mathrm{W},\mathrm{R}\rangle$. Let w_{s} denote the world in W_s which corresponds with $w \in \mathrm{W}$. $\langle \mathrm{W},\mathrm{R}\rangle$ itself may be regarded as $\langle \mathrm{W}_s,\mathrm{R}_s\rangle$ for the case where s is the identity substitution. Let W^+ be the disjoint union of all the W_ss, and R^+ be the disjoint union of all the R_ss. Then $\langle \mathrm{W}^+,\mathrm{R}^+\rangle$ will be a frame composed of a collection of isolated cells each having the structure of $\langle \mathrm{W},\mathrm{R}\rangle$. For any variable p, any $w \in \mathrm{W}$, and any substitution function s let $\mathrm{V}^+(p,w_{\mathrm{s}}) = \mathrm{V}(sp,w)$.

Lemma 13. $V^+(\alpha,w_{\mathrm{s}}) = V(s\alpha,w)$

Proof. By induction on the construction of α. □

Lemma 14. *$\langle W^+,R^+,V^+\rangle$ is generalisable.*

Proof. If $\langle \mathrm{W}^+,\mathrm{R}^+,\mathrm{V}^+\rangle$ is not generalisable there must be a Δ, β and s_1 such that $\Delta \models_{\langle W+,R+,V+\rangle} \alpha$ but $s_1\Delta \dashv_{\langle W+,R+,V+\rangle} s_1\alpha$. So there is some $w \in \mathrm{W}$ and some substitution function s_2 such that, for every $\delta \in \Delta$, $\mathrm{V}^+(s_1\delta,w_{\mathrm{s}2}) = 1$, but $\mathrm{V}^+(s_1\alpha,w_{\mathrm{s}2}) = 0$. Let s_3 be $s_1{}^*s_2$, the composition of s_1 and s_2 so that $(s_1{}^*s_2)\alpha = s_2s_1\alpha$. Then, by lemma 13, $\mathrm{V}^+(\delta,w_{\mathrm{s}3}) = \mathrm{V}(s_3\delta,w) = \mathrm{V}(s_2s_1\delta,w) = \mathrm{V}^+(s_1\delta,w_{\mathrm{s}2}) = 1$, and $\mathrm{V}^+(\alpha,w_{\mathrm{s}3}) = \mathrm{V}(s_3\alpha,w) = \mathrm{V}(s_2s_1\alpha,w) = \mathrm{V}^+(s_1\alpha,w_{\mathrm{s}2}) = 0$. But then $\Delta \dashv_{\langle W+,R+,V+\rangle} \alpha$. □

Theorem 15. *If Q is substitution-closed $\Gamma \models_{\mathrm{Q}} \alpha$ iff $\Gamma \models^{---}_{\mathrm{Q}} \alpha$.*

Proof. Since all generalisable models are models the only non-trivial direction is that if $\Gamma \models_{\mathrm{Q}} \alpha$ then $\Gamma \models^{---}_{\mathrm{Q}} \alpha$. If $\Gamma \dashv^{---}_{\mathrm{Q}} \alpha$ there is a model $\langle \mathrm{W},\mathrm{R},\mathrm{V}\rangle$ which respects Q (in the sense that if $\langle \Delta,\beta\rangle \in \mathrm{Q}$ then $\Delta \models_{\langle W,R,V\rangle} \beta$), but where $\Gamma \dashv_{\langle W,R,V\rangle} \alpha$. Because of the isolation of each cell in $\langle \mathrm{W}^+,\mathrm{R}^+\rangle$, if a sequent fails in $\langle \mathrm{W},\mathrm{R},\mathrm{V}\rangle$ it fails in $\langle \mathrm{W}^+,\mathrm{R}^+,\mathrm{V}^+\rangle$, and so $\Gamma \dashv_{\langle W+,R+,V+\rangle} \alpha$. By lemma 14 $\langle \mathrm{W}^+,\mathrm{R}^+,\mathrm{V}^+\rangle$ is generalisable, and so all that remains is to shew that if $\langle \Delta,\beta\rangle \in \mathrm{Q}$, $\Delta \models_{\langle W+,R+,V+\rangle} \beta$. If not then for some $w_{\mathrm{s}} \in \mathrm{W}^+$, $\mathrm{V}^+(\gamma,w_{\mathrm{s}}) = 1$ for every $\gamma \in \Delta$, but $\mathrm{V}^+(\beta,w_{\mathrm{s}}) = 0$, So by lemma 13, $\mathrm{V}(s\gamma,w) = 1$, but $\mathrm{V}(s\beta,w) = 0$. So $s\Delta \dashv_{\langle W,R,V\rangle} s\beta$. But $\langle \Delta,\beta\rangle \in \mathrm{Q}$ and so $\langle s\Delta,s\beta\rangle \in \mathrm{Q}$ and $\langle \mathrm{W},\mathrm{R},\mathrm{V}\rangle$ respects Q. □

Corollary 16. *If Q is substitution-closed $\Gamma \models^{---}_{\mathrm{Q}} \alpha$ iff $\Gamma \models_{\mathrm{Q}} \alpha$*

Proof. From theorems 15, 12 and 12a. □

The advantage of not assuming (SB) is that it is then up to Q whether or not propositional letters are regarded as variables or as constants. It would even be possible for Q to treat some propositional letters as variables, and some as constants.

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