*Journal of Universal Computer Science, vol. 15, no. 1 (2009), 304-354*
*submitted: 10/9/08, accepted: 12/12/08, appeared: 1/1/09 © J.UCS*

# A Characterisation of Coincidence Ideals for Complex Values

**Attila Sali**
(Alfréd Rényi Institute of Mathematics, Budapest, P.O.B.127, H-1364 Hungary
sali@renyi.hu)

**Klaus-Dieter Schewe**
(Information Science Research Centre, Palmerston North, New Zealand
kdschewe@acm.org)

**Abstract:** We investigate properties of coincidence ideals in subattribute lattices that occur in complex value datamodels, i.e. sets of subattributes, on which two complex values coincide. We let complex values be defined by constructors for records, sets, multisets, lists, disjoint union and optionality, i.e. the constructors cover the gist of all complex value data models. Such lattices carry the structure of a Brouwer algebra as long as the union-constructor is absent, and for this case sufficient and necessary conditions for coincidence ideals are already known. In this paper, we extend the characterisation of coincidence ideals to the most general case. The presence of the disjoint union constructor complicates all results and proofs significantly. The reason for this is that the union-constructor causes non-trivial restructuring rules to hold. The characterisation of coincidence ideal is of decisive importance for the axiomatisation of (weak) functional dependencies.

**Key Words:** complex values, restructuring, coincidence ideal
**Category:** F. 4.1, H. 2.1

## 1 Introduction

Complex values are around in database theory since the 1970's. First, so called semantic data models have been developed (see e.g. [Chen, 1976; Hull and King, 1987]), which were originally just meant to be used as design aids for relational databases, as application semantics was assumed to be easier captured by these models (see the argumentation in [Batini et al., 1992; Chen, 1983; Tjoa and Berger, 1993]). Later on some of these models, especially the nested relational model (see e.g. [Paredaens et al., 1989]), object oriented models (see e.g. [Schewe and Thalheim, 1993]) and object-relational models, the gist of which are captured by the higher-order Entity-Relationship model (HERM, see [Thalheim, 1992; Thalheim, 2000]) have become interesting as data models in their own right and some dependency and normalisation theory has been carried over to these advanced data models (see [Hartmann, 2001; Mok et al., 1996; Özsoyoglu and Yuan, 1987; Paredaens et al., 1989; Tari et al., 1997] as samples of the many work done on this so far). Most recently, the major research interest is on the model of semi-structured data and XML (see e.g. [Abiteboul et al., 2000]), which may also be regarded as some kind of object oriented model.

We refer to all these models as *higher-order data models*. This is, because the most important extension that came with these models was the introduction of constructors for complex values. These constructors usually comprise bulk constructors for sets, lists and multisets, a disjoint union constructor, and an optionality or null-constructor. In fact, all the structure of higher-order data models (including XML as far as XML can be considered a data model) is captured by the introduction of (some or all of) these constructors. This leads to lattices of subattributes, which even carry the structure of a Brouwer algebra as long as the union-constructor is absent.

A key problem is to develop dependency theories (or preferably a unified theory) for the higher-order data models. The development of such a dependency theory will have a significant impact on understanding application semantics and laying the grounds for a logically founded theory of well-designed non-relational databases. In doing so we come across the problem to characterise coincidence ideals, i.e. sets of subattributes, on which two complex values coincide. Such a characterisation is indeed essential for the completeness proofs for axiomatisations of functional dependencies.

For the relational model this was a triviality, but even if only few constructors are used, the characterisation of coincidence ideals is already non-trivial. The work in [Hartmann et al., 2006] covers the case of all constructors combined except the union constructor. This has been slighly extended in [Sali and Schewe, 2006] to capture also the union-constructor, provided that counter-attributes are excluded. In this paper we are now able to present sufficient and necessary conditions for the most general case, when all constructors are present simultaneously. The technical effort to achieve this characterisation compared with previous work is, however, enormous. In [Sali and Schewe, 2008] this result is used to extend the axiomatisation of weak functional dependencies to the most general case.

In Section 2 we define the preliminaries for our theory. We start with the definition of nested attributes that are composed of simple attributes using the constructors that have been mentioned above. Each nested attribute defines a set of complex values called its domain, and each complex value can be represented as a finite tree. We then define subattributes, which give rise to canonical projection maps on the domains. The presence of the union constructor leads to restructuring rules, which define non-trivial equivalences the set of subattributes of a given nested attribute. We obtain a lattice, which is even a Brouwer algebra, if the union constructor is absent. Nevertheless, also in the general case it is advantageous to define the notion of relative pseudo-complement.

In Section 3 we study certain ideals in such lattices of subattributes, focusing on the set of subattributes, on which two complex values coincide. These ideals are therefore called *coincidence ideals*. The objective is to obtain a precise characterisation in the sense that whenever an ideal satisfies the given set of properties, we can guarantee the existence of two complex values that coincide exactly on the given ideal. This leads to the *Central Theorem* on coincidence ideals.

## 2 Algebras of Nested Attributes

In this section we define our model of nested attributes, which covers the gist of higher-order data models including XML. In particular, we investigate the structure of the set $\mathcal{S}(X)$ of subattributes of a given nested attribute $X$. We show that we obtain a lattice, which in general is non-distributive. This lattice becomes a Brouwer algebra, if the union constructor is not used.

### 2.1 Nested Attributes

We start with a definition of simple attributes and values for them.

**Definition 1.** A *universe* is a finite set $\mathcal{U}$ together with domains (i.e. sets of values) $dom(A)$ for all $A \in \mathcal{U}$. The elements of $\mathcal{U}$ are called *simple attributes*.

For the relational model a universe was sufficient, as a relation schema could be defined by a subset $R \subseteq \mathcal{U}$. For higher-order data models, however, we need nested attributes. In the following definition we use a set $\mathcal{L}$ of labels, and tacitly assume that the symbol $\lambda$ is neither a simple attribute nor a label, i.e. $\lambda \notin \mathcal{U} \cup \mathcal{L}$, and that simple attributes and labels are pairwise different, i.e. $\mathcal{U} \cap \mathcal{L} = \emptyset$.

**Definition 2.** Let $\mathcal{U}$ be a universe and $\mathcal{L}$ a set of labels. The set $\mathcal{N}$ of *nested attributes* (over $\mathcal{U}$ and $\mathcal{L}$) is the smallest set with $\lambda \in \mathcal{N}$, $\mathcal{U} \subseteq \mathcal{N}$, and satisfying the following properties:

- for $X \in \mathcal{L}$ and $X'_1, \dots, X'_n \in \mathcal{N}$ we have $X(X'_1, \dots, X'_n) \in \mathcal{N}$;
- for $X \in \mathcal{L}$ and $X' \in \mathcal{N}$ we have $X\{X'\} \in \mathcal{N}$, $X[X'] \in \mathcal{N}$, and $X\langle X' \rangle \in \mathcal{N}$;
- for $X_1, \dots, X_n \in \mathcal{L}$ and $X'_1, \dots, X'_n \in \mathcal{N}$ we have $X_1(X'_1) \oplus \cdots \oplus X_n(X'_n) \in \mathcal{N}$.

We call $\lambda$ a *null attribute*, $X(X'_1, \dots, X'_n)$ a *record attribute*, $X\{X'\}$ a *set attribute*, $X[X']$ a *list attribute*, $X\langle X' \rangle$ a *multiset attribute* and $X_1(X'_1) \oplus \cdots \oplus X_n(X'_n)$ a *union attribute*.

In the following we will overload the use of symbols such as $X$, $Y$, etc. for nested attributes and labels. As record, set, list and multiset attributes have a unique leading label, this will not cause problems anyway. In all other cases it is clear from the context, whether a symbol denotes a nested attribute in $\mathcal{N}$ or a label. Usually, labels never appear as stand-alone symbols.

We also take the freedom to change the leading label $X$ in a set, list or multiset attribute to $X_{\{1,\dots,n\}}$, if the component attribute is a union attribute, say $X_1(X_1') \oplus \dots \oplus X_n(X_n')$. This emphasises the factors in the union attribute. We will see in the next two subsections that this notation will become important, when restructuring is considered.

We can now extend the association *dom* from simple to nested attributes, i.e. for each $X \in \mathcal{N}$ we will define a set of values $dom(X)$.

**Definition 3.** For each nested attribute $X \in \mathcal{N}$ we get a *domain* $dom(X)$ as follows:

- $dom(\lambda) = \{\top\}$;
- $dom(X(X_1', \dots, X_n')) = \{(v_1, \dots, v_n) \mid v_i \in dom(X_i') \text{ for } i = 1, \dots, n\}$;
- $dom(X\{X'\}) = \{\{v_1, \dots, v_k\} \mid k \in \mathbb{N} \text{ and } v_i \in dom(X') \text{ for } i = 1, \dots, k\}$, i.e. each element in $dom(X\{X'\})$ is a finite set with (pairwise different) elements in $dom(X')$;
- $dom(X[X']) = \{[v_1, \dots, v_k] \mid k \in \mathbb{N} \text{ and } v_i \in dom(X') \text{ for } i = 1, \dots, k\}$, i.e. each element in $dom(X[X'])$ is a finite (ordered) list with (not necessarily different) elements in $dom(X')$;
- $dom(X\langle X'\rangle) = \{\langle v_1, \dots, v_k\rangle \mid k \in \mathbb{N} \text{ and } v_i \in dom(X') \text{ for } i = 1, \dots, k\}$, i.e. each element in $dom(X\langle X'\rangle)$ is a finite multiset with elements in $dom(X')$, or in other words each $v \in dom(X')$ has a *multiplicity* $m(v) \in \mathbb{N}$ in a value in $dom(X\langle X'\rangle)$;
- $dom(X_1(X_1') \oplus \dots \oplus X_n(X_n')) = \{(X_i : v_i) \mid v_i \in dom(X_i') \text{ for } i = 1, \dots, n\}$.

Note that the relational model is covered, if only the record constructor is used. Thus, instead of a relation schema $R$ we will now consider a nested attribute $X$, assuming that the universe $\mathcal{U}$ and the set of labels $\mathcal{L}$ are fixed. Instead of an $R$-relation $r$ we will consider a finite set $r \subseteq dom(X)$.

### 2.2 Subattributes

In the relational model a functional dependency $X \to Y$ for $X, Y \subseteq R \subseteq \mathcal{U}$ is satisfied by an $R$-relation $r$ iff any two tuples $t_1, t_2 \in r$ that coincide on all the attributes in $X$ also coincide on the attributes in $Y$. Crucial to this definition is that we can project $R$-tuples to subsets of attributes.

Therefore, in order to define FDs on a nested attribute $X \in \mathcal{N}$ we need a notion of subattribute. For this we define a partial order $\geq$ on nested attributes in such a way that whenever $X \geq Y$ holds, we obtain a canonical projection $\pi_Y^X : dom(X) \to dom(Y)$. However, this partial order has to be defined on equivalence classes of attributes, as some domains may be identified.

**Definition 4.** $\equiv$ is the smallest *equivalence relation* on $\mathcal{N}$ satisfying the following properties:

- $\lambda \equiv X()$;
- $X(X'_1, \dots, X'_n) \equiv X(X'_1, \dots, X'_n, \lambda)$;
- $X(X'_1, \dots, X'_n) \equiv X(X'_{\sigma(1)}, \dots, X'_{\sigma(n)})$ for any permutation $\sigma \in \mathbf{S}_n$;
- $X_1(X'_1) \oplus \dots \oplus X_n(X'_n) \equiv X_{\sigma(1)}(X'_{\sigma(1)}) \oplus \dots \oplus X_{\sigma(n)}(X'_{\sigma(n)})$ for any permutation $\sigma \in \mathbf{S}_n$;
- $X(X'_1, \dots, X'_n) \equiv X(Y_1, \dots, Y_n)$ iff $X'_i \equiv Y_i$ for all $i = 1, \dots, n$;
- $X_1(X'_1) \oplus \dots \oplus X_n(X'_n) \equiv X_1(Y_1) \oplus \dots \oplus X_n(Y_n)$ iff $X'_i \equiv Y_i$ for all $i = 1, \dots, n$;
- $X\{X'\} \equiv X\{Y\}$ iff $X' \equiv Y$;
- $X[X'] \equiv X[Y]$ iff $X' \equiv Y$;
- $X\langle X'\rangle \equiv X\langle Y\rangle$ iff $X' \equiv Y$;
- $X(X'_1, \dots, Y_1(Y'_1) \oplus \dots \oplus Y_m(Y'_m), \dots, X'_n) \equiv Y_1(X'_1, \dots, Y'_1, \dots, X'_n) \oplus \dots \oplus Y_m(X'_1, \dots, Y'_m, \dots, X'_n)$;
- $X_{\{1,\dots,n\}}\{X_1(X'_1) \oplus \dots \oplus X_n(X'_n)\} \equiv X_{\{1,\dots,n\}}(X_1\{X'_1\}, \dots, X_n\{X'_n\})$;
- $X_{\{1,\dots,n\}}\langle X_1(X'_1) \oplus \dots \oplus X_n(X'_n)\rangle \equiv X_{\{1,\dots,n\}}(X_1\langle X'_1\rangle, \dots, X_n\langle X'_n\rangle)$.

Basically, the first four cases in this equivalence definition state that $\lambda$ in record attributes can be added or removed, and that order in record and union attributes does not matter. The last three cases in Definition 4 cover restructuring rules, two of which were already introduced by Abiteboul and Hull (see [Abiteboul and Hull, 1988]). Obviously, if we have a set of labelled elements with up to $n$ different labels, we can split this set into $n$ subsets, each of which contains just the elements with a particular label, and the union of these sets is the original set. The same holds for multisets. Of course, we can also split a list of labelled elements into lists containing only elements with the same label, thereby preserving the order, but in this case we cannot invert the splitting and thus cannot claim an equivalence.

In the following we identify $\mathcal{N}$ with the set $\mathcal{N}/_\equiv$ of equivalence classes. In particular, we will write $=$ instead of $\equiv$, and in the following definition we should say that $Y$ is a subattribute of $X$ iff $\tilde{X} \geq \tilde{Y}$ holds for some $\tilde{X} \equiv X$ and $\tilde{Y} \equiv Y$. In particular, for $X \equiv Y$ we obtain $X \geq Y$ and $Y \geq X$.

**Definition 5.** For $X, Y \in \mathcal{N}$ we say that $Y$ is a *subattribute* of $X$, iff $X \geq Y$ holds, where $\geq$ is the smallest partial order on $\mathcal{N}/_\equiv$ satisfying the following properties:

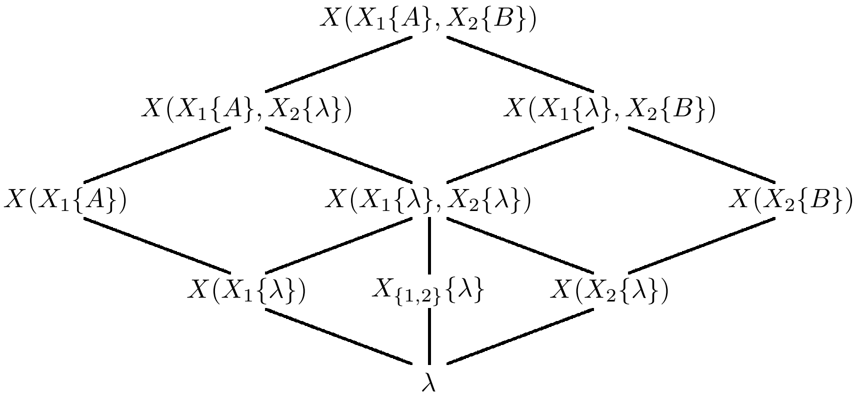


**Figure 1:** The lattice $\mathcal{S}(X\{X_1(A) \oplus X_2(B)\}) = \mathcal{S}(X(X_1\{A\}, X_2\{B\}))$

- $X \geq \lambda$ for all $X \in \mathcal{N}$;
- $X(Y_1, \dots, Y_n) \geq X(X'_{\sigma(1)}, \dots, X'_{\sigma(m)})$ for some injective $\sigma : \{1, \dots, m\} \to \{1, \dots, n\}$ and $Y_{\sigma(i)} \geq X'_{\sigma(i)}$ for all $i = 1, \dots, m$;
- $X_1(Y_1) \oplus \cdots \oplus X_n(Y_n) \geq X_{\sigma(1)}(X'_{\sigma(1)}) \oplus \cdots \oplus X_{\sigma(n)}(X'_{\sigma(n)})$ for some permutation $\sigma \in \mathbf{S}_n$ and $Y_i \geq X'_i$ for all $i = 1, \dots, n$;
- $X\{Y\} \geq X\{X'\}$ iff $Y \geq X'$;
- $X[Y] \geq X[X']$ iff $Y \geq X'$;
- $X\langle Y \rangle \geq X\langle X' \rangle$ iff $Y \geq X'$;
- $X_{\{1,\dots,n\}}[X_1(X'_1) \oplus \cdots \oplus X_n(X'_n)] \geq X(X_1[X'_1], \dots, X_n[X'_n])$;
- $X_{\{1,\dots,k\}}[X_1(X'_1) \oplus \cdots \oplus X_k(X'_k)] \geq X_{\{1,\dots,\ell\}}[X_1(X'_1) \oplus \cdots \oplus X_\ell(X'_\ell)]$ for $k \geq \ell$;
- $X(X_{i_1}\{\lambda\}, \dots, X_{i_k}\{\lambda\}) \geq X_{\{i_1,\dots,i_k\}}\{\lambda\}$;
- $X(X_{i_1}\langle\lambda\rangle, \dots, X_{i_k}\langle\lambda\rangle) \geq X_{\{i_1,\dots,i_k\}}\langle\lambda\rangle$;
- $X(X_{i_1}[\lambda], \dots, X_{i_k}[\lambda]) \geq X_{\{i_1,\dots,i_k\}}[\lambda]$.

Note that the last five cases in Definition 5 cover further restructuring rules due to the union constructor. Obviously, if we are given a list of elements labelled with $X_1, \dots, X_n$, we can take the individual sublists – preserving the order – that contain only those elements labelled by $X_i$ and build the tuple of these lists. In this case we can turn the label into a label for the whole sublist. This explains the first of the last five subattribute relationships.

For the other restructuring rules we have to add a little remark on notation here. As we identify $X\{X_1(X_1') \oplus \cdots \oplus X_n(X_n')\}$ with $X(X_1\{X_1'\}, \dots, X_n\{X_n'\})$, we obtain subattributes $X(X_{i_1}\{X_{i_1}'\}, \dots \; \dots, X_{i_k}\{X_{i_k}'\})$ for each subset $I = \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$. However, restructuring requires some care with labels. If we simply reused the label $X$ in the last property in Definition 5, we would obtain

$$X\{X_1(X_1') \oplus X_2(X_2')\} \equiv X(X_1\{X_1'\}, X_2\{X_2'\}) \geq$$
$$\geq X(X_1\{X_1'\}) \geq X(X_1\{\lambda\}) \geq X\{\lambda\}.$$

However, the last step here is wrong, as the left hand side is an indicator for the subset containing the elements with label $X_1$ being empty or not, whereas the right hand side is the corresponding indicator for the whole set, i.e. elements with labels $X_1$ or $X_2$. No such mapping can be claimed. In fact, what we really have to do is to mark the set label in an attribute of the form $X\{X_1(X_1') \oplus \cdots \oplus X_n(X_n')\}$ to indicate the inner union attribute, i.e. we should use $X_{\{1,\dots,n\}}$ (or even $X_{\{X_1,\dots,X_n\}}$) instead of $X$. As long as we are not dealing with subattributes of the form $X_{\{1,\dots,k\}}\{\lambda\}$, the additional index does not add any information and thus can be omitted to increase readability. The same applies to the multiset- and the list-constructor.

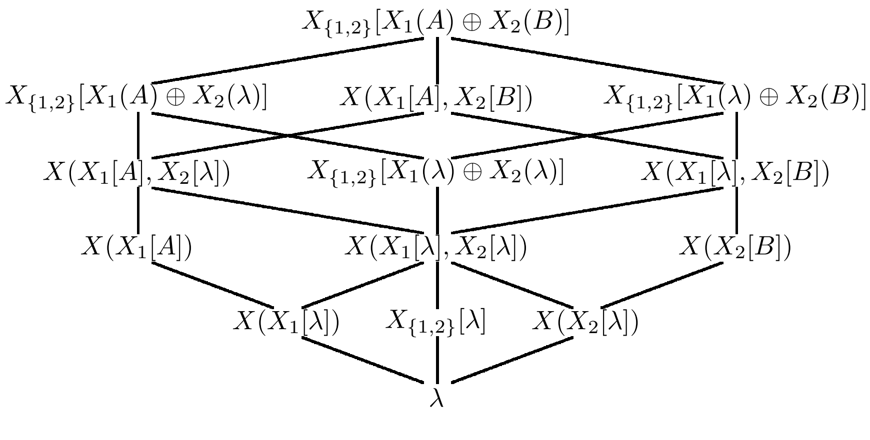


**Figure 2:** The lattice $\mathcal{S}(X[X_1(A) \oplus X_2(B)])$

Subattributes of the form $X_I\{\lambda\}$, $X_I[\lambda]$ and $X_I\{\lambda\}$ were called *counter attributes* in [Sali and Schewe, 2006], because they can be considered as counters for the number of elements in a list or multiset or as flags that tell, whether sets are empty or not. Note that $X_\emptyset\{\lambda\} = \lambda$, $X_{\{1,\dots,n\}}\{\lambda\} = X\{\lambda\}$ and $X_{\{i\}}\{\lambda\} = X(X_i\{\lambda\})$. Analogous conventions apply to list and multiset attributes.

Further note that due to the restructuring rules in Definitions 4 and 5 we may have the case that a record attribute is a subattribute of a set attribute and vice versa. This cannot be the case, if the union-constructor is absent. However, the presence of the restructuring rules allows us to assume that the union-constructor only appears inside a set-constructor or as the outermost constructor. This will be frequently exploited in our proofs.

Obviously, $X \geq Y$ induces a projection map $\pi^X_Y : dom(X) \to dom(Y)$. For $X \equiv Y$ we have $X \geq Y$ and $Y \geq X$ and the projection maps $\pi^X_Y$ and $\pi^Y_X$ are inverse to each other.

We use the notation $\mathcal{S}(X) = \{Z \in \mathcal{N} \mid X \geq Z\}$ to denote the *set of subattributes* of a nested attribute $X$. Figure 1 shows the subattributes of $X\{X_1(A) \oplus X_2(B)\} = X(X_1\{A\}, X_2\{B\})$ together with the relation $\geq$ on them. Note that the subattribute $X_{\{1,2\}}\{\lambda\}$ would not occur, if we only considered the record-structure, whereas other subattributes such as $X(X_i\{\lambda\})$ would not occur, if we only considered the set-structure. This is a direct consequence of the restructuring rules.

Figure 2 shows the subattributes of $X[X_1(A) \oplus X_2(B)]$ together with the relation $\geq$ on them. The subattributes $X_{\{1,2\}}[\lambda]$ would not occur, if we only considered the list-structure, whereas other subattributes such as $X(X_i[\lambda])$ would not occur, if we ignored the restructuring rules. Figure 3 shows the subattributes of $X\{X_1(A) \oplus X_2(B) \oplus X_3(C)\}$ together with the relation $\geq$ on them. The subattribute $X_I\{\lambda\}$ for $|I| \geq 2$ would not occur, if we only considered the record-structure.

### 2.3 The Lattice Structure

The set of subattributes $\mathcal{S}(X)$ of a nested attribute $X$ plays the same role in the dependency theory for higher-order data models as the powerset $\mathcal{P}(R)$ for a relation schema $R$ plays in the dependency theory for the relational model. $\mathcal{P}(R)$ is a Boolean algebra with order $\subseteq$, intersection $\cap$, union $\cup$ and the difference $-$. So, the question arises which algebraic structure $\mathcal{S}(X)$ carries.

**Definition 6.** Let $\mathcal{L}$ be a lattice with zero and one, partial order $\leq$, join $\sqcup$ and meet $\sqcap$. $\mathcal{L}$ has *relative pseudo-complements* iff for all $Y, Z \in \mathcal{L}$ the infimum $Y \leftarrow Z = \bigsqcap\{U \mid U \sqcup Y \geq Z\}$ exists. Then $Y \leftarrow 1$ (1 being the one in $\mathcal{L}$) is called the *relative complement* of $Y$.

If we have distributivity in addition, we call $\mathcal{L}$ a *Brouwer algebra*. In this case the relative pseudo-complements satisfy $U \geq (Y \leftarrow Z)$ iff $(U \sqcup Y \geq Z)$, but if we do not have distributivity this property may be violated though relative pseudo-complements exist.

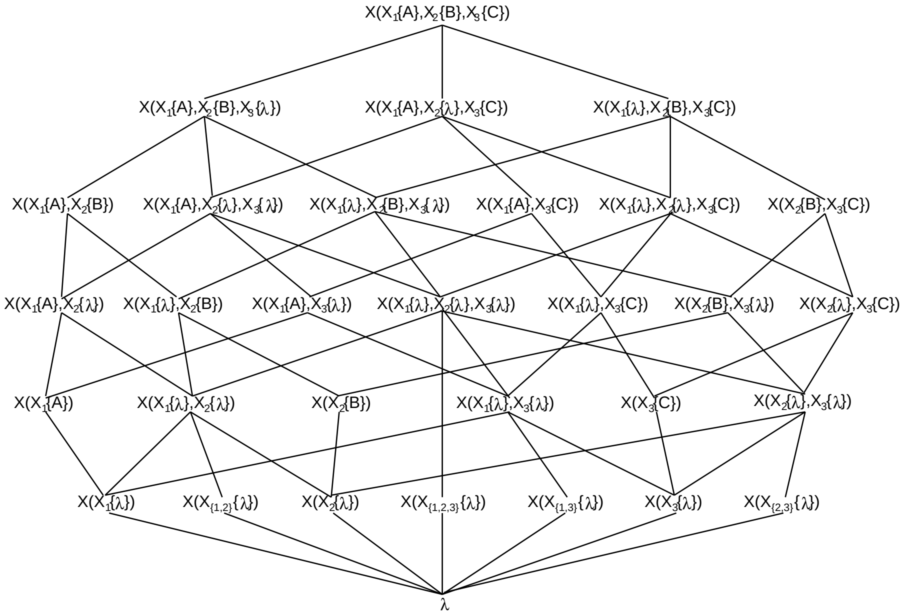


**Figure 3:** The subattribute lattice $\mathcal{S}(X\{X_1(A) \oplus X_2(B) \oplus X_3(C)\})$

**Theorem 7.** *The set $\mathcal{S}(X)$ of subattributes carries the structure of a lattice with zero and one and relative pseudo-complements, where the order $\geq$ is as defined in Definition 5, and $\lambda$ and $X$ are the zero and one, respectively. If $X$ does not contain the union constructor, $\mathcal{S}(X)$ defines a Brouwer algebra.*

*Proof.* For $X = \lambda$ and simple attributes $X = A$ we obtain trivial lattices with only one or two elements. Applying the record constructor leads to a cartesian product of lattices, while the set, list and multiset constructors add a new zero element to a lattice. These extensions preserve the properties of a Brouwer algebra.

In the case of set, list and multiset constructors applied to a union attribute we add counter attributes. This preserves the properties of a lattice and the existence of relative pseudo-complement, while distributivity may be lost.

*Example 1.* Let $X = X\{X_1(A) \oplus X_2(B)\}$ with $\mathcal{S}(X)$ as illustrated in Figure 1, $Y_1 = X\{\lambda\}$, $Y_2 = X(X_2\{B\})$, and $Z = X(X_1\{A\})$. Then we have

$$Z \sqcap (Y_1 \sqcup Y_2) = X(X_1\{A\} \sqcap (X\{\lambda\} \sqcup X(X_2\{B\})) =$$
$$X(X_1\{A\}) \sqcap X(X_1\{\lambda\}, X_2\{B\}) = X(X_1\{\lambda\}) \neq \lambda = \lambda \sqcup \lambda =$$
$$(X(X_1\{A\}) \sqcap X\{\lambda\}) \sqcup (X(X_1\{A\}) \sqcap X(X_2\{B\})) = (Z \sqcap Y_1) \sqcup (Z \sqcap Y_2)\ .$$

This shows that $\mathcal{S}(X)$ in general is not a distributive lattice. Furthermore, $Y' \sqcup Z \geq Y_1$ holds for all $Y'$ except $\lambda$, $X(X_1\{\lambda\})$ and $X(X_1\{A\})$. So $Z \leftarrow Y_1 = \lambda$, but not all $Y' \geq \lambda$ satisfy $Y' \sqcup Z \geq Y_1$. $\square$

It is easy to determine explicit inductive definitions of the operations $\sqcap$ (meet), $\sqcup$ (join) and $\leftarrow$ (relative pseudo-complement). This can be done by boring technical verification of the properties of meets, joins and relative pseudo-complements and is therefore omitted here.

## 3 Coincidence Ideals

In this section we investigate sets of subattributes, on which two complex values coincide. It is rather easy to see that these turn out to be ideals in the lattice $\mathcal{S}(X)$, i.e. they are non-empty and downward-closed. Therefore, we will call them *coincidence ideals*. However, there are many other properties that hold for coincidence ideals.

There are two major reasons for looking at coincidence ideals. The first one is that properties of subattributes, on which two complex values coincide, may give rise to axioms for functional dependencies. Indeed, the properties of coincidence ideals in Definition 10 are very closely related to the sound axioms and rules for (weak) functional dependencies in [Sali and Schewe, 2008].

The second reason is that in the completeness proof in [Sali and Schewe, 2008] we have to construct two complex values that coincide exactly on a given set of attributes, so that a set of dependencies is satisfied by these values, while a non-derivable dependency is not. This step appears also in the corresponding completeness proof for the RDM, but in that case it is trivial, because it simply amounts to getting two tuples that coincide on a given set of attributes, but differ on all others.

Thus, what we want to achieve is a characterisation of a coincidence ideal that allows us to construct two complex values that coincide exactly on it. This will be the main result of this section, called the Central Theorem 17 on coincidence ideals. The proof of this result, however, will be very technical.

### 3.1 Necessary Properties of Coincidence Ideals

Let us start doing the first step, i.e. introducing coincidence ideal as sets of subattributes, on which two complex values coincide, and derive necessary conditions for such ideals. For one of the properties dealing with the join $Y \cup Z$ we will need the notion of reconsilable subattributes, which was already used in the axiomatisations of restricted cases (see [Hartmann et al., 2005; Hartmann et al., 2006]). The following Definition 8 extends this notion to capture all constructors, in particular the union constructor.

**Definition 8.** Two subattributes $Y, Z \in \mathcal{S}(X)$ are called *reconsilable* iff one of the following holds:

1. $Y \geq Z$ or $Z \geq Y$;
2. $X = X[X']$, $Y = X[Y']$, $Z = X[Z']$ and $Y', Z' \in \mathcal{S}(X')$ are reconsilable;
3. $X = X(X_1, \dots, X_n)$, $Y = X(Y_1, \dots, Y_n)$, $Z = X(Z_1, \dots, Z_n)$ and $Y_i, Z_i \in \mathcal{S}(X_i)$ are reconsilable for all $i = 1, \dots, n$;
4. $X = X_1(X_1') \oplus \cdots \oplus X_n(X_n')$, $Y = X_1(Y_1') \oplus \cdots \oplus X_n(Y_n')$, $Z = X_1(Z_1') \oplus \cdots \oplus X_n(Z_n')$ and $Y_i', Z_i' \in \mathcal{S}(X_i')$ are reconsilable for all $i = 1, \dots, n$;
5. $X = X[X_1(X_1') \oplus \cdots \oplus X_n(X_n')]$, $Y = X(Y_1, \dots, Y_n)$ with $Y_i = X_i[Y_i']$ or $Y_i = \lambda = Y_i'$, $Z = X[X_1(Z_1') \oplus \cdots \oplus X_n(Z_n')]$, and $Y_i'$, $Z_i'$ are reconsilable for all $i = 1, \dots, n$.

Note that for the set- and multiset-constructor we can only obtain reconsilability for subattributes in a $\geq$-relation.

**Definition 9.** Let $X \in \mathcal{N}$ be a nested attribute. A set $\mathcal{F} \subseteq \mathcal{S}(X)$ is called a *coincidence ideal* iff there exists two complex values $t_1, t_2 \in dom(X)$ with $\mathcal{F} = \{Y \in \mathcal{S}(X) \mid \pi_Y^X(t_1) = \pi_Y^X(t_2)\}$.

The following theorem shows necessary properties of coincidence ideals. Showing that these properties are also sufficient is the theme of the next subsection.

**Theorem 10.** *Let $X \in \mathcal{N}$ be a nested attribute, and $\mathcal{F} = \{Y \in \mathcal{S}(X) \mid \pi_Y^X(t_1) = \pi_Y^X(t_2)\}$ be a coincidence ideal. Then $\mathcal{F}$ satisfies the following properties:*

1. *$\lambda \in \mathcal{F}$;*
2. *if $Y \in \mathcal{F}$ and $Z \in \mathcal{S}(X)$ with $Y \geq Z$, then $Z \in \mathcal{F}$;*
3. *if $Y, Z \in \mathcal{F}$ are reconsilable, then $Y \sqcup Z \in \mathcal{F}$;*
4. *with $I^+ = \{i \in \{1, \dots, n\} \mid X(X_i\{\lambda\}) \in \mathcal{F}\}$ and $I^- = \{i \in \{1, \dots, n\} \mid X(X_i\{\lambda\}) \notin \mathcal{F}\}$*
   - *(a) for $I = \{i_1, \dots, i_k\}$, if $X_I\{\lambda\} \in \mathcal{F}$ and $X_J\{\lambda\} \notin \mathcal{F}$ for $I \subsetneq J$, then $X(X_{i_1}\{X_{i_1}'\}, \dots, X_{i_k}\{X_{i_k}'\}) \in \mathcal{F}$;*
   - *(b) if $X_I\{\lambda\} \in \mathcal{F}$ and $X(X_i\{\lambda\}) \notin \mathcal{F}$ for all $i \in I$, then there is a partition $I = I_1 \dot\cup I_2$ with $X_{I_1}\{\lambda\} \notin \mathcal{F}$, $X_{I_2}\{\lambda\} \notin \mathcal{F}$ and $X_{I'}\{\lambda\} \in \mathcal{F}$ for all $I' \subseteq I$ with $I' \cap I_1 \neq \emptyset \neq I' \cap I_2$;*
   - *(c) if $X_{\{1,\dots,n\}}\{\lambda\} \in \mathcal{F}$ and $X_{I^-}\{\lambda\} \notin \mathcal{F}$, then there exists some $i \in I^+$ such that for all $J \subseteq I^-$ $X_{J \cup \{i\}}\{\lambda\} \in \mathcal{F}$ holds;*

(d) *if* $X_J\{\lambda\} \notin \mathcal{F}$ *and* $X_{\{j\}}\{\lambda\} \notin \mathcal{F}$ *for all* $j \in J$ *and for all* $i \in I$ *there is some* $J_i \subseteq J$ *with* $X_{J_i \cup \{i\}}\{\lambda\} \notin \mathcal{F}$, *then* $X_{I \cup J}\{\lambda\} \notin \mathcal{F}$, *provided* $I \cap J = \emptyset$;

(e) *if* $X_{I^-}\{\lambda\} \in \mathcal{F}$ *and* $I' \subseteq I^+$ *such that for all* $i \in I'$ *there is some* $J \subseteq I^-$ *with* $X_{J \cup \{i\}}\{\lambda\} \notin \mathcal{F}$, *then* $X_{I' \cup J'}\{\lambda\} \notin \mathcal{F}$ *for all* $J' \subseteq I^-$ *with* $X_{J'}\{\lambda\} \notin \mathcal{F}$;

5. (a) *if* $X_I\{\lambda\} \in \mathcal{F}$ *and* $X_J\{\lambda\} \in \mathcal{F}$ *with* $I \cap J = \emptyset$, *then* $X_{I \cup J}\{\lambda\} \in \mathcal{F}$;

(b) *if* $X_I[\lambda] \in \mathcal{F}$ *and* $X_J[\lambda] \in \mathcal{F}$ *with* $I \cap J = \emptyset$, *then* $X_{I \cup J}[\lambda] \in \mathcal{F}$;

(c) *if* $X_I\langle\lambda\rangle \in \mathcal{F}$ *and* $X_J\langle\lambda\rangle \in \mathcal{F}$ *with* $I \cap J = \emptyset$, *then* $X_{I \cup J}\langle\lambda\rangle \in \mathcal{F}$;

(d) *if* $X_I[\lambda] \in \mathcal{F}$ *and* $X_J[\lambda] \in \mathcal{F}$ *with* $J \subseteq I$, *then* $X_{I-J}[\lambda] \in \mathcal{F}$;

(e) *if* $X_I\langle\lambda\rangle \in \mathcal{F}$ *and* $X_J\langle\lambda\rangle \in \mathcal{F}$ *with* $J \subseteq I$, *then* $X_{I-J}\langle\lambda\rangle \in \mathcal{F}$;

(f) *if* $X_I[\lambda] \in \mathcal{F}$ *and* $X_J[\lambda] \in \mathcal{F}$, *then* $X_{I \cap J}[\lambda] \in \mathcal{F}$ *iff* $X_{(I-J)\cup(J-I)}[\lambda] \in \mathcal{F}$;

(g) *if* $X_I\langle\lambda\rangle \in \mathcal{F}$ *and* $X_J\langle\lambda\rangle \in \mathcal{F}$, *then* $X_{I \cap J}\langle\lambda\rangle \in \mathcal{F}$ *iff* $X_{(I-J)\cup(J-I)}\langle\lambda\rangle \in \mathcal{F}$;

6. (a) *for* $X = X\{\bar{X}\{X_1(X_1') \oplus \cdots \oplus X_n(X_n')\}\}$, *whenever* $I \subseteq \{1, \dots, n\}$, *there is a partition* $I = I^- \cup I_{+-} \cup I_+ \cup I_-$ *such that*

   i. $X\{\bar{X}_{\{i\}}\{\lambda\}\} \in \mathcal{F}$ *iff* $i \notin I^-$,

   ii. $X\{\bar{X}_{I'}\{\lambda\}\} \in \mathcal{F}$, *whenever* $I' \cap I_+ \neq \emptyset$,

   iii. $X\{\bar{X}_{I'}\{\lambda\}\} \in \mathcal{F}$ *iff* $X\{\bar{X}_{I' \cap (I_{+-} \cup I^-)}\{\lambda\}\} \in \mathcal{F}$, *whenever* $I' \subseteq I_{+-} \cup I^- \cup I_-$;

(b) *for* $X = X\langle\bar{X}\{X_1(X_1') \oplus \cdots \oplus X_n(X_n')\}\rangle$, *whenever* $I \subseteq \{1, \dots, n\}$, *there is a partition* $I = I^- \cup I_{+-} \cup I_+ \cup I_-$ *such that*

   i. $X\langle\bar{X}_{\{i\}}\{\lambda\}\rangle \in \mathcal{F}$ *iff* $i \notin I^-$,

   ii. $X\langle\bar{X}_{I'}\{\lambda\}\rangle \in \mathcal{F}$, *whenever* $I' \cap I_+ \neq \emptyset$,

   iii. $X\langle\bar{X}_{I'}\{\lambda\}\rangle \in \mathcal{F}$ *iff* $X\langle\bar{X}_{I' \cap (I_{+-} \cup I^-)}\{\lambda\}\rangle \in \mathcal{F}$, *whenever* $I' \subseteq I_{+-} \cup I^- \cup I_-$;

7. (a) *if* $X = X(X_1', \dots, X_n')$, *then* $\mathcal{F}_i = \{Y_i \in \mathcal{S}(X_i') \mid X(\lambda, \dots, Y_i, \dots, \lambda) \in \mathcal{F}\}$ *is a coincidence ideal;*

(b) *if* $X = X[X']$, *such that* $X'$ *is not a union attribute, and* $\mathcal{F} \neq \{\lambda\}$, *then* $\mathcal{G} = \{Y \in \mathcal{S}(X') \mid X[Y] \in \mathcal{F}\}$ *is a coincidence ideal;*

(c) *If* $X = X_1(X_1') \oplus \cdots \oplus X_n(X_n')$ *and* $\mathcal{F} \neq \{\lambda\}$, *then the set* $\mathcal{F}_i = \{Y_i \in \mathcal{S}(X_i') \mid X_1(\lambda) \oplus \cdots \oplus X_i(Y_i) \oplus \cdots \oplus X_n(\lambda) \in \mathcal{F}\}$ *is a coincidence ideal;*

(d) *if* $X = X\{X'\}$, *such that* $X'$ *is not a union attribute, and* $\mathcal{F} \neq \{\lambda\}$, *then* $\mathcal{G} = \{Y \in \mathcal{S}(X') \mid X\{Y\} \in \mathcal{F}\}$ *is a defect coincidence ideal;*

(e) *if* $X = X\langle X'\rangle$, *such that* $X'$ *is not a union attribute, and* $\mathcal{F} \neq \langle\lambda\rangle$, *then* $\mathcal{G} = \{Y \in \mathcal{S}(X') \mid X\langle Y\rangle \in \mathcal{F}\}$ *is a defect coincidence ideal.*

*A* defect coincidence ideal *on* $\mathcal{S}(X)$ *is a subset* $\mathcal{F} \subseteq \mathcal{S}(X)$ *satisfying properties 1, 2, 4(a)-(d), 6(a),(b), 7(d)-(e) and*

8. (a) *if* $X = X(X'_1, \dots, X'_n)$, *then* $\mathcal{F}_i = \{Y_i \in \mathcal{S}(X'_i) \mid X(\lambda, \dots, Y_i, \dots, \lambda) \in \mathcal{F}\}$ *is a defect coincidence ideal;*

(b) *if* $X = X[X']$, *such that* $X'$ *is not a union attribute, and* $\mathcal{F} \neq \{\lambda\}$, *then* $\mathcal{G} = \{Y \in \mathcal{S}(X') \mid X[Y] \in \mathcal{F}\}$ *is a defect coincidence ideal;*

(c) *If* $X = X_1(X'_1) \oplus \cdots \oplus X_n(X'_n)$ *and* $\mathcal{F} \neq \{\lambda\}$, *then the set* $\mathcal{F}_i = \{Y_i \in \mathcal{S}(X'_i) \mid X_1(\lambda) \oplus \cdots \oplus X_i(Y_i) \oplus \cdots \oplus X_n(\lambda) \in \mathcal{F}\}$ *is a defect coincidence ideal.*

In [Hartmann et al., 2004] and in [Sali, 2004] the term "SHL-ideal" was used instead; in [Hartmann et al., 2005] in a restricted setting the term "HL-ideal" was used. In all these cases the definition was given by means of properties as in the theorem, but not all the conditions from Definition 10 were yet present.

*Proof.* Let $\mathcal{F} = \{Y \in \mathcal{S}(X) \mid \pi_Y^X(t_1) = \pi_Y^X(t_2)\} \subseteq \mathcal{S}(X)$. The ideal properties 1 and 2 are trivial.

For property 3 let $t_1, t_2 \in dom(X)$ with $\pi_Y^X(t_1) = \pi_Y^X(t_2)$ and $\pi_Z^X(t_1) = \pi_Z^X(t_2)$ for reconsilable subattributes $Y, Z \in \mathcal{F}$.

- In case $Y \geq Z$ we immediately get $Y \sqcup Z = Y \in \mathcal{F}$.
- In case $X = X[X']$ we must have $Y = X[Y']$ and $Z = X[Z']$ with reconsilable subattributes $Y', Z' \in \mathcal{S}(X')$. Furthermore, $t_1 = [t_{1,1}, \dots, t_{1,n}]$ and $t_2 = [t_{2,1}, \dots, t_{2,m}]$. This gives $n = m$, $\pi_{Y'}^{X'}(t_{1,j}) = \pi_{Y'}^{X'}(t_{2,j})$ and $\pi_{Z'}^{X'}(t_{1,j}) = \pi_{Z'}^{X'}(t_{2,j})$ for all $j = 1, \dots, n$, hence $Y', Z' \in \mathcal{F}_j$ for all $j = 1, \dots, n$, where $\mathcal{F}_j = \{U \in \mathcal{S}(X') \mid \pi_U^{X'}(t_{1,j}) = \pi_U^{X'}(t_{2,j})\}$ is the set of subattributes, on which $t_{1,j}$ and $t_{2,j}$ coincide. By induction $\mathcal{F}_j$ is a coincidence ideal, so $Y' \sqcup Z' \in \mathcal{F}_j$ holds for all $j = 1, \dots, n$. This gives $\pi_{Y' \sqcup Z'}^{X'}(t_{1,j}) = \pi_{Y' \sqcup Z'}^{X'}(t_{2,j})$ for all $j = 1, \dots, n$ and hence also $\pi_{Y \sqcup Z}^X(t_1) = \pi_{Y \sqcup Z}^X(t_2)$, which implies $Y \sqcup Z \in \mathcal{F}$ as desired.
- In case $X = X(X_1, \dots, X_n)$ we must have $Y = X(Y_1, \dots, Y_n)$ and $Z = X(Z_1, \dots, Z_n)$ with reconsilable subattributes $Y_i, Z_i \in \mathcal{S}(X_i)$ for $i = 1, \dots, n$. Furthermore, $t_1 = (t_{1,1}, \dots, t_{1,n})$ and $t_2 = (t_{2,1}, \dots, t_{2,n})$, which implies $\pi_{Y_i}^{X_i}(t_{1,i}) = \pi_{Y_i}^{X_i}(t_{2,i})$ and $\pi_{Z_i}^{X_i}(t_{1,i}) = \pi_{Z_i}^{X_i}(t_{2,i})$ for all $i = 1, \dots, n$. This

gives $Y_i, Z_i \in \mathcal{F}_i = \{U \in \mathcal{S}(X_i) \mid \pi_U^{X_i}(t_{1,i}) = \pi_U^{X_i}(t_{2,i})\}$. By induction these sets $\mathcal{F}_i$ for $i = 1, \dots, n$ are coincidence ideals, hence $Y_i \sqcup Z_i \in \mathcal{F}_i$ holds. This implies $\pi_{Y_i \sqcup Z_i}^{X_i}(t_{1,i}) = \pi_{Y_i \sqcup Z_i}^{X_i}(t_{2,i})$ for all $i = 1, \dots, n$. With $Y \sqcup Z = X(Y_1 \sqcup Z_1, \dots, Y_n \sqcup Z_n)$ follows also $\pi_{Y \sqcup Z}^X(t_1) = \pi_{Y \sqcup Z}^X(t_2)$, which implies $Y \sqcup Z \in \mathcal{F}$ as desired.

– In case $X = X_1(X_1') \oplus \dots \oplus X_n(X_n')$ we must have $Y = X_1(Y_1) \oplus \dots \oplus X_n(Y_n)$ and $Z = X_1(Z_1) \oplus \dots \oplus X_n(Z_n)$ with reconsilable subattributes $Y_i, Z_i \in \mathcal{S}(X_i')$ for $i = 1, \dots, n$. Furthermore $t_1 = (X_i : t_1')$ and $t_2 = (X_i : t_2')$ for some $i \in \{1, \dots, n\}$, which implies $\pi_{Y_i}^{X_i'}(t_1') = \pi_{Y_i}^{X_i'}(t_2')$ and $\pi_{Z_i}^{X_i'}(t_1') = \pi_{Z_i}^{X_i'}(t_2')$. This gives $Y_i, Z_i \in \mathcal{F}_i = \{U \in \mathcal{S}(X_i') \mid \pi_U^{X_i'}(t_1') = \pi_U^{X_i'}(t_2')\}$. By induction $\mathcal{F}_i$ is a coincidence ideal, hence $Y_i \sqcup Z_i \in \mathcal{F}_i$ follows, which gives $\pi_{Y_i \sqcup Z_i}^{X_i'}(t_1') = \pi_{Y_i \sqcup Z_i}^{X_i'}(t_2')$ and further $\pi_{Y \sqcup Z}^X(t_1) = \pi_{Y \sqcup Z}^X(t_2)$. This implies $Y \sqcup Z \in \mathcal{F}$ as desired.

– In case $X = X[X_1(X_1') \oplus \dots \oplus X_n(X_n')]$ the case not covered by 2 and 3 is $Y = X(Y_1, \dots, Y_n)$ with $Y_i = X_i[Y_i']$ or $Y_i = \lambda = Y_i'$, and $Z = X[X_1(Z_1') \oplus \dots \oplus X_n(Z_n')]$, such that $Y_i'$, $Z_i'$ are reconsilable for all $i = 1, \dots, n$. We get $Y \sqcup Z = X[X_1(Y_1' \sqcup Z_1') \oplus \dots \oplus X_n(Y_n' \sqcup Z_n')]$. Now let $t_j = [t_{j1}, \dots, t_{jm}]$ for $j = 1, 2$ and $t_{jk} = (X_\ell : t_{jk}'')$ for some label $\ell$. The lists must have equal length, because the coincide on $Z$, hence also on $X[\lambda]$. The coincidence on $Z$ implies $\pi_{Z_\ell'}^{X_\ell'}(t_{1k}'') = \pi_{Z_\ell'}^{X_\ell'}(t_{2k}'')$ for all those elements with label $X_\ell$. As we also have $\pi_Y^X(t_j) = (\dots, [\dots, \pi_{Y_\ell'}^{X_\ell'}(t_{jk}''), \dots], \dots)$, we also get $\pi_{Y_\ell'}^{X_\ell'}(t_{1k}'') = \pi_{Y_\ell'}^{X_\ell'}(t_{2k}'')$ for $Y_\ell \neq \lambda$. Thus $Y_\ell'$ and $Z_\ell'$ are elements of the coincidence ideal defined by $t_{1k}''$ and $t_{2k}''$ for all those elements with label $X_\ell$. By induction $Y_\ell' \sqcup Z_\ell'$ must be in that coincidence ideal, too. This holds for all $\ell$, as in the remaining cases we defined $Y_\ell' = \lambda$. This implies $\pi_{Y_\ell' \sqcup Z_\ell'}^{X_\ell'}(t_{1k}'') = \pi_{Y_\ell' \sqcup Z_\ell'}^{X_\ell'}(t_{2k}'')$ for all $k$, such that $t_{1k}$ and $t_{2k}$ have the label $X_\ell$. Hence we get $\pi_{Y \sqcup Z}^X(t_1) = \pi_{Y \sqcup Z}^X(t_2)$ and therefore $Y \sqcup Z \in \mathcal{F}$ as desired.

For property 4(a) let $X = X\{X_1(X_1') \oplus \dots \oplus X_n(X_n')\} = X(X_1\{X_1'\}, \dots, X_n\{X_n'\})$, $Y = X_I\{\lambda\}$, $Z_1 = X_J\{\lambda\}$ and $Z_2 = X(X_{i_1}\{X_{i_1}'\}, \dots, X_{i_k}\{X_{i_k}'\})$. For $Y \in \mathcal{F}$ and $Z_1 \notin \mathcal{F}$ we have $\pi_Y^X(t_1) = \pi_Y^X(t_2)$ and $\pi_{Z_1}^X(t_1) \neq \pi_{Z_1}^X(t_2)$. Thus, one of $t_1$ or $t_2$ — without loss of generality let this be $t_2$ — must not contain elements of the form $(X_j : v_j)$ with $j \in J$. On the other hand, either $t_1$ and $t_2$ both contain elements of the form $(X_i : v_i)$ with $i \in I$ or both do not. As $I \subsetneq J$, it follows $\pi_{X(X_i\{\lambda\})}^X(t_1) = \pi_{X(X_i\{\lambda\})}^X(t_2) = \emptyset$ for all $i \in I$, which implies $\pi_{Z_2}^X(t_1) = \pi_{Z_2}^X(t_2)$, so $Z_2 \in \mathcal{F}$.

For property 4(b) let $X_I\{\lambda\} \in \mathcal{F}$, but $X_{\{i\}}\{\lambda\} \notin \mathcal{F}$ for all $i \in I$, that is $\pi_{X_I\{\lambda\}}^X(t_1) = \pi_{X_I\{\lambda\}}^X(t_2)$ and $\pi_{X(X_i\{\lambda\})}^X(t_1) \neq \pi_{X(X_i\{\lambda\})}^X(t_2)$ for all $i \in I$. Let $I_j \subseteq I$ be such that $t_j$ contains an element of the form $(X_i : v_i)$ for all

$i \in I_j$ $(j = 1, 2)$. Obviously, $I = I_1 \dot{\cup} I_2$ and $\pi^X_{X_{I'}\{\lambda\}}(t_1) = \pi^X_{X_{I'}\{\lambda\}}(t_2)$ for all $I' \subseteq I$ with $I' \cap I_1 \neq \emptyset \neq I' \cap I_2$, so $X_{I'}\{\lambda\} \in \mathcal{F}$ for these $I'$. Furthermore, $\pi^X_{X_{I_j}\{\lambda\}}(t_1) \neq \pi^X_{X_{I_j}\{\lambda\}}(t_2)$ for $j = 1, 2$, so $X_{I_1}\{\lambda\}, X_{I_2}\{\lambda\} \notin \mathcal{F}$.

For property 4(c) we have $\pi^X_{X_j\{\lambda\}}(t_1) = \pi^X_{X_j\{\lambda\}}(t_2)$ and $\pi^X_{X_k\{\lambda\}}(t_1) \neq \pi^X_{X_k\{\lambda\}}(t_2)$ for all $j \in I^+$ and $k \in I^-$. Assume that for all $i \in I^+$ there is some $J \subseteq I^-$ with $X_{J\cup\{i\}}\{\lambda\} \notin \mathcal{F}$, i.e. $\pi^X_{X_{J\cup\{i\}}\{\lambda\}}(t_1) \neq \pi^X_{X_{J\cup\{i\}}\{\lambda\}}(t_2)$. Hence one of these projections must be $\emptyset$. As we have $\pi^X_{X_i\{\lambda\}}(t_1) = \pi^X_{X_i\{\lambda\}}(t_2)$, these must both be $\emptyset$, which implies $\pi^X_{X_{I^+}\{\lambda\}}(t_j) = \emptyset$ for $j = 1, 2$. Now $\pi^X_{X_k\{\lambda\}}(t_1) \neq \pi^X_{X_k\{\lambda\}}(t_2)$ for all $k \in I^-$, so if $\pi^X_{X_{I^-}\{\lambda\}}(t_1) \neq \pi^X_{X_{I^-}\{\lambda\}}(t_2)$ holds, one of these projections must be $\emptyset$ again, which implies that one $t_j$ is $\emptyset$, the other not empty. That is $\pi^X_{X_{\{1,\dots,n\}}\{\lambda\}}(t_1) \neq \pi^X_{X_{\{1,\dots,n\}}\{\lambda\}}(t_2)$ contradicting $X_{\{1,\dots,n\}}\{\lambda\} \in \mathcal{F}$.

For property 4(d) assume $X_J\{\lambda\} \notin \mathcal{F}$, $X_{\{j\}}\{\lambda\} \notin \mathcal{F}$ for all $j \in J$, and for all $i \in I$ there is some $J_i \subseteq J$ with $X_{J_i\cup\{i\}}\{\lambda\} \notin \mathcal{F}$. It follows that one of the two complex values – without loss of generality let this be $t_1$ – contains values $(X_j : \tau_j)$ for all $j \in J$, while the other one does not contain such values. From this we derive $\pi^X_{X_{J'\cup\{i\}}\{\lambda\}}(t_1) \neq \emptyset$ for all $J' \subseteq J$ and all $i \in I$. If we also had $X_{I\cup J}\{\lambda\} \in \mathcal{F}$, $t_1, t_2$ would coincide on $X_{I\cup J}\{\lambda\}$, which gives $\pi^X_{X_{J'\cup\{i\}}\{\lambda\}}(t_2) \neq \emptyset$ for all $J' \subseteq J$ and at least one $i \in I$ contradicting the assumption that for at least one such $J' = J_i$ we have $\pi^X_{X_{J'\cup\{i\}}\{\lambda\}}(t_1) \neq \pi^X_{X_{J'\cup\{i\}}\{\lambda\}}(t_2)$.

For property 4(e) assume $X_{I^-}\{\lambda\} \in \mathcal{F}$ and that for each $\ell \in I'$ there is some $J_\ell \subseteq I^-$ with $X_{J_\ell\cup\{\ell\}}\{\lambda\} \notin \mathcal{F}$. Let $X_{J'}\{\lambda\} \notin \mathcal{F}$ for $J' \subseteq I^-$, and assume $X_{I'\cup J'}\{\lambda\} \in \mathcal{F}$. Define $I^-_j = \{i \in I^- \mid \pi^X_{X_{\{i\}}\{\lambda\}}(t_j) \neq \emptyset\}$ $(j = 1, 2)$ to define a partition $I^- = I^-_1 \dot{\cup} I^-_2$. As $t_1, t_2$ differ on $X_{J'}\{\lambda\}$, this implies $J' \subseteq I'_1$ or $J' \subseteq I'_2$. Without loss of generality we can assume the first of these possibilities. As $t_1, t_2$ coincide on $X_{I'\cup J'}\{\lambda\}$, we must have $\pi^X_{X_{I'}\{\lambda\}}(t_2) \neq \emptyset$, so also $\pi^X_{X_{\{i\}}\{\lambda\}}(t_2) \neq \emptyset$ for some $i \in I'$. Then also $\pi^X_{X_{\{i\}}\{\lambda\}}(t_1) \neq \emptyset$ due to $I' \subseteq I^+$. Hence we get $\pi^X_{X_{J\cup\{i\}}\{\lambda\}}(t_j) \neq \emptyset$ for $j = 1, 2$ and all $J \subseteq I^-$ contradicting the assumption that at least one such $J = J_i$ exists, such that $t_1, t_2$ differ on $X_{J_i\cup\{i\}}\{\lambda\}$. Hence $X_{I'\cup J'}\{\lambda\} \notin \mathcal{F}$ follows.

For property 5(a) assume $X_I\{\lambda\}, X_J\{\lambda\} \in \mathcal{F}$ with $I \cap J = \emptyset$, i.e. $\pi^X_{X_I\{\lambda\}}(t_1) = \pi^X_{X_I\{\lambda\}}(t_2)$ and $\pi^X_{X_J\{\lambda\}}(t_1) = \pi^X_{X_J\{\lambda\}}(t_2)$. In case $\pi^X_{X_I\{\lambda\}}(t_1) = \pi^X_{X_J\{\lambda\}}(t_1) = \emptyset$ there are no values of the form $(X_i : v_i)$ with $i \in I \cup J$ in $t_1$, hence also not in $t_2$. In case at least one of these projections leads to a non-empty set we must have $(X_i : v_i) \in t_1$ for at least one $i \in I \cup J$ and one value $v_i \in dom(X'_i)$. The same holds for $t_2$, hence in both cases $\pi^X_{X_{I\cup J}\{\lambda\}}(t_1) = \pi^X_{X_{I\cup J}\{\lambda\}}(t_2)$, i.e. $X_{I\cup J}\{\lambda\} \in \mathcal{F}$.

For property 5(b) let $X_I[\lambda] \in \mathcal{F}$, i.e. $\pi^X_{X_I[\lambda]}(t_1) = \pi^X_{X_I[\lambda]}(t_2)$, which means that $t_1$ and $t_2$ contain the same number of elements of the form $(X_i : v_i)$ with $i \in I$. If the same holds for $J$ with $I \cap J = \emptyset$, then $t_1$ and $t_2$ must also contain the same number of elements of the form $(X_i : v_i)$ with $i \in I \cup J$, i.e.

$\pi^X_{X_{I\cup J}[\lambda]}(t_1) = \pi^X_{X_{I\cup J}[\lambda]}(t_2)$ and hence $X_{I\cup J}[\lambda] \in \mathcal{F}$. Property 5(d) follows from the same argument.

For property 5(f) let $\pi^X_Y(t_1) = \pi^X_Y(t_2)$ for $Y \in \{X_I[\lambda], X_J[\lambda], X_{I\cap J}[\lambda]\}$, which means that $t_1$, $t_2$ contain the same number of elements with labels in $I$, $J$ and $I\cap J$, respectively. So they also contain the same number of elements with labels in $(I-J)\cup(J-I)$ and vice versa.

The proof of properties 5(c),(e) and (g) dealing with multisets is completely analogous to the proof for properties 5(b),(d) and (f) dealing with lists.

For property 6(a) we can assume $t_1 \neq \emptyset \neq t_2$. Otherwise, in case $t_1 = t_2 = \emptyset$ we simple choose $I_+ = I$, while in case exactly one of the $t_i$ is empty, we choose $I^- = I$, which both lead immediately to the desired result. For $t_1 \neq \emptyset \neq t_2$ define

$$\begin{aligned}
I_+ &= \{i \in I \mid \pi^X_{X\{\bar{X}_{\{i\}}\{\lambda\}\}}(t_1) = \{\{\top\}\} = \pi^X_{\bar{X}\{X_{\{i\}}\{\lambda\}\}}(t_2)\},\\
I_- &= \{i \in I \mid \pi^X_{X\{\bar{X}_{\{i\}}\{\lambda\}\}}(t_1) = \{\emptyset\} = \pi^X_{\bar{X}\{X_{\{i\}}\{\lambda\}\}}(t_2)\},\\
I^- &= \{i \in I \mid \pi^X_{X\{\bar{X}_{\{i\}}\{\lambda\}\}}(t_1) \neq \pi^X_{\bar{X}\{X_{\{i\}}\{\lambda\}\}}(t_2)\},
\end{aligned}$$

and $I_{+-} = I - I^- - I_+ - I_-$. Then $t_1, t_2$ obviously coincide on all $X\{\bar{X}_{I'}\{\lambda\}\}$ with $I'\cap I_+ \neq \emptyset$, which gives property ii. Property i holds by definition of $I^-$. For $I' \subseteq I_{+-} \cup I^- \cup I_-$ we get $\pi^X_{X\{\bar{X}_{I'}\{\lambda\}\}}(t_j) = \pi^X_{X\{\bar{X}_{I'\cap(I_{+-}\cup I^-)}\{\lambda\}\}}(t_j)$ for $j = 1, 2$, which gives property iii.

For property 6(b) we can assume $t_1 \neq \langle\rangle \neq t_2$. Otherwise, in case $t_1 = t_2 = \langle\rangle$ we simple choose $I_+ = I$, while in case exactly one of the $t_i$ is the empty multiset, we choose $I^- = I$, which both lead immediately to the desired result. For $t_1 \neq \langle\rangle \neq t_2$ define

$$\begin{aligned}
I_+ &= \{i \in I \mid \pi^X_{X\langle\bar{X}_{\{i\}}\{\lambda\}\rangle}(t_1) = \langle \underbrace{\{\top\}}_{x \text{ times}} \rangle = \pi^X_{X\langle\bar{X}_{\{i\}}\{\lambda\}\rangle}(t_2)\},\\
I_- &= \{i \in I \mid \pi^X_{X\langle\bar{X}_{\{i\}}\{\lambda\}\rangle}(t_1) = \langle \underbrace{\emptyset}_{x \text{ times}} \rangle = \pi^X_{X\langle\bar{X}_{\{i\}}\{\lambda\}\rangle}(t_2)\},\\
I^- &= \{i \in I \mid \pi^X_{X\langle\bar{X}_{\{i\}}\{\lambda\}\rangle}(t_1) \neq \pi^X_{X\langle\bar{X}_{\{i\}}\{\lambda\}\rangle}(t_2)\},
\end{aligned}$$

and $I_{+-} = I - I^- - I_+ - I_-$. Then $t_1, t_2$ obviously coincide on all $X\langle\bar{X}_{I'}\{\lambda\}\rangle$ with $I' \cap I^+ \neq \emptyset$, which gives property ii. Property i follows from the definition of $I^-$. For $I' \subseteq I_{+-} \cup I^- \cup I_-$ we get $\pi^X_{X\langle\bar{X}_{I'}\{\lambda\}\rangle}(t_j) = \pi^X_{X\langle\bar{X}_{I'\cap(I_{+-}\cup I^-)}\{\lambda\}\rangle}(t_j)$ for $j = 1, 2$, which gives property iii.

For property 7(a) let $t_j = (t_{j1}, \dots, t_{jn})$ for $j = 1, 2$. Then $X(\lambda, \dots, \lambda, Y_i, \lambda, \dots$ $\dots, \lambda) \in \mathcal{F}$ implies $\pi^{X'_i}_{Y_i}(t_{1i}) = \pi^{X'_i}_{Y_i}(t_{2i})$ and vice versa, so $\mathcal{F}_i$ is the ideal defined by coincidence of $t_{1i}$ and $t_{2i}$. Proceeding by induction on the nesting depth we conclude that $\mathcal{F}_i$ is a coincidence ideal.

Similarly, for property 7(b) let $t_j = [t_{j1}, \dots, t_{jk}]$ $(j = 1, 2)$. Both lists must have the same length, because we assume $\mathcal{F} \neq \{\lambda\}$. Then $X[Y] \in \mathcal{F}$ implies $\pi_Y^{X'}(t_{1i}) = \pi_Y^{X'}(t_{2i})$ for all $i = 1, \dots, k$ and vice versa. By induction on the nesting depth we conclude that $\mathcal{G}$ is the intersection of coincidence ideals, hence a coincidence ideal, as we assumed that $X'$ is not a union attribute.

For property 7(c) we may assume $t_j = (X_i : t'_j)$ $(j = 1, 2)$. Then $X_1(\lambda) \oplus \cdots \oplus X_i(Y_i) \oplus \cdots \oplus X_n(\lambda) \in \mathcal{F}$ implies $\pi_{Y_i}^{X'_i}(t'_1) = \pi_{Y_i}^{X'_i}(t'_2)$ and vice versa, so by induction on the nesting depth $\mathcal{F}_i$ is the coincidence ideal defined by $t'_1$ and $t'_2$.

For property 7(d) $t_1$ and $t_2$ are finite sets with elements in $dom(X')$ and we have $\mathcal{G} = \{Y \in \mathcal{S}(X') \mid \{\pi_Y^{X'}(\tau) \mid \tau \in t_1\} = \{\pi_Y^{X'}(\tau) \mid \tau \in t_2\}\}$. In this case we can repeat the arguments above to show properties 1, 2, 4(a)-(d) and 6(a),(b) for $\mathcal{G}$. By induction on the nesting depth we obtain 7(d),(e) and 8(a)-(c).

Analogously, for property 7(e) $t_1$ and $t_2$ are finite multisets with elements in $dom(X')$ and we have $\mathcal{G} = \{Y \in \mathcal{S}(X') \mid \langle \pi_Y^{X'}(\tau) \mid \tau \in t_1 \rangle = \langle \pi_Y^{X'}(\tau) \mid \tau \in t_2 \rangle\}$. In this case we can repeat the arguments above to show properties 1, 2, 4(a)-(d) and 6(a),(b) for $\mathcal{G}$. By induction on the nesting depth we obtain 7(d),(e) and 8(a)-(c).

In the proof we did indeed show a bit more than claimed in Theorem 10, as we also dealt with defect coincidence ideals. The additional results are formalised in the following corollary.

**Corollary 11.** *Let $X \in \mathcal{N}$ be a nested attribute, but not a union attribute.*

1. *For finite sets $S_1$ and $S_2$ with elements in $dom(X)$ let $\mathcal{G} = \{Y \in \mathcal{S}(X) \mid \{\pi_Y^X(\tau) \mid \tau \in S_1\} = \{\pi_Y^X(\tau) \mid \tau \in S_2\}\} \subseteq \mathcal{S}(X)$. Then $\mathcal{G}$ is a defect coincidence ideal.*
2. *For finite multisets $M_1$ and $M_2$ with elements in $dom(X)$ let $\mathcal{G} = \{Y \in \mathcal{S}(X) \mid \langle \pi_Y^X(\tau) \mid \tau \in M_1 \rangle = \langle \pi_Y^X(\tau) \mid \tau \in M_2 \rangle\} \subseteq \mathcal{S}(X)$. Then $\mathcal{G}$ is a defect coincidence ideal.*

### 3.2 Sufficiency of the Coincidence Ideal Characterisation

We now proceed with showing the converse of the result in Theorem 10. The general idea is to proceed by structural induction extending the corresponding proofs in [Hartmann et al., 2005] and in [Hartmann et al., 2006]. However, a difficulty arises with the set and multiset constructors, as for them we will have to deal with defect coincidence ideals, which will request a different treatment.

**Theorem 12.** *Let $\mathcal{G} \subseteq \mathcal{S}(X)$ be a defect coincidence ideal for the nested attribute $X \in \mathcal{N}$ such that the union constructor appears in $X$ only directly inside a set-, list or multiset-constructor. Then the following holds:*

1. *There exist two finite sets $S_1, S_2 \subseteq dom(X)$ such that $\{\pi_Y^X(\tau) \mid \tau \in S_1\} = \{\pi_Y^X(\tau) \mid \tau \in S_2\}$ holds iff $Y \in \mathcal{G}$. For $\mathcal{G} \neq \{\lambda\}$ both sets are non-empty.*

2. *There exist two finite multisets $M_1, M_2 \subseteq dom(X)$ such that $\langle \pi_Y^X(\tau) \mid \tau \in M_1\rangle = \langle \pi_Y^X(\tau) \mid \tau \in M_2\rangle$ holds iff $Y \in \mathcal{G}$. For $\mathcal{G} \neq \{\lambda\}$ both multisets are non-empty.*

The work in [Hartmann et al., 2006, Lemmata 21 and 24] contains a proof of this theorem for the case that the union constructor does not appear at all. This has been generalised in [Sali and Schewe, 2006, Lemma 4.3] to the general case but excluding counter attributes, i.e. attributes of the form $X_I\{\lambda\}$, $X_I\langle\lambda\rangle$ or $X_I[\lambda]$ with $|I| \geq 2$. We will refer to this proof as part of the proof in the general case, i.e. the proof of Theorem 12.

In the general case we have to take into account that the union constructor may appear directly inside a set-, list or multiset-constructor. Therefore, an attribute $X'$ occurring inside an attribute $X$ will be called an *embedded attribute*, and we write $emb(X)$ for the set of all embedded attributes of $X$. The nesting of embedded attributes gives rise to the notion of *degeneration depth* (see Definition 13 below), and the proof in [Hartmann et al., 2006] covers the basic case of degeneration depth 0. We therefore proceed using induction over the degeneration depth to prove Theorem 12.

**Definition 13.** Let $X \in \mathcal{N}$ be a nested attribute, such that the union constructor appears in $X$ only directly inside a set-, list or multiset-constructor. For an embedded attribute $X' = X'\{X_1(X_1') \oplus \cdots \oplus X_n(X_n')\}$ or $X' = X'\langle X_1(X_1') \oplus \cdots \oplus X_n(X_n')\rangle$ or $X' = X'[X_1(X_1') \oplus \cdots \oplus X_n(X_n')]$ in $emb(X)$ the *degeneration depth* $dd(X')$ of $X'$ is 1, if the union constructor does not appear in any $X_i'$ $(i = 1, \dots, n)$, and $\max\{dd(X_i') \mid i = 1, \dots, n\} + 1$ otherwise.

The *degeneration depth* $dd(X)$ of $X$ is the maximum of all $dd(X')$ for attributes $X'$ of the given form that appear in $X$.

For the basic case of Theorem 12 with $dd(X) = 0$ let $\mathcal{S}^r(X) \subseteq \mathcal{S}(X)$ denote the sublattice of $\mathcal{S}(X)$, in which all subattributes containing some $X_I\{\lambda\}$, $X_I[\lambda]$ or $X_I\langle\lambda\rangle$ with $|I| \geq 2$ are omitted. As remarked in [Sali and Schewe, 2006] this gives rise to a Brouwer algebra. We will establish the claimed result by a direct construction, for which we will use *distinguished values.*

**Definition 14.** Let $X$ be a nested attribute such that the union-constructor only appears in $X$ inside a list-constructor. For each $Y \in \mathcal{S}^r(X)$ we define the *distinguished value* $\tau_Y^X \in dom(X)$ as follows:

1. $\tau_\lambda^\lambda = \top$;

2. $\tau_A^A = a$ and $\tau_\lambda^A = a'$ for a simple attribute $A$ and $a, a' \in dom(A)$, $a \neq a'$;

3. $\tau^{X(X_1,\dots,X_n)}_{X(Y_1,\dots,Y_n)} = (X_1 : \tau^{X_1}_{Y_1}, \dots, X_n : \tau^{X_n}_{Y_n});$

4. $\tau^{X\{X'\}}_{X\{Y\}} = \{\tau^{X'}_{Y}\}$, if $X'$ is not a union attribute, and $\tau^{X\{X'\}}_{\lambda} = \emptyset$;

5. $\tau^{X\langle X'\rangle}_{X\langle Y\rangle} = \langle\tau^{X'}_{Y}\rangle$, if $X'$ is not a union attribute, and $\tau^{X\langle X'\rangle}_{\lambda} = \langle\rangle$;

6. $\tau^{X[X']}_{X[Y]} = [\tau^{X'}_{Y}]$, if $X'$ is not a union attribute, and $\tau^{X[X']}_{\lambda} = []$;

7. $\tau^{X[X_1(X'_1)\oplus\cdots\oplus X_n(X'_n)]}_{X(Y_1,\dots,Y_n)} = [(X_{i_1} : \tau^{X'_{i_1}}_{Y'_{i_1}}), \dots, (X_{i_k} : \tau^{X'_{i_k}}_{Y'_{i_k}})]$ with $1 \le i_1 < \cdots < i_k \le n$ such that $\{i_1, \dots, i_k\} = \{i \mid Y_i \neq \lambda\}$ – that is the list contains only those $(X_i : \tau^{X'_i}_{Y'_i})$, for which $Y_i \neq \lambda$, i.e. $Y_i = X_i[Y'_i]$;

8. $\tau^{X[X_1(X'_1)\oplus\cdots\oplus X_n(X'_n)]}_{X[X_1(Y'_1)\oplus\cdots\oplus X_n(Y'_n)]} = [(X_2 : \tau^{X'_2}_{Y'_2}), \dots, (X_n : \tau^{X'_n}_{Y'_n}), (X_1 : \tau^{X'_1}_{Y'_1})]$, where the list contains only those $(X_i : \tau^{X'_i}_{Y'_i})$, for which $Y'_i \neq \lambda$.

Using these distinguished values we first show some elementary properties for them, which are used in a second step to prove the base case of Theorem 12.

**Lemma 15.** *Let $X$ be a nested attribute such that the union-constructor appears in $X$ only immediately inside a list-constructor. Then we have:*

*1. We have $\pi^X_Y(\tau^X_Z) = \pi^X_Y(\tau^X_Y)$ iff $Z \ge Y$.*

*2. For $Y, Z \in \mathcal{S}^r(X)$ and $Z^\sharp = (Y \leftarrow Z) \leftarrow (Y \sqcap Z)$ we have $\pi^X_Y(\tau^X_Z) = \pi^X_Y(\tau^X_{Z^\sharp})$.*

*3. For all $Y \neq \lambda$ there is some $Z$ with $\pi^X_Y(\tau^X_Z) \neq \pi^X_Y(\tau^X_\lambda)$.*

*Proof (see Lemma 4.3 in [Sali and Schewe, 2006]).* For the only-if-part of the first statement there is nothing to show for $Y = \lambda$, $Z \ge Y$, $Y = X\{\lambda\}$, $Y = X\langle\lambda\rangle$ or $Y = X[\lambda]$. We then use structural induction on $X$:

For a simple attribute $X = A$ we have $Y = A$ and $Z = \lambda$, so $\pi^X_Y(\tau^X_Z) = a' \neq a = \pi^X_Y(\tau^X_Y)$. For $X = X(X_1, \dots, X_n)$, $Y = X(Y_1, \dots, Y_n)$ and $Z = X(Z_1, \dots, Z_n)$ we have by induction $Z_i \ge Y_i$ for all $i = 1, \dots, n$, thus $Z \ge Y$. For $X = X\{X'\}$, $Y = X\{Y'\}$ and $Z = X\{Z'\}$ we get $Z' \ge Y'$ by induction, hence also $Z \ge Y$. The same argument applies to multisets and lists. Finally, for $X = X[X_1(X'_1) \oplus \cdots \oplus X_n(X'_n)]$ we have to consider four cases for $Y$ and $Z$:

– Let $Y = X(X_1[Y_1], \dots, X_n[Y_n])$ and $Z = X(X_1[Z_1], \dots, X_n[Z_n])$. Then we have

$$\pi^X_Y(\tau^X_Y) = (X_1 : [\pi^{X'_1}_{Y_1}(\tau^{X'_1}_{Y_1})], \dots, X_n : [\pi^{X'_n}_{Y_n}(\tau^{X'_n}_{Y_n})])$$

and

$$\pi^X_Y(\tau^X_Z) = (X_1 : [\pi^{X'_1}_{Y_1}(\tau^{X'_1}_{Z_1})], \dots, X_n : [\pi^{X'_n}_{Y_n}(\tau^{X'_n}_{Z_n})]) .$$

By induction we must have $Z_i \ge Y_i$ for all $i = 1, \dots, n$, hence also $Z \ge Y$.

– Let $Y = X(X_1[Y_1], \dots, X_n[Y_n])$ and $Z = X[X_1(Z_1) \oplus \cdots \oplus X_n(Z_n)]$. Then $\pi_Y^X(\tau_Y^X)$ and $\pi_Y^X(\tau_Z^X)$ are the same as in the previous case, so by induction $Z_i \geq Y_i$ for all $i = 1, \dots, n$. This implies

$$Z \geq X[X_1(Y_1) \oplus \cdots \oplus X_n(Y_n)] \geq X(X_1[Y_1], \dots, X_n[Y_n]) = Y \ .$$

– Let $Y = X[X_1(Y_1) \oplus \cdots \oplus X_n(Y_n)]$ and $Z = X[X_1(Z_1) \oplus \cdots \oplus X_n(Z_n)]$. Then we have

$$\pi_Y^X(\tau_Y^X) = [(X_2 : \pi_{Y_2}^{X_2'}(\tau_{Y_2}^{X_2'})), \dots, (X_1 : \pi_{Y_1}^{X_1'}(\tau_{Y_1}^{X_1'}))]$$

and

$$\pi_Y^X(\tau_Z^X) = [(X_2 : \pi_{Y_2}^{X_2'}(\tau_{Z_2}^{X_2'})), \dots, (X_1 : \pi_{Y_1}^{X_1'}(\tau_{Z_1}^{X_1'}))] \ .$$

By induction we must have $Z_i \geq Y_i$ for all $i = 1, \dots, n$, hence also $Z \geq X[X_1(Y_1) \oplus \cdots \oplus X_n(Y_n)] = Y$.

– Let $Y = X[X_1(Y_1) \oplus \cdots \oplus X_n(Y_n)]$ and $Z = X(X_1[Z_1], \dots, X_n[Z_n])$. Then we have

$$\pi_Y^X(\tau_Y^X) = [(X_2 : \pi_{Y_2}^{X_2'}(\tau_{Y_2}^{X_2'})), \dots, (X_1 : \pi_{Y_1}^{X_1'}(\tau_{Y_1}^{X_1'}))] \neq \\ [(X_1 : \pi_{Y_1}^{X_1'}(\tau_{Z_1}^{X_1'})), \dots, (X_n : \pi_{Y_n}^{X_n'}(\tau_{Z_n}^{X_n'}))] \ .$$

For the if-part of the first statement it is sufficient to show $\pi_Y^X(\tau_X^X) = \pi_Y^X(\tau_Y^X)$ for all $Y \in \mathcal{S}^r(X)$. From this for $Z \geq Y$ we obtain immediately $\pi_Y^X(\tau_Z^X) = \pi_Y^Z(\pi_Z^X(\tau_Z^X)) = \pi_Y^Z(\pi_Z^X(\tau_X^X)) = \pi_Y^X(\tau_X^X) = \pi_Y^X(\tau_Y^X)$ as desired.

Apply again structural induction on $X$ ignoring the trivial cases $X = \lambda$, $X = A$ and $Y = \lambda$. For $X = X(X_1, \dots, X_n)$ and $Y = X(Y_1, \dots, Y_n)$ we have $\pi_{Y_i}^{X_i}(\tau_{X_i}^{X_i}) = \pi_{Y_i}^{X_i}(\tau_{Y_i}^{X_i})$ by induction for all $i = 1, \dots, n$, hence also

$$\pi_Y^X(\tau_X^X) = (\pi_{Y_1}^{X_1}(\tau_{X_1}^{X_1}), \dots, \pi_{Y_n}^{X_n}(\tau_{X_n}^{X_n})) = (\pi_{Y_1}^{X_1}(\tau_{Y_1}^{X_1}), \dots, \pi_{Y_n}^{X_n}(\tau_{Y_n}^{X_n})) = \pi_Y^X(\tau_Y^X),$$

which closes the record case. For $X = X\{X'\}$ and $Y = X\{Y'\}$ we get $\pi_Y^X(\tau_X^X) = \{\pi_{Y'}^{X'}(\tau_{X'}^{X'})\} = \{\pi_{Y'}^{X'}(\tau_{Y'}^{X'})\} = \pi_Y^X(\tau_Y^X)$, which closes the set case. The cases for lists and multisets are analogous.

Finally, let $X = X[X_1(X_1') \oplus \cdots \oplus X_n(X_n')]$ and $Y = X(Y_{i_1}, \dots, Y_{i_k})$ with $Y_{i_j} \neq \lambda$ for $j = 1, \dots, k$. Then we get

$$\pi_Y^X(\tau_X^X) = \pi_Y^X([X_2 : \tau_{X_2'}^{X_2'}, \dots, X_n : \tau_{X_n'}^{X_n'}, X_1 : \tau_{X_1'}^{X_1'}]) = \\ ([\pi_{Y_{i_1}'}^{X_{i_1}'}(\tau_{X_{i_1}'}^{X_{i_1}'})], \dots, [\pi_{Y_{i_k}'}^{X_{i_k}'}(\tau_{X_{i_k}'}^{X_{i_k}'})]) = ([\pi_{Y_{i_1}'}^{X_{i_1}'}(\tau_{Y_{i_1}'}^{X_{i_1}'})], \dots, [\pi_{Y_{i_k}'}^{X_{i_k}'}(\tau_{Y_{i_k}'}^{X_{i_k}'})]) = \\ \pi_Y^X([X_2 : \tau_{Y_2'}^{X_2'}, \dots, X_n : \tau_{Y_n'}^{X_n'}, X_1 : \tau_{Y_1'}^{X_1'}]) = \pi_Y^X(\tau_Y^X).$$

Similarly, for $Y = X[X_{i_1}(Y'_{i_1}) \oplus \dots \oplus X_{i_k}(Y'_{i_k})]$ we obtain

$$\pi_Y^X(\tau_X^X) = \pi_Y^X([X_2 : \tau_{X'_2}^{X'_2}, \dots, X_n : \tau_{X'_n}^{X'_n}, X_1 : \tau_{X'_1}^{X'_1}]) =$$
$$[X_2 : \pi_{Y'_2}^{X'_2}(\tau_{X'_2}^{X'_2}), \dots, X_1 : \pi_{Y'_1}^{X'_1}(\tau_{X'_1}^{X'_1})](\text{omit indices different from } i_1, \dots, i_k) =$$
$$[X_2 : \pi_{Y'_2}^{X'_2}(\tau_{Y'_2}^{X'_2}), \dots, X_1 : \pi_{Y'_1}^{X'_1}(\tau_{Y'_1}^{X'_1})](\text{omit indices different from } i_1, \dots, i_k) =$$
$$\pi_Y^X([X_2 : \tau_{Y'_2}^{X'_2}, \dots, X_n : \tau_{Y'_n}^{X'_n}, X_1 : \tau_{Y'_1}^{X'_1}]) = \pi_Y^X(\tau_Y^X).$$

For the second statement there is nothing to prove for $Y = \lambda$ or $Y \geq Z$, which gives $Z^\sharp = (Y \leftarrow Z) \leftarrow (Y \sqcap Z) = \lambda \leftarrow Z = Z$. Now proceed by induction on $X$ and assume $\lambda \neq Y \ngeq Z$. Note that the cases $X = \lambda$ and $X$ a simple attribute are already covered.

For $X = X(X_1, \dots, X_n)$, $Y = X(Y_1, \dots, Y_n)$ and $Z = X(Z_1, \dots, Z_n)$ we have by induction $\pi_{Y_i}^{X_i}(\tau_{Z_i}^{X_i}) = \pi_{Y_i}^{X_i}(\tau_{Z_i^\sharp}^{X_i})$ for all $i = 1, \dots, n$ with $Z_i^\sharp = (Y_i \leftarrow Z_i) \leftarrow (Y_i \sqcap Z_i)$. This implies

$$\begin{aligned} \pi_Y^X(\tau_Z^X) &= (X_1 : \pi_{Y_1}^{X_1}(\tau_{Z_1}^{X_1}), \dots, X_n : \pi_{Y_n}^{X_n}(\tau_{Z_n}^{X_n})) \\ &= (X_1 : \pi_{Y_1}^{X_1}(\tau_{Z_1^\sharp}^{X_1}), \dots, X_n : \pi_{Y_n}^{X_n}(\tau_{Z_n^\sharp}^{X_n})) = \pi_Y^X(\tau_{Z^\sharp}^X) \ . \end{aligned}$$

For $X = X\{X'\}$ with $X'$ not being a union attribute, $Y = X\{Y'\}$ and $Z = X\{Z'\}$ with $Y' \ngeq Z'$ we get by induction $\pi_{Y'}^{X'}(\tau_{Z'}^{X'}) = \pi_{Y'}^{X'}(\tau_{Z'^\sharp}^{X'})$ with ${Z'}^\sharp = (Y' \leftarrow Z') \leftarrow (Y' \sqcap Z')$. This implies

$$\pi_Y^X(\tau_Z^X) = \{\pi_{Y'}^{X'}(\tau_{Z'}^{X'})\} = \{\pi_{Y'}^{X'}(\tau_{Z'^\sharp}^{X'})\} \ .$$

The same argument applies for $X = X\langle X' \rangle$ or $X = X[X']$ with $X'$ not being a union attribute.

Finally, let $X = X[X_1(X'_1) \oplus \dots \oplus X_n(X'_n)]$ and $\lambda \neq Y \ngeq Z$. Then we have to consider three different cases for $Y$ and $Z$:

- Let $Y = X[Y_1 \oplus \dots \oplus Y_n]$ with $Y_i = X_i(Y'_i)$, and $Z = X(Z_1, \dots, Z_n)$ with $Z_i = X_i[Z'_i]$ or $Z_i = \lambda = Z'_i$. Then $Z^\sharp = X(Z_1^\sharp, \dots, Z_n^\sharp)$ with $Z_i^\sharp = X_i[{Z'_i}^\sharp]$, ${Z'_i}^\sharp = (Y'_i \leftarrow Z'_i) \leftarrow (Y'_i \sqcap Z'_i)$ for $Z_i \neq \lambda$, and $Z_i^\sharp = \lambda$ for $Z_i = \lambda$.

  We have $\pi_Y^X(\tau_Z^X) = [\dots, (X_i : \pi_{Y'_i}^{X'_i}(\tau_{Z'_i}^{X'_i})), \dots]$ with only such $i$ in the list, for which $Y_i \neq \lambda \neq Z_i$ holds. By induction $\pi_{Y'_i}^{X'_i}(\tau_{Z'_i}^{X'_i}) = \pi_{Y'_i}^{X'_i}(\tau_{Z'^\sharp_i}^{X'_i})$, which implies the equality

  $$\pi_Y^X(\tau_Z^X) = [\dots, (X_i : \pi_{Y'_i}^{X'_i}(\tau_{Z'^\sharp_i}^{X'_i})), \dots] = \pi_Y^X(\tau_{Z^\sharp}^X) \ .$$

- Let $Y = X[Y_1 \oplus \dots \oplus Y_n]$ with $Y_i = X_i(Y'_i)$, and $Z = X[Z_1 \oplus \dots \oplus Z_n]$ with $Z_i = X_i(Z'_i)$. In this case $Z^\sharp = X[X_1(W_1) \oplus \dots \oplus X_n(W_n)]$ with $W_i =$

$\begin{cases} Z_i'^\sharp & \text{for } Z_i \neq \lambda \neq Y_i \\ \lambda & \text{else} \end{cases}$ and $Z_i'^\sharp = (Y_i' \leftarrow Z_i') \leftarrow (Y_i' \sqcap Z_i')$. Then $\pi_Y^X(\tau_Z^X) = [(X_2 : \pi_{Y_2'}^{X_2'}(\tau_{Z_2'}^{X_2'})), \dots, (X_1 : \pi_{Y_1'}^{X_1'}(\tau_{Z_1'}^{X_1'}))]$ with only such $i$ in the list, for which $Y_i \neq \lambda \neq Z_i$ holds. By induction, this is equal to $[(X_2 : \pi_{Y_2'}^{X_2'}(\tau_{Z_2'^\sharp}^{X_2'})), \dots, (X_1 : \pi_{Y_1'}^{X_1'}(\tau_{Z_1'^\sharp}^{X_1'}))] = \pi_Y^X(\tau_{Z^\sharp}^X)$.

– Let $Y = X(Y_1, \dots, Y_n)$ with $Y_i = X_i[Y_i']$ or $Y_i = \lambda = Y_i'$, and $Z = X[Z_1 \oplus \dots \oplus Z_n]$ with $Z_i = X_i(Z_i')$. Then $Z^\sharp = X(W_1, \dots, W_n)$ with $W_i = \begin{cases} X_i[Z_i'^\sharp] & \text{for } Y_i \neq \lambda \\ \lambda & \text{else} \end{cases}$, and $Z_i'^\sharp = (Y_i' \leftarrow Z_i') \leftarrow (Y_i' \sqcap Z_i')$.

  This gives $\pi_Y^X(\tau_Z^X) = (\dots, (X_i : L_i), \dots)$ with

$$L_i = \begin{cases} [\pi_{Y_i'}^{X_i'}(\tau_{Z_i'}^{X_i'})] & \text{for } Y_i \neq \lambda \neq Z_i \\ [] & \text{else} \end{cases} .$$

  By induction we have $\pi_{Y_i'}^{X_i'}(\tau_{Z_i'}^{X_i'}) = \pi_{Y_i'}^{X_i'}(\tau_{Z_i'^\sharp}^{X_i'})$, which implies $\pi_Y^X(\tau_Z^X) = \pi_Y^X(\tau_{Z^\sharp}^X)$.

For the third statement we proceed again by induction on $X \neq \lambda$. For a simple attribute $X = A$ we must have $Y = A$. Take $Z = A$, so we get $\pi_Y^X(\tau_Z^X) = a \neq a' = \pi_Y^X(\tau_\lambda^X)$.

For $X = X(X_1, \dots, X_n)$ and $Y = X(Y_1, \dots, Y_n)$ there must be some $Y_i \neq \lambda$. By induction we find some $Z_i$ with $\pi_{Y_i}^{X_i}(\tau_{Z_i}^{X_i}) \neq \pi_{Y_i}^{X_i}(\tau_\lambda^{X_i})$. For $Z = X(Z_1, \dots, Z_n)$ with $Z_j = Y_j$ for all $j \neq i$ it follows $\pi_Y^X(\tau_Z^X) \neq \pi_Y^X(\tau_\lambda^X)$.

For $X = X\{X'\}$ and $Y = X\{Y'\}$ with $Y' \neq \lambda$ we take $Z = X\{Z'\}$, where $Z'$ satisfies $\pi_{Y'}^{X'}(\tau_{Z'}^{X'}) \neq \pi_{Y'}^{X'}(\tau_\lambda^{X'})$ by induction. Then we get $\pi_Y^X(\tau_Z^X) = \{\pi_{Y'}^{X'}(\tau_{Z'}^{X'})\} \neq \{\pi_{Y'}^{X'}(\tau_\lambda^{X'})\} = \pi_Y^X(\tau_\lambda^X)$. The argument for multisets and lists in the last case is completely analogous.

With this lemma we can complete the proof of Theorem 12.

*Proof of Theorem 12.* Let us first assume $dd(X) = 0$, for which the proof was given in [Sali and Schewe, 2006].

Then for the first statement define $t_1 = \{\tau_Y^X \mid Y \in \mathcal{S}^r(X)\}$ and $t_2 = \{\tau_Y^X \mid Y \in \mathcal{G}\}$ and apply Lemma 15. Statement 3 in that lemma gives the result for the trivial case $\mathcal{G} = \{\lambda\}$. For $\mathcal{G} \neq \{\lambda\}$ statement 2 in Lemma 15 implies the equality for all $Y \in \mathcal{G}$, as for any $Z \in \mathcal{S}^r(X)$ we obtain $Y \geq Z^\sharp$ and thus $Z^\sharp \in \mathcal{G}$. Statement 1 in Lemma 15 is used for the inequality for $Y \notin \mathcal{G}$, for if we had equality, there would exist some $Z \in \mathcal{G}$ with $\pi_Y^X(\tau_Z^X) = \pi_Y^X(\tau_Y^X)$, hence $Z \geq Y$, which gives the contradiction $Y \in \mathcal{G}$.

For the second statement the construction is a bit more tricky (see [Hartmann et al., 2006]). Take the complement of $\mathcal{G}$, i.e. the filter $\mathcal{H} = \mathcal{S}^r(X) - \mathcal{G}$. For each minimal element $Y \in \mathcal{H}$ there is a maximal Boolean algebra $\mathbb{B}(Y) \subseteq \mathcal{S}^r(X)$ with maximal element $Y$. More precisely, let $Y'_1, \dots, Y'_x$ be the maximal proper subattributes of $Y$. Then the Boolean algebra $\mathbb{B}(Y)$ has the top element $Y$, bottom element $Y'_1 \sqcap \dots \sqcap Y'_x$, and contains all $Y'_j$ $(j = 1, \dots, x)$ (see [Hartmann et al., 2006, Lemma 22]). Take subsets $\mathbb{B}(Y)_1, \mathbb{B}(Y)_2$ consisting of all $Z_1, Z_2 \in \mathbb{B}(Y)$ with an odd or even distance (in the lattice) from $Y$, respectively, and define $t_{Y,i} = \langle \tau^X_{Z_i} \mid Z_i \in \mathbb{B}(Y)_i \rangle$. This exploits the fact that in a finite Boolean lattice each element $x$ has a unique distance from the top element 1. For this take a maximal chain $x = x_0 < x_1 < \dots < x_n = 1$ and define $n$ to be the distance between $x$ and 1. Finally, build the multiset union $t_i = \biguplus_{Y \in \mathcal{H} \text{minimal}} t_{Y,i}$. Using again statement 2 of Lemma 15 gives the equality for all $Y \in \mathcal{G}$. Analogously, statement 1 of Lemma 15 is used for the inequality for $Y \notin \mathcal{G}$, and statement 3 of Lemma 15 covers the case of $\mathcal{G} = \{\lambda\}$.

Let us now assume that Theorem 12 holds for nested attributes $X'$ with $dd(X') \leq i$. Let $dd(X) = i + 1$.

For the set case we could write $S_1 = \{\tau^X_Y \mid Y \in \mathcal{S}(X), Y \neq \lambda\}$ and $S_2 = \{\tau^X_Y \mid Y \in \mathcal{G}, Y \neq \lambda\}$ in the base case. In general, we will construct similar sets with the following differences:

1. Instead of $X$ we consider a subattribute $\tilde{X} \in \mathcal{S}(X)$ with $dd(\tilde{X}) = 0$.
2. Instead of $\mathcal{G}$ we consider a defect coincidence ideal $\tilde{\mathcal{G}}$ on $\mathcal{S}(\tilde{X})$.
3. Instead of having just one distinguished value $\tau^{\tilde{X}}_Y$ for $Y \in \mathcal{S}(\tilde{X})$ we consider several such values ${}^j\sigma^{\tilde{X}}_Y$ $(j = 1, \dots, o)$ and ${}^j\tau^{\tilde{X}}_Y$ $(j = 1, \dots, p)$ such that the sets become $S_1 = \{{}^j\sigma^{\tilde{X}}_Y \mid Y \in \mathcal{S}(\tilde{X}), j \in \{1, \dots, o\}, Y \neq \lambda\}$ and $S_2 = \{{}^j\tau^{\tilde{X}}_Y \mid Y \in \mathcal{S}(\tilde{X}), j \in \{1, \dots, p\}, Y \neq \lambda\}$. Of course, for $dd(X) = 0$ we had $o = p = 1$, $\tilde{X} = X$, $\tilde{\mathcal{G}} = \mathcal{G}$, and ${}^1\sigma^X_Y = {}^1\tau^X_Y$.
4. The modified distinguished values ${}^j\sigma^{\tilde{X}}_Y$ and ${}^j\tau^{\tilde{X}}_Y$ depend on the defect coincidence ideal $\tilde{\mathcal{G}}$, hence on $\mathcal{G}$.

So in particular, for $X' \in emb(X)$ with $dd(X') \leq i$ we assume that sets $S_1$, $S_2$ have the form described above.

First let $\bar{X} = \bar{X}\{X_1(X'_1) \oplus \dots \oplus X_n(X'_n)\} \in emb(X)$ be such that $dd(\bar{X}) = i+1$, i.e. $\bar{X}$ indicates an outermost occurrence of an embedded set attribute with a component union attribute. Using properties 7(d)-(e) and 8(a)-(c) of Theorem 10 $\mathcal{G}$ induces a defect coincidence ideal $\bar{\mathcal{G}}$ on $\mathcal{S}(\bar{X})$. Let $I^+ = \{i \in \{1, \dots, n\} \mid \bar{X}_{\{i\}}\{\lambda\} \in \bar{\mathcal{G}}\}$ and $I^- = \{i \in \{1, \dots, n\} \mid \bar{X}_{\{i\}}\{\lambda\} \notin \bar{\mathcal{G}}\}$. We now distinguish three subcases:

1. $\bar{X}_{\{1,\dots,n\}}\{\lambda\} \in \bar{\mathcal{G}}$ and $\bar{X}_{I^-}\{\lambda\} \notin \bar{\mathcal{G}}$;

2. $\bar{X}_{\{1,\dots,n\}}\{\lambda\} \in \bar{\mathcal{G}}$ and $\bar{X}_{I^-}\{\lambda\} \in \bar{\mathcal{G}}$;

3. $\bar{X}_{\{1,\dots,n\}}\{\lambda\} \notin \bar{\mathcal{G}}$.

In subcase 1 we obtain a partition $I^+ = I_+ \cup I_- \cup I_{+-}$ with $\bar{X}_{I'}\{\lambda\} \in \bar{\mathcal{G}}$ for all $I'$ with $I' \cap I_+ \neq \emptyset$ and $\bar{X}_{J_- \cup J}\{\lambda\} \in \bar{\mathcal{G}}$ iff $\bar{X}_J\{\lambda\} \in \bar{\mathcal{G}}$ for all $J \subseteq I^- \cup I_{+-}$ and all $J_- \subseteq I_-$. Taking $I_+$ and $I_-$ maximal with these properties gives rise to the following properties for the counter-attributes in $\mathcal{S}(\bar{X})$:

– $\bar{X}_{I'}\{\lambda\} \in \bar{\mathcal{G}}$, whenever $I' \cap I_+ \neq \emptyset$.

– $\bar{X}_{I'}\{\lambda\} \in \bar{\mathcal{G}}$, whenever $I' \subseteq I_-$.

– $\bar{X}_{I'}\{\lambda\} \notin \bar{\mathcal{G}}$, whenever $I' \subseteq I^-$ due to $\bar{X}_{I^-}\{\lambda\} \notin \bar{\mathcal{G}}$, the definition of $I^-$ and property 4(a) of Theorem 10.

– $\bar{X}_{I'}\{\lambda\} \in \bar{\mathcal{G}}$ iff $\bar{X}_{I' \cap (I_{+-} \cup I^-)}\{\lambda\} \in \bar{\mathcal{G}}$ for all $I' \subseteq I^- \cup I_{+-} \cup I_-$ due to property 6(a)iii of Theorem 10.

– $\bar{X}_{I' \cup J'}\{\lambda\} \notin \bar{\mathcal{G}}$, whenever $I' \subseteq I_{+-} \cup I_-$ and $\emptyset \neq J' \subseteq I^-$ hold. Otherwise, if for $i \in I_{+-}$ we had $\bar{X}_{\{i\} \cup J'}\{\lambda\} \in \bar{\mathcal{G}}$ for all $J' \subseteq I^-$, then also $\bar{X}_{\{i\} \cup J'}\{\lambda\} \in \bar{\mathcal{G}}$ for all $J' \subseteq I^- \cup I_-$ due to property 6(a)iii of Theorem 10. Then due to property 6(a)ii of Theorem 10 we get $\bar{X}_{I'}\{\lambda\} \in \bar{\mathcal{G}}$ for all $I'$ with $i \in I'$, which means we could add $i$ to $I_+$ contradicting the maximality of $I_+$. Therefore, for each $i \in I_{+-}$ there exists some $J_i \subseteq I^-$ with $\bar{X}_{\{i\} \cup J_i}\{\lambda\} \notin \bar{\mathcal{G}}$. Then property 4(d) of Theorem 10 implies $\bar{X}_{I' \cup J'}\{\lambda\} \notin \bar{\mathcal{G}}$ for all $I' \subseteq I_{+-}$ and $\emptyset \neq J' \subseteq I^-$, so finally the claimed property follows from property 6(a)iii of Theorem 10.

These properties of counter-attributes are illustrated in Figure 4. Furthermore, due to property 4(a) of Theorem 10 we get whenever $\bar{X}_J\{\lambda\} \in \bar{\mathcal{G}}$ for $J \subseteq I_{+-}$, then also $\bar{X}_{J'}\{\lambda\} \in \bar{\mathcal{G}}$ for all $J' \subseteq J$.

Take $J_1, \dots, J_\ell \subseteq I_{+-}$ maximal with $\bar{X}_{J_i}\{\lambda\} \in \bar{\mathcal{G}}$. Then also $\bar{X}(X_{j_1}\{X'_{j_1}\}, \dots, X_{j_x}\{X'_{j_x}\}) \in \bar{\mathcal{G}}$ for $J_i = \{j_1, \dots, j_x\}$. Then for $i = 1, \dots, \ell$ define

$$\varrho_i^+ = \{(X_j : v_j) \mid j \in I^-\} \cup \{(X_j : v_j) \mid j \in I_{+-} - J_i\}$$

and $\varrho_i^- = \{(X_j : v_j) \mid j \in I_{+-} - J_i\}$ using arbitrary fixed values $v_j \in dom(X'_j)$ for $j \in I_{+-} \cup I^-$.

Now consider $\bar{X}^+ = \bar{X}(X_{i_1}\{X'_{i_1}\}, \dots, X_{i_k}\{X'_{i_k}\})$ for $I_+ = \{i_1, \dots, i_k\}$. Ignoring that this is equivalent to $\bar{X}\{X_{i_1}(X'_{i_1}) \oplus \dots \oplus X_{i_k}(X'_{i_k})\}$, i.e. ignoring counter-attributes, we have $dd(\bar{X}^+) \leq i$. Furthermore, $\bar{\mathcal{G}}^+ = \{Y \in \bar{\mathcal{G}} \mid \bar{X}^+ \geq Y, Y \text{ not a counter-attribute}\}$ is a defect coincidence ideal on $\mathcal{S}(\bar{X}^+)$.

By induction we find $S_1^+ = \{{}^j\sigma_Y^{\tilde{\bar{X}}} \mid Y \in \mathcal{S}(\tilde{\bar{X}}) - \{\lambda\}, j \in \{1, \dots, o\}\} \subseteq dom(\bar{X}^+)$ and $S_2^+ = \{{}^j\tau_Y^{\tilde{\bar{X}}} \mid Y \in \tilde{\bar{\mathcal{G}}} - \{\lambda\}, j \in \{1, \dots, p\}\}$ with $\{\pi_Y^{\bar{X}^+}(\tau) \mid \tau \in$

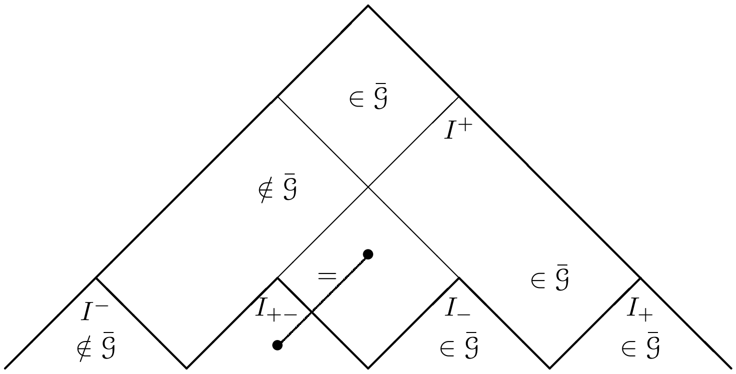


Figure 4: Counter Attributes for $\bar{X} = \bar{X}\{X_1(X_1') \oplus \cdots \oplus X_n(X_n')\}$ with $dd(\bar{X}) = i+1$ in case 1

$S_1^+\} = \{\pi_Y^{\bar{X}^+}(\tau) \mid \tau \in S_2^+\}$ iff $Y \in \bar{\mathcal{G}}^+$. In particular, as ${}^j\tau_{X_{i_j}\{\lambda\}}^{X_{i_j}\{X'_{i_j}\}} = \{{}^j\tau_\lambda^{X'_{i_j}}\} \neq \emptyset$, ${}^j\tau_\lambda^{X_{i_j}\{X'_{i_j}\}} = \emptyset$ and $I_+ \neq \emptyset$, we also get $\{\pi_{\bar{X}_{I'}\{\lambda\}}^{\bar{X}^+}(\tau) \mid \tau \in S_1^+\} = \{\{\top\}\} = \{\pi_{\bar{X}_{I'}\{\lambda\}}^{\bar{X}^+}(\tau) \mid \tau \in S_2^+\}$ for all $I' \subseteq I_+$. Now define

$$
\begin{aligned}
{}^{(j-1)\cdot\ell+i}\bar{\sigma}_Y^{\tilde{\bar{X}}} &= {}^j\sigma_Y^{\tilde{\bar{X}}} \cup \varrho_i^+ \text{ (for } j = 1,\dots,o; i = 1,\dots,\ell)\\
{}^{(j-1)\cdot(\ell+1)+i+1}\bar{\tau}_Y^{\tilde{\bar{X}}} &= {}^j\tau_Y^{\tilde{\bar{X}}} \cup \varrho_i^- \text{ (for } j = 1,\dots,p; i = 1,\dots,\ell)
\end{aligned}
$$

and

$$
{}^{(j-1)\cdot(\ell+1)+1}\bar{\tau}_Y^{\tilde{\bar{X}}} = {}^j\tau_Y^{\tilde{\bar{X}}} \text{ (for } j = 1,\dots,p).
$$

Let $\tilde{X} \in \mathcal{S}(X)$ result from $X$ by replacing $\bar{X}$ by $\tilde{\bar{X}}$. Let $\tilde{\mathcal{G}} \subseteq \mathcal{G}$ be the defect coincidence ideal on $\mathcal{S}(\tilde{(X)})$ that induces $\tilde{\bar{\mathcal{G}}}$ on $\mathcal{S}(\tilde{\bar{X}})$. Without loss of generality we may assume that there is no other embedded attribute $X' \in emb(X)$ with degeneration depth $i+1$, and all embedded attributes $X'' \in emb(X)$ with $dd(X'') \neq 0$ are embedded attributes of $\bar{X}$ – otherwise we have to simultaneously replace $X'$ by $\tilde{X}'$, and use similarly constructed distinguished values ${}^j\bar{\sigma}_Y^{\tilde{X}'}$ and ${}^j\bar{\tau}_Y^{\tilde{X}'}$ as defined above or by one of the remaining cases. Define ${}^j\sigma_Y^{\tilde{X}}$ and ${}^j\tau_Y^{\tilde{X}}$ for $Y \in \mathcal{S}^r(\tilde{X})$ using properties 3-6 of Definition 14 and the values ${}^g\bar{\sigma}_{\bar{Y}}^{\tilde{\bar{X}}}$, ${}^g\bar{\tau}_{\bar{Y}}^{\tilde{\bar{X}}}$ constructed above, then take $S_1 = \{{}^j\sigma_Y^{\tilde{\bar{X}}} \mid Y \in \mathcal{S}(\tilde{X}), Y \neq \lambda, j \in \{1,\dots,o\cdot\ell\}\}$ and $S_2 = \{{}^j\tau_Y^{\tilde{\bar{X}}} \mid Y \in \tilde{\mathcal{G}}, Y \neq \lambda, j \in \{1,\dots,p\cdot(\ell+1)\}\}$.

For $Z \in \mathcal{S}(X)$ the projected values $\pi_Z^X({}^j\sigma_Y^{\tilde{\bar{X}}})$ and $\pi_Z^X({}^{j'}\tau_Y^{\tilde{\bar{X}}})$ involve $\pi_{\bar{Z}}^{\bar{X}}({}^j\bar{\sigma}_{\bar{Y}}^{\tilde{\bar{X}}})$ or $\pi_{\bar{Z}}^{\bar{X}}({}^{j'}\bar{\tau}_{\bar{Y}}^{\tilde{\bar{X}}})$, respectively, with $Z \in \mathcal{G}$ iff $\bar{Z} \in \bar{\mathcal{G}}$.

1. Let $\bar{Z} = \bar{X}(Y_1, \dots, Y_n)$. Then $\bar{Z} \in \bar{\mathcal{G}}$ iff $Y_i = \lambda$ for all $i \in I^-$, $\{j \in I_{+-} \mid Y_j \neq \lambda\} \subseteq J_x$ for some $x \in \{1, \dots, \ell\}$ and $\bar{X}(Y_{i_1}, \dots, Y_{i_k}) \in \bar{\mathcal{G}}^+$. Then for $Y_j = X_j\{Y_j'\}$ we get

$$\pi^{\bar{X}}_{\bar{Z}}({}^{(j-1)\ell+i}\bar{\sigma}^{\bar{\tilde{X}}}_{\bar{Y}}) =$$
$$\pi^{\bar{X}^+}_{\bar{X}(Y_{i_1},\dots,Y_{i_k})}({}^{j}\sigma^{\bar{\tilde{X}}}_{\bar{Y}}) \cup \{(X_j : \pi^{X_j'}_{Y_j'}(v_j)) \mid j \in (I^- \cup I_{+-}) - J_i, Y_j \neq \lambda\}$$

and

$$\pi^{\bar{X}}_{\bar{Z}}({}^{(j-1)(\ell+1)+i+1}\bar{\tau}^{\bar{\tilde{X}}}_{\bar{Y}}) =$$
$$\pi^{\bar{X}^+}_{\bar{X}(Y_{i_1},\dots,Y_{i_k})}({}^{j}\tau^{\bar{\tilde{X}}}_{\bar{Y}}) \cup \{(X_j : \pi^{X_j'}_{Y_j'}(v_j)) \mid j \in I_{+-} - J_i, Y_j \neq \lambda\}.$$

For $\bar{Z} \in \bar{\mathcal{G}}$ these values are equal. Furthermore,

$$\pi^{\bar{X}}_{\bar{Z}}({}^{(j-1)(\ell+1)+1}\bar{\tau}^{\bar{\tilde{X}}}_{\bar{Y}}) = \pi^{\bar{X}^+}_{\bar{X}(Y_{i_1},\dots,Y_{i_k})}({}^{j}\tau^{\bar{\tilde{X}}}_{\bar{Y}}) = \pi^{\bar{X}}_{\bar{Z}}({}^{(j'-1)\ell+i}\bar{\sigma}^{\bar{\tilde{X}}}_{\bar{Y}})$$

iff $\{j \in I_{+-} \mid Y_j \neq \lambda\} \subseteq J_i$ and $\pi^{\bar{X}^+}_{\bar{X}(Y_{i_1},\dots,Y_{i_k})}({}^{j'}\sigma^{\bar{\tilde{X}}}_{\bar{Y}}) = \pi^{\bar{X}^+}_{\bar{X}(Y_{i_1},\dots,Y_{i_k})}({}^{j}\tau^{\bar{\tilde{X}}}_{\bar{Y}})$.

2. Let $\bar{Z} = \bar{X}_I\{\lambda\}$. If $I \cap I_+ \neq \emptyset$, then we have already seen that $\{\pi^{\bar{X}}_{\bar{Z}}({}^{j}\sigma^{\bar{\tilde{X}}}_{\bar{Y}})\} = \{\{\top\}\} = \{\pi^{\bar{X}}_{\bar{Z}}({}^{j'}\tau^{\bar{\tilde{X}}}_{\bar{Y}})\}$, and in this case $\bar{Z} \in \bar{\mathcal{G}}$ holds.
If $I \cap I_+ = \emptyset$, but $I \cap I^- \neq \emptyset$ holds, then $\bar{X}_I\{\lambda\} \notin \bar{\mathcal{G}}$. In this case $\pi^{\bar{X}}_{\bar{Z}}({}^{(j-1)(\ell+1)+1}\bar{\tau}^{\bar{\tilde{X}}}_{\bar{Y}}) = \emptyset$, but $\pi^{\bar{X}}_{\bar{Z}}({}^{j'}\bar{\sigma}^{\bar{\tilde{X}}}_{\bar{Y}}) \neq \emptyset$ for all $j'$.
Now let $I \subseteq I_{+-} \cup I_-$. Then $\bar{X}_I\{\lambda\} \in \bar{\mathcal{G}}$ iff $I \cap I_{+-} \subseteq J_i$ for some $i \in \{1, \dots, \ell\}$. For this $i$ we get

$$\pi^{\bar{X}}_{\bar{Z}}({}^{(j-1)\ell+i}\bar{\sigma}^{\bar{\tilde{X}}}_{\bar{Y}}) = \emptyset = \pi^{\bar{X}}_{\bar{Z}}({}^{(j-1)(\ell+1)+i+1}\bar{\tau}^{\bar{\tilde{X}}}_{\bar{Y}}) = \pi^{\bar{X}}_{\bar{Z}}({}^{(j-1)(\ell+1)+1}\bar{\tau}^{\bar{\tilde{X}}}_{\bar{Y}}),$$

while for any $i' \neq i$ we have

$$\pi^{\bar{X}}_{\bar{Z}}({}^{(j-1)\ell+i'}\bar{\sigma}^{\bar{\tilde{X}}}_{\bar{Y}}) = \{\top\} = \pi^{\bar{X}}_{\bar{Z}}({}^{(j-1)(\ell+1)+i'+1}\bar{\tau}^{\bar{\tilde{X}}}_{\bar{Y}}).$$

If no such $i$ exists, we have $\pi^{\bar{X}}_{\bar{Z}}({}^{(j-1)\ell+i'}\bar{\sigma}^{\bar{\tilde{X}}}_{\bar{Y}}) = \{\top\}$ for all $i'$, while still $\pi^{\bar{X}}_{\bar{Z}}({}^{(j-1)(\ell+1)+1}\bar{\tau}^{\bar{\tilde{X}}}_{\bar{Y}}) = \emptyset$ holds.

In subcase 3 we must have $\bar{X}_{I^-}\{\lambda\} \notin \bar{\mathcal{G}}$ due to property 4(a) of Theorem 10 and in particular $I^- \neq \emptyset$. Furthermore, $\bar{X}_I\{\lambda\} \notin \bar{\mathcal{G}}$, whenever $I \cap I^- \neq \emptyset$ follows from the same property. If we now partition $I^+$ into $I_+$, $I_{+-}$ and $I_-$ according to property 6(a) of Theorem 10, then we get immediately $I_+ = \emptyset$ due to 6(a)ii. Furthermore, we must have $\bar{X}_{I'}\{\lambda\} \in \bar{\mathcal{G}}$, whenever $I' \subseteq I_-$ and $\bar{X}_{I'}\{\lambda\} \in \bar{\mathcal{G}}$ iff

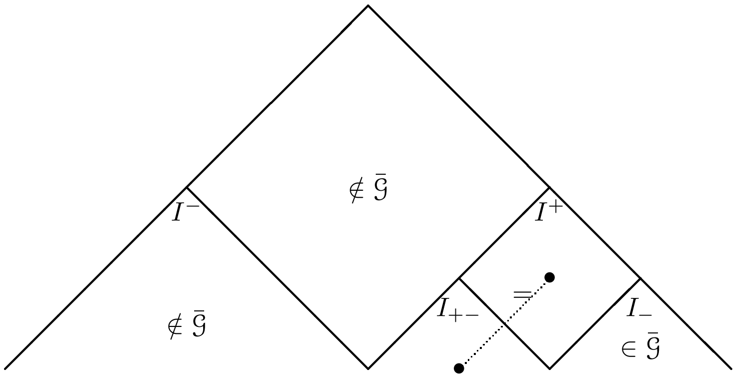


Figure 5: Counter Attributes for $\bar{X} = \bar{X}\{X_1(X_1') \oplus \cdots \oplus X_n(X_n')\}$ with $dd(\bar{X}) = i+1$ in case 3

$\bar{X}_{I' \cap I_{+-}}\{\lambda\} \in \bar{\mathcal{G}}$ for all $I' \subseteq I_- \cup I_{+-}$. These properties are illustrated in Figure 5.

As in subcase 1 take $J_1, \dots, J_\ell \subseteq I_{+-}$ maximal with $\bar{X}_{J_i}\{\lambda\} \in \bar{\mathcal{G}}$. Then for $i = 1, \dots, \ell$ define ${}^i\bar{\sigma}^\lambda_\lambda = \{(X_j : v_j) \mid j \in I^-\} \cup \{(X_j : v_j) \mid j \in I_{+-} - J_i\}$ and ${}^{i+1}\bar{\tau}^\lambda_\lambda = \{(X_j : v_j) \mid j \in I_{+-} - J_i\}$ with arbitrary $v_j \in dom(X_j')$ for $j \in I_{+-} \cup I_-$, and ${}^1\bar{\tau}^\lambda_\lambda = \emptyset$. As before extend these values using properties 3–6 of Definition 14 to define the distinguished values ${}^i\bar{\sigma}^{\tilde{\bar{X}}}_Y$ and ${}^i\bar{\tau}^{\tilde{\bar{X}}}_Y$. Then define $S_1 = \{{}^i\bar{\sigma}^{\tilde{\bar{X}}}_Y \mid Y \in \mathcal{S}(\tilde{\bar{X}}), i = 1, \dots, \ell, Y \neq \lambda\}$ and $S_2 = \{{}^i\bar{\tau}^{\tilde{\bar{X}}}_Y \mid Y \in \tilde{\mathcal{G}}, i = 1, \dots, \ell+1, Y \neq \lambda\}$.

In order to show $\{\pi^X_Z(\tau) \mid \tau \in S_1\} = \{\pi^X_Z(\tau) \mid \tau \in S_2\}$ iff $Z \in \mathcal{G}$ we can use the same argument as in subcase 1, so it suffices to show $\{\pi^{\bar{X}}_{\bar{Z}}({}^i\bar{\sigma}^\lambda_\lambda) \mid i = 1, \dots, \ell\} = \{\pi^{\bar{X}}_{\bar{Z}}({}^i\bar{\tau}^\lambda_\lambda) \mid i = 1, \dots, \ell+1\}$ iff $\bar{Z} \in \bar{\mathcal{G}}$.

- If $\bar{Z} = \bar{X}(Y_1, \dots, Y_n)$, then $\bar{Z} \in \bar{\mathcal{G}}$ iff $Y_i = \lambda$ for all $i \in I^-$ and $\{j \in I_{+-} \mid Y_j \neq \lambda\} \subseteq J_i$ for some $i \in \{1, \dots, \ell\}$. As we have $\pi^{\bar{X}}_{\bar{Z}}({}^i\bar{\sigma}^\lambda_\lambda) = \emptyset = \pi^{\bar{X}}_{\bar{Z}}({}^{i+1}\bar{\tau}^\lambda_\lambda)$, only if $\{j \in I_{+-} \mid Y_j \neq \lambda\} \subseteq J_i$ and $Y_j = \lambda$ for all $j \in I^-$, the claim follows immediately.

- If $\bar{Z} = \bar{X}_I\{\lambda\}$, then $\bar{Z} \in \bar{\mathcal{G}}$ iff $I = J \cup I'$ with $I' \subseteq I_-$ and $J \subseteq J_i$ for some $i \in \{1, \dots, \ell\}$, in which case $\pi^{\bar{X}}_{\bar{Z}}({}^i\bar{\sigma}^\lambda_\lambda) = \emptyset = \pi^{\bar{X}}_{\bar{Z}}({}^{i+1}\bar{\tau}^\lambda_\lambda)$ holds again.

This completes subcase 3.

In the remaining subcase 2 we also take a partition $I^+ = I_+ \cup I_- \cup I_{+-}$ with $\bar{X}_{I'}\{\lambda\} \in \bar{\mathcal{G}}$ for all $I'$ with $I' \cap I_+ \neq \emptyset$, and $\bar{X}_{J_- \cup J}\{\lambda\} \in \bar{\mathcal{G}}$ iff $\bar{X}_J\{\lambda\} \in \bar{\mathcal{G}}$ for all $J \subseteq I^- \cup I_{+-}$ and all $J_- \subseteq I_-$. Take $I_+$ and $I_-$ maximal with these properties.

Furthermore, due to property 4(b) of Theorem 10 we obtain a partition of $I^-$ into $I_1^-$ and $I_2^-$ with $\bar{X}_{I_j^-}\{\lambda\} \notin \bar{\mathcal{G}}$ $(j = 1, 2)$, but $\bar{X}_{I'}\{\lambda\} \in \bar{\mathcal{G}}$, whenever $I' \subseteq I^-$

with $I' \cap I_1^- \neq \emptyset \neq I' \cap I_2^-$. This gives rise to the following additional properties for the counterattributes in $\mathcal{S}(\bar{X})$ (these are illustrated in Figure 6):

– $\bar{X}_{I'}\{\lambda\} \in \bar{\mathcal{G}}$, whenever $I' \subseteq I_-$ holds – this is due to property 6(a)iii of Theorem 10.

– $\bar{X}_{I'}\{\lambda\} \notin \bar{\mathcal{G}}$, whenever $I' \neq \emptyset$ and $I' \subseteq I_1^-$ or $I' \subseteq I_2^-$ holds.

– For each $i \in I_{+-}$ there exists some $J_i \subseteq I^-$ with $\bar{X}_{\{i\}\cup J_i}\{\lambda\} \notin \bar{\mathcal{G}}$ due to the maximality of $I_+$. Then also $\bar{X}_{J_i}\{\lambda\} \notin \bar{\mathcal{G}}$ due to property 4(a) of Theorem 10, and thus $J_i \subseteq I_1^-$ or $J_i \subseteq I_2^-$.

  Therefore, define $I_{+-}^1 = \{i \in I_{+-} \mid \exists J_i \subseteq I_1^-.\bar{X}_{\{i\}\cup J_i}\{\lambda\} \notin \bar{\mathcal{G}}\}$ and analogously $I_{+-}^2 = \{i \in I_{+-} \mid \exists J_i \subseteq I_2^-.\bar{X}_{\{i\}\cup J_i}\{\lambda\} \notin \bar{\mathcal{G}}\}$. Then $\bar{X}_{J\cup I'}\{\lambda\} \notin \bar{\mathcal{G}}$, whenever $I' \subseteq I_{+-}^j$ and $\emptyset \neq J \subseteq I_j^-$ $(j = 1, 2)$ due to property 4(d) of Theorem 10.

  Furthermore, for $J' \not\subseteq I_{+-}^j$ $(j = 1, 2)$ we obtain $\bar{X}_{J'\cup K}\{\lambda\} \in \bar{\mathcal{G}}$ for all $K \subseteq I^-$, $K \neq \emptyset$. This can be seen as follows: Suppose the statement were false, say $\bar{X}_{J'\cup K}\{\lambda\} \notin \bar{\mathcal{G}}$. Then we also have $\bar{X}_K\{\lambda\} \notin \bar{\mathcal{G}}$; otherwise property 4(a) of Theorem 10 would give an immediate contradiction to $K \subseteq I^-$. So $K \subseteq I_j^-$ for $j = 1$ or $j = 2$. For the same reason we get $\bar{X}_{\{k\}\cup K}\{\lambda\} \notin \bar{\mathcal{G}}$ for all $k \in J'$, hence the contradiction $J' \subseteq I_{+-}^j$.

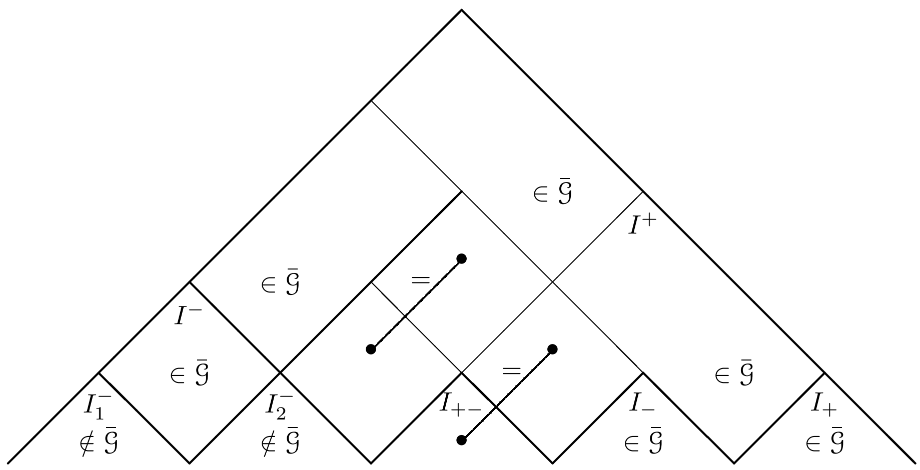


Figure 6: Counter Attributes for $\bar{X} = \bar{X}\{X_1(X_1') \oplus \cdots \oplus X_n(X_n')\}$ with $dd(\bar{X}) = i + 1$ in case 2

Now we consider two cases 2.1, in which $\bar{X}_{I_{+-}}\{\lambda\} \notin \bar{\mathcal{G}}$ holds, and 2.2, in which $\bar{X}_{I_{+-}}\{\lambda\} \in \bar{\mathcal{G}}$ holds.

In case 2.1, whenever $\bar{X}_J\{\lambda\} \in \bar{\mathcal{G}}$ holds, then also $\bar{X}_{J'}\{\lambda\} \in \bar{\mathcal{G}}$ must hold for all $J' \subseteq J$ and even $\bar{X}(X_{j_1}\{X'_{j_1}\}, \dots, X_{j_p}\{X'_{j_p}\}) \in \bar{\mathcal{G}}$ for $J' = \{j_1, \dots, j_p\}$. Now take maximal $J_1, \dots, J_\ell \subseteq I_{+-}$ with $\bar{X}_{J_i}\{\lambda\} \in \bar{\mathcal{G}}$. In particular, $\bar{X}_{J_i \cap I^j_{+-}}\{\lambda\} \in \bar{\mathcal{G}}$. Then for $i = 1, \dots, \ell$ define

$$\begin{aligned}
\varrho^+_{3i-1} &= \{(X_j : v_j) \mid j \in I_{+-} - (J_i \cap I^1_{+-})\} \cup \{(X_j : v_j) \mid j \in I^-\} \\
\varrho^+_{3i-2} &= \{(X_j : v_j) \mid j \in I_{+-} - J_i\} \cup \{(X_j : v_j) \mid j \in I^-\} \\
\varrho^+_{3i} &= \{(X_j : v_j) \mid j \in I_{+-} - (J_i \cap I^2_{+-})\} \cup \{(X_j : v_j) \mid j \in I^-\}
\end{aligned}$$

with arbitrary $v_j \in dom(X'_j)$ for $j \in I^- \cup I_{+-}$. Similarly, define

$$\begin{aligned}
\varrho^-_1 &= \{(X_j : v_j) \mid j \in I^-\} \\
\varrho^-_2 &= \{(X_j : v_j) \mid j \in I_{+-} - I^1_{+-}\} \cup \{(X_j : v_j) \mid j \in I^-_2\} \\
\varrho^-_3 &= \{(X_j : v_j) \mid j \in I_{+-} - I^2_{+-}\} \cup \{(X_j : v_j) \mid j \in I^-_1\} \\
\varrho^-_{3i+1} &= \{(X_j : v_j) \mid j \in I_{+-} - J_i\} \cup \{(X_j : v_j) \mid j \in I^-\} \\
\varrho^-_{3i+2} &= \{(X_j : v_j) \mid j \in I_{+-} - (J_i \cap I^1_{+-})\} \cup \{(X_j : v_j) \mid j \in I^-_2\} \\
\varrho^-_{3i+3} &= \{(X_j : v_j) \mid j \in I_{+-} - (J_i \cap I^2_{+-})\} \cup \{(X_j : v_j) \mid j \in I^-_1\}
\end{aligned}$$

Now define $\bar{X}^+$ and $\bar{\mathcal{G}}^+$ as in subcase 1, so by induction we find $S^+_1 = \{{}^j\sigma^{\tilde{\tilde{X}}}_Y \mid Y \in \mathcal{S}(\tilde{\tilde{X}}), j = 1, \dots, o\}$ and $S^+_2 = \{{}^j\tau^{\tilde{\tilde{X}}}_Y \mid Y \in \tilde{\tilde{\mathcal{G}}}, j = 1, \dots, p\}$ with $\{\pi^{\bar{X}^+}_Y(\tau) \mid \tau \in S^+_1\} = \{\pi^{\bar{X}^+}_Y(\tau) \mid \tau \in S^+_2\}$ iff $Y \in \bar{\mathcal{G}}^+$. We extend $S^+_1$ and $S^+_2$, respectively, to $S_1$ and $S_2$ analogously to subcase 1, i.e. we obtain $S_1 = \{{}^j\sigma^{\tilde{X}}_Y \mid Y \in \mathcal{S}(\tilde{X}), Y \neq \lambda, j = 1, \dots, 3 \cdot o \cdot \ell\}$ and $S_2 = \{{}^j\tau^{\tilde{X}}_Y \mid Y \in \tilde{\mathcal{G}}, Y \neq \lambda, j = 1, \dots, 3 \cdot p \cdot (\ell + 1)\}$ using

$$\begin{aligned}
{}^{3(j-1)\ell+i}\bar{\sigma}^{\tilde{\tilde{X}}}_Y &= {}^j\sigma^{\tilde{\tilde{X}}}_Y \cup \varrho^+_i \quad (j = 1, \dots, o\,, i = 1, \dots, 3\ell) \text{ and} \\
{}^{3(j-1)(\ell+1)+i}\bar{\tau}^{\tilde{\tilde{X}}}_Y &= {}^j\tau^{\tilde{\tilde{X}}}_Y \cup \varrho^-_i \quad (j = 1, \dots, p\,, i = 1, \dots, 3\ell + 3).
\end{aligned}$$

For $Z \in \mathcal{S}(X)$ the projected values $\pi^X_Z({}^j\sigma^{\tilde{X}}_Y)$ and $\pi^X_Z({}^j\tau^{\tilde{X}}_Y)$ involve $\pi^{\bar{X}}_{\bar{Z}}({}^j\bar{\sigma}^{\tilde{\tilde{X}}}_Y)$ and $\pi^{\bar{X}}_{\bar{Z}}({}^j\bar{\tau}^{\tilde{\tilde{X}}}_Y)$, respectively, and if $Z \in \mathcal{G}$, then also $\bar{Z} \in \bar{\mathcal{G}}$.

First let $\bar{Z} = \bar{X}(Y_1, \dots, Y_n)$. Then $\bar{Z} \in \bar{\mathcal{G}}$ holds iff $Y_i = \lambda$ for all $i \in I^-$, $\{j \in I_{+-} \mid Y_j \neq \lambda\} \subseteq J_x$ for some $x \in \{1, \dots, \ell\}$ and $\bar{X}(Y_{i_1}, \dots, Y_{i_k}) \in \bar{\mathcal{G}}^+$ hold (for $I^+ = \{i_1, \dots, i_k\}$). Then we get

$$\pi^{\bar{X}}_{\bar{Z}}({}^{3(j-1)\ell+i}\bar{\sigma}^{\tilde{\tilde{X}}}_Y) = \pi^{\bar{X}^+}_{\bar{X}(Y_{i_1},\dots,Y_{i_k})}({}^j\sigma^{\tilde{\tilde{X}}}_Y)\} \cup \{(X_j : \pi^{X'_j}_{Y'_j}(v_j)) \mid j \in I^- \cup I^{(i)}_{+-}, Y_j \neq \lambda\}$$

for $Y_j = X_j\{Y'_j\}$ and $I^{(i)}_{+-} = \begin{cases} I_{+-} - J_{\frac{i+2}{3}} & \text{for } i \equiv 1(3) \\ I_{+-} - (J_{\frac{i+1}{3}} \cap I^1_{+-}) & \text{for } i \equiv 2(3) \\ I_{+-} - (J_{\frac{i}{3}} \cap I^2_{+-}) & \text{for } i \equiv 0(3) \end{cases}.$

Analogously, for $i > 3$ we obtain

$$\pi^{\bar{X}}_{\bar{Z}}({}^{3(j-1)(\ell+1)+i}\bar{\tau}^{\bar{\bar{X}}}_{\bar{Y}}) = \pi^{\bar{X}^+}_{\bar{X}(Y_{i_1},\dots,Y_{i_k})}({}^{j}\tau^{\bar{\bar{X}}}_{\bar{Y}})\} \cup \{(X_j : \pi^{X'_j}_{Y'_j}(v_j)) \mid j \in I^-_{(i)} \cup I^{(i)}_{+-}, Y_j \neq \lambda\}$$

with $I^-_{(i)} = \begin{cases} I^- & \text{if } i \equiv 1(3) \\ I^-_1 & \text{if } i \equiv 0(3) \\ I^-_2 & \text{if } i \equiv 2(3) \end{cases}$

For $i \leq 3$ we have to replace the definition of $I^-_{(i)}$ by $I^-_{(i)} = \begin{cases} \emptyset & \text{if } i = 1 \\ I_{+-} - I^1_{+-} & \text{if } i = 2 \\ I_{+-} - I^2_{+-} & \text{if } i = 3 \end{cases}$

These values are equal for $\bar{Z} \in \bar{\mathcal{G}}$. In case $\bar{Z} \notin \bar{\mathcal{G}}$ we have either $\bar{X}(Y_{i_1},\dots,Y_{i_k}) \notin \bar{\mathcal{G}}^+$, which gives inequality for the first components of the unions due to the construction of $S^+_1$ and $S^+_2$, or $Y_i \neq \lambda$ for some $i \in I^-$, which implies that $(X_i : \pi^{X'_i}_{Y'_i}(v_i))$ always appears in the projection of $S_1$, but not so for $S_2$, or $\{j \in I_{+-} \mid Y_j \neq \lambda\} \not\subseteq J_x$ for all $x = 1,\dots,\ell$, which implies that some $(X_j : \pi^{X'_j}_{Y'_j}(v_j))$ always appears in the projection of $S_1$, but not so for $S_2$ due to the definition of $\varrho^-_1$. So we get inequality in all cases.

Now consider $\bar{Z} = \bar{X}_I\{\lambda\}$. For $I \cap I_+ \neq \emptyset$, which implies $\bar{X}_I\{\lambda\} \in \bar{\mathcal{G}}$, we have already seen the equality. Therefore, assume $I \cap I_+ = \emptyset$. If $I \cap I^- \neq \emptyset$, then $\bar{X}_I\{\lambda\} \notin \bar{\mathcal{G}}$ iff $I \cap I_{+-} \subseteq I^j_{+-}$ and $I \cap I^- \subseteq I^-_j$ for $j = 1$ or $j = 2$. In this case we get $\pi^{\bar{X}}_{\bar{X}_I\{\lambda\}}(\tau) = \{\top\}$ for all $\tau \in S_1$, but using either $\varrho^-_2$ or $\varrho^-_3$ for $j = 1$ or $j = 2$, respectively, we obtain $\pi^{\bar{X}}_{\bar{X}_I\{\lambda\}}(\tau) = \emptyset$ for some $\tau \in S_2$ iff $\bar{X}_I\{\lambda\} \notin \bar{\mathcal{G}}$.

Therefore, we can further assume $I \cap I^- = \emptyset$. Now $\bar{X}_I\{\lambda\} \in \bar{\mathcal{G}}$ iff $\bar{X}_{I \cap I_{+-}}\{\lambda\} \in \bar{\mathcal{G}}$, so we may even assume $I \subseteq I_{+-}$. Then $\bar{X}_I\{\lambda\} \in \bar{\mathcal{G}}$ holds iff $I \subseteq J_x$ for some $x \in \{1,\dots,\ell\}$. For $J_x = \{x_1,\dots,x_y\}$ we have $\bar{X}(X_{x_1}\{X'_{x_1}\},\dots,X_{x_y}\{X'_{x_y}\}) \in \bar{\mathcal{G}}$, so we have already shown the equality for $I \subseteq J_x$. For $I \not\subseteq J_x$ for all $x = 1,\dots,\ell$ we obtain $\pi^{\bar{X}}_{\bar{X}_I\{\lambda\}}({}^1\bar{\tau}^{\bar{\bar{X}}}_{\bar{Y}}) = \emptyset$, but $\pi^{\bar{X}}_{\bar{X}_I\{\lambda\}}(\tau) \neq \emptyset$ for all $\tau \in S_1$, which shows the desired inequality.

Next consider case 2.2, i.e. $\bar{X}_{I_{+-}}\{\lambda\} \in \bar{\mathcal{G}}$. In this case let $\varrho^-_0 = \{(X_i : \tau^{X'_i}_{X'_i}) \mid i \in I^-\}$, $\varrho^-_1 = \{(X_i : \tau^{X'_i}_{X'_i}) \mid i \in I^-_2\}$, and $\varrho^-_2 = \{(X_i : \tau^{X'_i}_{X'_i}) \mid i \in I^-_1\}$.

For $I_{+-} = \{j_1,\dots,j_q\}$ let $X_{+-} = \bar{X}(X_{j_1}\{X'_{j_1}\},\dots,X_{j_q}\{X'_{j_q}\})$ and $\mathcal{G}_{+-} = \{Y \in \bar{\mathcal{G}} \mid X_{+-} \geq Y\}$. For $Y = \bar{X}(Y_1,\dots,Y_q) \leq X_{+-}$ let $ind(Y) = \{j \mid Y_j \neq \lambda\}$. Let $Y^{(1)},\dots,Y^{(\varkappa)}$ be the maximal elements in $\mathcal{G}_{+-}$ such that $\bar{X}_K\{\lambda\} \in \mathcal{G}$ holds for all $K \subseteq I_{+-} - ind(Y^{(j)})$ $(j = 1,\dots,\varkappa)$. Define $\varrho^+_{j+1} = \tau^{X_{+-}}_{Y^{(j)}} \cup \varrho^-_0$ for $j = 1,\dots,\varkappa$, and $\varrho^+_1 = \tau^{X_{+-}}_{X_{+-}} \cup \varrho^-_0$.

Let $K_1, \dots, K_\mu \subseteq I_{+-}$ be maximal with $\bar{X}_{K_j}\{\lambda\} \notin \bar{\mathcal{G}}$. For $j = 1, \dots, \mu$ define

$$\begin{aligned}
\varrho_j^0 &= \{(X_\alpha : \tau_{X'_\alpha}^{X'_\alpha}) \mid \alpha \in I_{+-} - K_j\} \cup \varrho_0^- \ , \\
\varrho_j^1 &= \{(X_\alpha : \tau_{X'_\alpha}^{X'_\alpha}) \mid \alpha \in I_{+-} - (K_j \cap I_{+-}^1)\} \cup \varrho_1^- \quad \text{and} \\
\varrho_j^2 &= \{(X_\alpha : \tau_{X'_\alpha}^{X'_\alpha}) \mid \alpha \in I_{+-} - (K_j \cap I_{+-}^2)\} \cup \varrho_2^- \ .
\end{aligned}$$

For $j \in \{1, \dots, \mu\}$ let $J_1, \dots, J_{\mu_j} \subseteq K_j$ be maximal with $\bar{X}_{J_i}\{\lambda\} \in \bar{\mathcal{G}}$. For $i = 1, \dots, \mu_j$ define

$$\begin{aligned}
\varrho_{j,i}^0 &= \{(X_\alpha : \tau_{X'_\alpha}^{X'_\alpha}) \mid \alpha \in I_{+-} - J_i\} \cup \varrho_0^- \ , \\
\varrho_{j,i}^1 &= \{(X_\alpha : \tau_{X'_\alpha}^{X'_\alpha}) \mid \alpha \in I_{+-} - (J_i \cap I_{+-}^1)\} \cup \varrho_1^- \ , \\
\varrho_{j,i}^2 &= \{(X_\alpha : \tau_{X'_\alpha}^{X'_\alpha}) \mid \alpha \in I_{+-} - (J_i \cap I_{+-}^2)\} \cup \varrho_2^- \ , \\
\varrho_{j,i}^{10} &= \{(X_\alpha : \tau_{X'_\alpha}^{X'_\alpha}) \mid \alpha \in I_{+-} - J_i\} \cup \varrho_0^- \ , \\
\varrho_{j,i}^{11} &= \{(X_\alpha : \tau_{X'_\alpha}^{X'_\alpha}) \mid \alpha \in I_{+-} - (J_i \cap I_{+-}^1)\} \cup \varrho_0^- \ \text{and} \\
\varrho_{j,i}^{12} &= \{(X_\alpha : \tau_{X'_\alpha}^{X'_\alpha}) \mid \alpha \in I_{+-} - (J_i \cap I_{+-}^2)\} \cup \varrho_0^- \ .
\end{aligned}$$

As before, we now extend these complex values to values in $dom(\bar{X})$. For this define $\bar{X}^+$, $\bar{\mathcal{G}}^+$, $S_1^+$ and $S_2^+$ as in the first subcase. Then, using $\tilde{X}$ and $\tilde{\mathcal{G}}$ as before, define

$$\begin{aligned}
{}^{(j-1)(\varkappa+1)+i}\bar{\sigma}_Y^{\bar{\tilde{X}}} &= {}^j\sigma_Y^{\bar{\tilde{X}}} \cup \varrho_i^+ \quad (j = 1, \dots, o \quad , i = 1, \dots, \varkappa + 1) \\
\text{and } {}^{(j-1)\varkappa+i-1}\bar{\tau}_Y^{\bar{\tilde{X}}} &= {}^j\tau_Y^{\bar{\tilde{X}}} \cup \varrho_i^+ \quad (j = 1, \dots, p \quad , i = 2, \dots, \varkappa + 1)
\end{aligned}$$

omitting in both cases those values, for which $Y \sqcup Y^{(i)} \notin \bar{\mathcal{G}}$ (or $Y \sqcup Y^{(i-1)} \notin \bar{\mathcal{G}}$, respectively). Next define

$${}^{p(\varkappa+3(j-1)\mu)+3(i-1)+x+1}\bar{\tau}_Y^{\bar{\tilde{X}}} = {}^j\tau_Y^{\bar{\tilde{X}}} \cup \varrho_i^x$$

for $j = 1, \dots, p$, $i = 1, \dots, \mu$ and $x = 0, \dots, 2$,

$${}^{p(\varkappa+3\mu)+3(j-1)\mu\mu_k+3(k-1)\mu_k+3(i-1)+x+1}\bar{\tau}_Y^{\bar{\tilde{X}}} = {}^j\tau_Y^{\bar{\tilde{X}}} \cup \varrho_{k,i}^x$$

for $j = 1, \dots, p$, $k = 1, \dots, \mu$, $i = 1, \dots, \mu_k$ and $x = 0, \dots, 2$,

and analogously

$${}^{o(\varkappa+1)+3(j-1)\mu\mu_k+3(k-1)\mu_k+3(i-1)+x+1}\bar{\sigma}_Y^{\bar{\tilde{X}}} = {}^j\sigma_Y^{\bar{\tilde{X}}} \cup \varrho_{k,i}^{1x}$$

for $j = 1, \dots, o$, $k = 1, \dots, \mu$, $i = 1, \dots, \mu_k$ and $x = 0, \dots, 2$.

First consider $\bar{Z} = \bar{X}(Y_1, \dots, Y_n)$. If $\bar{Z} \in \bar{\mathcal{G}}$, then $Y_i = \lambda$ for all $i \in I^-$. So we can ignore the part in the definition of ${}^j\sigma_Y^{\tilde{\bar{X}}}$ and ${}^j\tau_Y^{\tilde{\bar{X}}}$ that arises from $\varrho_x^-$ $(x = 0, \dots, 2)$. This reduces the attention to $\tau_{X_{+-}}^{X_{+-}}$ appearing with a complex value in $S_1$ and $\varrho_j^x$ $(x = 0, \dots, 2,\ j = 1, \dots, \mu)$ appearing only in values in $S_2$. By induction for each $j$ and $Y$ there is a $j'$ and a $Y'$ with $\pi_{\bar{\bar{Z}}}^{\bar{X}}({}^j\sigma_Y^{\bar{\tilde{\bar{X}}}}) = \pi_{\bar{\bar{Z}}}^{\bar{X}}({}^{j'}\tau_{Y'}^{\bar{\tilde{\bar{X}}}})$ and vice versa.

First let $\bar{Z} \in \bar{\mathcal{G}}$ be maximal. Then $\bar{Z} \sqcap X_{+-}$ is maximal in $\mathcal{G}_{+-}$, and from Lemma 15(1.) we obtain $\pi_{\bar{Z} \sqcap X_{+-}}^{\bar{X}}(\varrho_1^+) = \pi_{\bar{Z} \sqcap X_{+-}}^{\bar{X}}(\varrho_{j+1}^+)$ in case $\bar{Z} \sqcap X_{+-} = Y^{(j)}$. Otherwise, there is some $Z' \geq \bar{Z} \sqcap X_{+-}$ such that $\tau_{Z'}^{X_{+-}} = \{(X_\alpha : \tau_{X'_\alpha}^{X'_\alpha}) \mid \alpha \in I_{+-} - K_z\}$ holds for some $z \in \{1, \dots, \mu\}$, which implies $\pi_{\bar{Z} \sqcap X_{+-}}^{\bar{X}}(\varrho_1^+) = \pi_{\bar{Z} \sqcap X_{+-}}^{\bar{X}}(\varrho_z^0)$ due to Lemma 15(1.) Due to our construction this implies that for each $j = 1, \dots, o$ and each $Y$ there exist $j' \in \{1, \dots, p\}$, $Y'$ and $i \in \{2, \dots, \varkappa + 1\}$ with $\pi_{\bar{\bar{Z}}}^{\bar{X}}({}^{(j-1)(\varkappa+1)+1}\bar{\sigma}_Y^{\bar{\tilde{\bar{X}}}}) = \pi_{\bar{\bar{Z}}}^{\bar{X}}({}^{(j'-1)\varkappa+i-1}\bar{\tau}_{Y'}^{\bar{\tilde{\bar{X}}}})$ or $\pi_{\bar{\bar{Z}}}^{\bar{X}}({}^{(j-1)(\varkappa+1)+1}\bar{\sigma}_Y^{\bar{\tilde{\bar{X}}}}) = \pi_{\bar{\bar{Z}}}^{\bar{X}}({}^{p\varkappa+3(j'-1)\mu+3(i'-1)+1}\bar{\tau}_{Y'}^{\bar{\tilde{\bar{X}}}})$ for some $i' \in \{1, \dots, \mu\}$.

Further, $ind(\bar{Z} \sqcap X_{+-}) \cap K_j \subseteq J_i$ for some $i \in \{1, \dots, \mu_j\}$, so $\pi_{\bar{Z} \sqcap X_{+-}}^{\bar{X}}(\varrho_j^x) = \pi_{\bar{Z} \sqcap X_{+-}}^{\bar{X}}(\varrho_{j,i}^{1x})$, which gives the desired equality for all maximal $\bar{Z} \in \bar{\mathcal{G}}$, hence for all $\bar{Z} \in \bar{\mathcal{G}}$.

If $\bar{Z} \notin \bar{\mathcal{G}}$ holds, then assuming equality of the projected sets would imply that $\pi_{\bar{Z} \sqcap X_{+-}}^{\bar{X}}(\tau_{X_{+-}}^{X_{+-}})$ equals either $\pi_{\bar{Z} \sqcap X_{+-}}^{\bar{X}}(\tau_{Y^{(j)}}^{X_{+-}})$ for $j \in \{1, \dots, \varkappa\}$ or $\pi_{\bar{Z} \sqcap X_{+-}}^{\bar{X}}(\tau_Y^{X_{+-}})$ with $Y = \bar{X}(X_{\alpha_1}\{X'_{\alpha_1}\}, \dots, X_{\alpha_x}\{X'_{\alpha_x}\})$ such that $I_{+-} - \{\alpha_1, \dots \dots, \alpha_x\}$ is one of $K_j$, $K_j \cap I_{+-}^1$ or $K_j \cap I_{+-}^2$ for $j \in \{1, \dots, \mu\}$ or $J_i$, $J_i \cap I_{+-}^1$ or $J_i \cap I_{+-}^2$ for $i \in \{1, \dots, \mu_j\}$. On the other hand we have $\pi_{\bar{Z} \sqcap X_{+-}}^{\bar{X}}(\tau_{X_{+-}}^{X_{+-}}) = \pi_{\bar{Z} \sqcap X_{+-}}^{\bar{X}}(\tau_{\bar{Z} \sqcap X_{+-}}^{X_{+-}})$, so by Lemma 15(1.) we obtain $Y^{(j)} \geq \bar{Z} \sqcap X_{+-}$ or $Y \geq \bar{Z} \sqcap X_{+-}$.

In the first of these two cases we have $Y^{(j)} \in \mathcal{G}_{+-}$, which implies the contradiction $\bar{Z} \in \bar{\mathcal{G}}$. In the second case we must have $\mu \geq 1$, so let $I_{\bar{Z}} = ind(\bar{Z} \sqcap X_{+-})$. If $I_{\bar{Z}} = I_{+-}$, then the projection of $\tau_{Y^{(j)}}^{X_{+-}}$ contains $\emptyset$ for all indices in $K_j$, but this cannot happen for any $\tau \in S_1$. Hence we obtain the desired inequality in this case. If $I_{\bar{Z}} \neq I_{+-}$, then $\bar{X}_{I_{\bar{Z}}}\{\lambda\} \notin \bar{\mathcal{G}}$, otherwise property 4(a) in Theorem 10 gives $\bar{X}_I\{\lambda\} \notin \bar{\mathcal{G}}$ for all $I \subseteq I_{+-}$ with $I_{\bar{Z}} \subsetneq I$. In particular, we could take $I$ to be one $K_j$ $(j \in \{1, \dots, \mu\})$, so $I_{\bar{Z}} = J_i$ for some $i \in \{1, \dots, \mu_j\}$, but then $\bar{Z} \sqcap X_{+-} \in \mathcal{G}_{+-}$ due to property 4(a) of Theorem 10, which contradicts our assumption $\bar{Z} \notin \bar{\mathcal{G}}$. Hence $I_{\bar{Z}} \subseteq K_j$ for some $j \in \{1, \dots, \mu\}$ and $I_{\bar{Z}} \not\subseteq I_{+-} - ind(Y^{(j)})$. Then the projection onto $\bar{Z} \sqcap X_{+-}$ yields one tuple with only $\emptyset$ for some $\tau \in S_2$, while for each $\tau \in S_1$ we always get at least one non-empty component. Hence the desired inequality for $\bar{Z} \notin \bar{\mathcal{G}}$.

Now consider $\bar{Z} = \bar{X}_I\{\lambda\}$. If $I \cap I_+ \neq \emptyset$ and thus $\bar{Z} \in \bar{\mathcal{G}}$, then we already know that $\pi^{\bar{X}}_{\bar{Z}}({}^j\sigma^{\tilde{\bar{X}}}_Y) = \{\top\} = \pi^{\bar{X}}_{\bar{Z}}({}^{j'}\sigma^{\tilde{\bar{X}}}_{Y'})$, which gives the desired equality in this case. So we can assume $I \cap I_+ = \emptyset$. Due to the construction of $S_1$ and $S_2$ and the fact that $\bar{Z} \in \bar{\mathcal{G}}$ holds iff $\bar{X}_{I-I_-}\{\lambda\} \in \bar{\mathcal{G}}$ holds, we may even assume $I \subseteq I_{+-} \cup I^-$.

Assume $I \cap I^- \neq \emptyset$. Then $\bar{Z} \in \bar{\mathcal{G}}$ iff $I \cap I_{+-} \subseteq I^j_{+-}$ and $I \cap I^- \subseteq I^-_j$ hold for $j = 1$ or $j = 2$. In this case we get $\pi^{\bar{X}}_{\bar{X}_I\{\lambda\}}(\tau) = \{\top\}$ for all $\tau \in S_1$, but using either $\varrho^1_j$ or $\varrho^2_j$ for $j = 1$ or $j = 2$, respectively, we obtain $\pi^{\bar{X}}_{\bar{X}_I\{\lambda\}}(\tau) = \emptyset$ for some $\tau \in S_2$ iff $\bar{X}_I\{\lambda\} \notin \bar{\mathcal{G}}$.

So finally we may assume $I \subseteq I_{+-}$. Then $\bar{X}_I\{\lambda\} \notin \bar{\mathcal{G}}$ iff $I \subseteq K_j$ for some $j \in \{1, \dots, \mu\}$ and $I \not\subseteq J_i$ for $i = 1, \dots, \mu_j$. In this case we obtain $\pi^{\bar{X}}_{\bar{X}_I\{\lambda\}}(\tau) = \emptyset$ for at least one $\tau \in S_2$, but not so for $S_1$, while in case $\bar{X}_I\{\lambda\} \in \bar{\mathcal{G}}$ we either always obtain $\{\top\}$ or both $\{\top\}$ and $\emptyset$. This completes the proof for subcase 2.

Now let $\bar{X} = \bar{X}\langle X_1(X'_1) \oplus \dots \oplus X_n(X'_n)\rangle \in emb(X)$ with $dd(\bar{X}) = i + 1$. Using a similar construction as for the case above, where $\bar{X}$ was a set attribute, we may assume without loss of generality that $X = X\{\bar{X}\}$ holds – this avoids dealing with too many indices. Let $\bar{\mathcal{G}} = \{Y \in \mathcal{S}(\bar{X}) \mid X\{Y\} \in \mathcal{G}\}$ be the defect coincidence ideal on $\mathcal{S}(\bar{X})$ induced by the defect coincidence ideal $\mathcal{G}$ on $\mathcal{S}(X)$. As $\bar{X} \equiv \bar{X}(X_1\langle X'_1\rangle, \dots, X_n\langle X'_n\rangle)$ holds, we may ignore the fact that we have a multiset attribute – thus also ignore the corresponding counter-attributes – and first consider defect coincidence ideals $\bar{\mathcal{H}} = \{Y \in \bar{\mathcal{G}} \mid Y = \bar{X}(Y_1, \dots, Y_n)$ for some $Y_i, i \in \{1, \dots, n\}\}$ and $\mathcal{H} = \{X\{Y\} \in \mathcal{S}(X) \mid Y \in \bar{\mathcal{H}}\}$. In doing so we get $dd(X) = i$, hence by induction there exist $S^+_1, S^+_2 \subseteq dom(X)$ with the desired property for $\mathcal{H}$. In particular, for $\bar{X}(X_{i_1}\langle\lambda\rangle, \dots, X_{i_k}\langle\lambda\rangle) \in \bar{\mathcal{H}}$ we have $\{\pi^{\bar{X}}_{\bar{X}_{i_1,\dots,i_k}\langle\lambda\rangle}(\tau) \mid \tau \in S^+_1\} = \{\pi^{\bar{X}}_{\bar{X}_{i_1,\dots,i_k}\langle\lambda\rangle}(\tau) \mid \tau \in S^+_2\}$.

We may modify all occurring mutlisets by choosing a suitable power of 2 for each $i \in \{1, \dots, n\}$ such that whenever $I \neq J$ holds, the projections $\pi^{\bar{X}}_{\bar{X}_I\langle\lambda\rangle}(\tau)$ and $\pi^{\bar{X}}_{\bar{X}_J\langle\lambda\rangle}(\tau)$ yields completely different multiplicities, and $\{\pi^{\bar{X}}_{\bar{X}_I\langle\lambda\rangle}(\tau) \mid \tau \in S^+_1\} \neq \{\pi^{\bar{X}}_{\bar{X}_I\langle\lambda\rangle}(\tau) \mid \tau \in S^+_2\}$ holds for $I = \{i_1, \dots, i_k\}$ and $\bar{X}(X_{i_1}\langle\lambda\rangle, \dots$ $\dots, X_{i_k}\langle\lambda\rangle) \notin \bar{\mathcal{H}}$.

Now look at those $I = \{i_1, \dots, i_k\}$ , for which $\bar{X}_I\langle\lambda\rangle \in \bar{\mathcal{G}}$ holds, but $\bar{X}(X_{i_1}\langle\lambda\rangle,$ $\dots, X_{i_k}\langle\lambda\rangle) \notin \bar{\mathcal{G}}$. Let $M_I = \{|\pi^{\bar{X}}_{\bar{X}_I\langle\lambda\rangle}(\tau)| \mid \tau \in S^+_1 \cup S^+_2\}$ be the set of corresponding multiplicities. Choose an unused $v_{i_1} \in dom(X'_{i_1})$ and define $t^m_I = \langle\underbrace{(X_{i_1} : v_{i_1})}_{m \text{ times}}\rangle$ for $m \in M_I$. Adding all $t^m_I$ to $S^+_1$ and $S^+_2$ gives $\{\pi^{\bar{X}}_{\bar{X}_I\langle\lambda\rangle}(\tau) \mid \tau \in S^+_1\} = \{\pi^{\bar{X}}_{\bar{X}_I\langle\lambda\rangle}(\tau) \mid \tau \in S^+_2\}$ without disturbing any of the other previously established equalities and inequalities. Hence, doing this for each $I$ for each $I$ in question we obtain the desired sets $S_1$ and $S_2$.

The case of $\bar{X} = \bar{X}[X_1(X'_1) \oplus \dots \oplus X_n(X'_n)] \in emb(X)$ with $dd(\bar{X}) = i+1$ is handled analogously to the multiset case, i.e. first ignore the counter-attributes, then manipulate the number of occurrences of elements in the sublists, finally append lists with repeated entries of just one element. This completes the set case.

For the multiset case we proceed almost analogously exploiting the construction of $M_1$ and $M_2$ by using minimal elements in the filter $\mathcal{H} = \mathcal{S}^r(X) - \mathcal{G}$ and Boolean algebras $\mathbb{B}(Y) \subseteq \mathcal{S}^r(X)$ with top element $Y \in \mathcal{H}$. Again, we consider one-by-one embedded set, multiset and list attributes with degeneration depth $i+1$ starting with the set case.

So let $\bar{X} = \bar{X}\{X_1(X_1') \oplus \cdots \oplus X_n(X_n')\} \in emb(X)$ be such that $dd(\bar{X}) = i+1$. Using the same arguments as in the set case we can assume without loss of generality that $X = X\langle \bar{X} \rangle$ holds. Thus take $\bar{\mathcal{G}} = \{Y \in \mathcal{G} \mid X\langle Y \rangle \in \mathcal{G}\}$, which is a defect coincidence ideal on $\mathcal{S}(\bar{X})$. We distinguish the following three subcases:

1. $\bar{X}_{\{1,\dots,n\}}\{\lambda\} \in \bar{\mathcal{G}}$ and $\bar{X}_{I^-}\{\lambda\} \notin \bar{\mathcal{G}}$;
2. $\bar{X}_{\{1,\dots,n\}}\{\lambda\} \in \bar{\mathcal{G}}$ and $\bar{X}_{I^-}\{\lambda\} \in \bar{\mathcal{G}}$;
3. $\bar{X}_{\{1,\dots,n\}}\{\lambda\} \notin \bar{\mathcal{G}}$.

In subcase 1 we obtain a partition $I^+ = I_+ \cup I_- \cup I_{+-}$ with the following properties:

- $\bar{X}_{I'}\{\lambda\} \in \bar{\mathcal{G}}$ for all $I'$ with $I' \cap I_+ \neq \emptyset$,
- $\bar{X}_{J_- \cup J}\{\lambda\} \in \bar{\mathcal{G}}$ holds iff $\bar{X}_J\{\lambda\} \in \bar{\mathcal{G}}$ for all $J \subseteq I^- \cup I_{+-}$ and $J_- \subseteq I_-$.

Taking $I_+$ and $I_-$ maximal with these properties, we obtain (as in the set case, subcase 1) the following additional properties of counter-attributes in $\mathcal{S}(\bar{X})$:

- $\bar{X}_{I'}\{\lambda\} \in \bar{\mathcal{G}}$ for all $I' \subseteq I_-$,
- $\bar{X}_{I'}\{\lambda\} \notin \bar{\mathcal{G}}$ for all $I' \subseteq I^-$,
- $\bar{X}_{I' \cup J'}\{\lambda\} \notin \bar{\mathcal{G}}$, whenever $I' \subseteq I_{+-} \cup I_-$ and $\emptyset \neq J' \subseteq I^-$ holds,
- $\bar{X}_{J'}\{\lambda\} \in \bar{\mathcal{G}}$, whenever $J' \subseteq J \subseteq I_{+-}$ and $\bar{X}_J\{\lambda\} \in \bar{\mathcal{G}}$ hold.

These properties were already illustrated by Figure 4. Now take $J_1, \dots, J_\nu \subseteq I_{+-}$ maximal with $\bar{X}_{J_i}\{\lambda\} \in \bar{\mathcal{G}}$. For $J_i = \{j_1, \dots, j_\ell\}$ we also get $\bar{X}(X_{j_1}\{X_{j_1}'\}, \dots$ $\dots, X_{j_\ell}\{X_{j_\ell}'\}) \in \bar{\mathcal{G}}$. These $J_1, \dots, J_\nu$ generate an ideal $\mathcal{I}$ in the Boolean powerset algebra $\mathfrak{P}(I_{+-})$. Let $\mathcal{J} = \mathfrak{P}(I_{+-}) - \mathcal{I}$ be the complementary filter, $K_1, \dots, K_\mu$ the minimal elements in $\mathcal{J}$ and $\mathcal{I}_i = K_i \downarrow$ the corresponding principal ideals. For $K \subseteq I_{+-}$ define $\varrho_K^+ = \{(X_j : v_j) \mid j \in K\} \cup \{(X_j : v_j) \mid j \in I^-\}$ and $\varrho_K^- = \{(X_j : v_j) \mid j \in K\}$ using arbitrary fixed elements $v_j \in dom(X_j')$. For $i = 1, \dots, \mu$ define $M_{1i} = \{\varrho_K^+ \mid K \in \mathcal{I}_i$ with odd distance to $\emptyset\}$ and $M_{2i} = \{\varrho_K^- \mid K \in \mathcal{I}_i$ with even distance to $\emptyset\}$. Further for $j = 1, 2$ let $M_j^-$ be the multiset union of all $M_{ji}$ for $i = 1, \dots, \mu$.

For $I_+ = \{i_1, \dots, i_k\}$ consider again $\bar{X}^+ = \bar{X}(X_{i_1}\{X'_{i_1}\}, \dots, X_{i_k}\{X'_{i_k}\})$. Ignoring the equivalenve arising from restructuring we get $dd(\bar{X}^+) \leq i$, so by induction with respect to the defect coincidence ideal $\bar{\mathcal{G}}^+ = \{Y \in \bar{\mathcal{G}} \mid \bar{X}^+ \geq Y\}$ we obtain multisets $M_1^+$, $M_2^+$ with $\langle \pi_Y^{\bar{X}^+}(\tau) \mid \tau \in M_1^+ \rangle = \langle \pi_Y^{\bar{X}^+}(\tau) \mid \tau \in M_2^+ \rangle$ iff $Y \in \bar{\mathcal{G}}^+$ holds. Finally, for $j = 1, 2$ define $M_j = \langle \tau^+ \cup \tau^- \mid \tau^+ \in M_j^+, \tau^- \in M_j^- \rangle$.

Now look at $Z = X\langle \bar{Z} \rangle \in \mathcal{S}(X)$. First assume $\bar{Z} = \bar{X}(Z_1, \dots, Z_n)$. Then $\bar{Z} \in \bar{\mathcal{G}}$ holds iff $Z_i = \lambda$ for all $i \in I^-$, $\{j \in I_{+-} \mid Z_j \neq \lambda\} \subseteq J_i$ for some $i \in \{1, \dots, \nu\}$, and $\bar{X}(Z_{i_1}, \dots, Z_{i_k}) \in \bar{\mathcal{G}}^+$ hold. Furthermore, $\pi_{\bar{Z}}^{\bar{X}}(\tau^+ \cup \tau^-) = \pi_{\bar{X}(Z_{i_1}, \dots, Z_{i_k})}^{\bar{X}}(\tau^+) \cup \pi_{\bar{X}(Z_{j_1}, \dots, Z_{j_\ell})}^{\bar{X}_{+-}}(\tau^-)$ using $I_{+-} = \{j_1, \dots, j_\ell\}$ and $\bar{X}_{+-} = \bar{X}(X_{j_1}\{X'_{j_1}\}, \dots, X_{j_\ell}\{X'_{j_\ell}\})$. So all we need to show is $\langle \pi_{\bar{X}(Z_{j_1}, \dots, Z_{j_\ell})}^{\bar{X}_{+-}}(\tau) \mid \tau \in M_1^- \rangle = \langle \pi_{\bar{X}(Z_{j_1}, \dots, Z_{j_\ell})}^{\bar{X}_{+-}}(\tau) \mid \tau \in M_2^- \rangle$ iff $\{j \in I_{+-} \mid Z_j \neq \lambda\} \subseteq J_i$ for some $i \in \{1, \dots, \nu\}$. This follows immediately from our construction of $M_j^-$, which is the same construction as for the base case $dd(U) = 0$ applied to $U = \bar{X}(X_{j_1}\{\lambda\}, \dots, X_{j_\ell}\{\lambda\})$.

Next let $\bar{Z} = \bar{X}_I\{\lambda\}$. If $I \cap I_+ \neq \emptyset$ holds, we already have seen $\langle \pi_{\bar{Z}}^{\bar{X}}(\tau^+) \mid \tau^+ \in M_1^+ \rangle = \langle \underbrace{\{\top\}}_{|M_1^+| \text{ times}} \rangle = \langle \pi_{\bar{Z}}^{\bar{X}}(\tau^+) \mid \tau^+ \in M_2^+ \rangle$, hence also $\langle \pi_{\bar{Z}}^{\bar{X}}(\tau) \mid \tau \in M_1 \rangle = \langle \underbrace{\{\top\}}_{|M_1| \text{ times}} \rangle = \langle \pi_{\bar{Z}}^{\bar{X}}(\tau) \mid \tau \in M_2 \rangle$. Therefore, we can assume $I \cap I_+ = \emptyset$. Due to our construction and the fact that $\bar{X}_I\{\lambda\} \in \bar{\mathcal{G}}$ iff $\bar{X}_{I-I_-}\{\lambda\} \in \bar{\mathcal{G}}$ holds, we may even assume $I \subseteq I_{+-} \cup I^-$. In case $I \cap I^- \neq \emptyset$, thus in particular $\bar{X}_I\{\lambda\} \notin \bar{\mathcal{G}}$, we have $\pi_{\bar{Z}}^{\bar{X}}(\tau) \neq \emptyset$ for all $\tau \in M_1^-$, while $\pi_{\bar{Z}}^{\bar{X}}(\tau) = \emptyset$ for at least one $\tau \in M_2^-$, so we have the desired inequality in this case. Finally, assume $I \subseteq I_{+-}$. Then $\bar{X}_I\{\lambda\} \in \bar{\mathcal{G}}$ holds iff $I \subseteq J_i$ for some $i \in \{1, \dots, \nu\}$, or $K_j \not\subseteq I$ for all $j = 1, \dots, \mu$. In case $K_j \subseteq I$ we obtain $\langle \pi_{\bar{Z}}^{\bar{X}}(\tau) \mid \tau \in M_{1j} \rangle = \langle \underbrace{\{\top\}}_{|M_{1j}| \text{ times}} \rangle$, while $\langle \pi_{\bar{Z}}^{\bar{X}}(\tau) \mid \tau \in M_{2j} \rangle = \langle \emptyset, \underbrace{\{\top\}}_{|M_{2j}| \text{ times}} \rangle$ holds, which implies the desired inequality, and completes subcase 1.

In subcase 3 we must have $\bar{X}_{I^-}\{\lambda\} \notin \bar{\mathcal{G}}$ due to property 4(a) of Theorem 10 and in particular $I^- \neq \emptyset$. If we now partition $I^+$ into $I_+$, $I_{+-}$ and $I_-$ as in the previous subcase using property 6(a) of Theorem 10, then property 6(a)(ii.) of Theorem 10 and the fact that $\bar{X}_I\{\lambda\} \notin \bar{\mathcal{G}}$ holds for $I \cap I^- \neq \emptyset$ implies that $I_+ = \emptyset$. Then we can apply the same construction as in the subcase 1 with the only difference that we now have to take $M_j = M_j^-$. The proof of the desired equalities and inequalities remains the same.

So let us concentrate on the remaining subcase 2, in which we partition $I^+ = I_+ \cup I_- \cup I_{+-}$ with maximal $I_+$ and $I_-$, so that we can establish the following properties in the same way as in subcase 2 of the set case (illustrated in Figure 6):

- $\bar{X}_{I'}\{\lambda\} \in \bar{\mathcal{G}}$ holds for all $I'$ with $I \cap I_+ \neq \emptyset$,
- $\bar{X}_{J_- \cup J}\{\lambda\} \in \bar{\mathcal{G}}$ holds iff $\bar{X}_J\{\lambda\} \in \bar{\mathcal{G}}$ for all $J \subseteq I^- \cup I_{+-}$ and all $J_- \subseteq I_-$,
- $I^- = I_1^- \dot{\cup} I_2^-$ and for all $I' \subseteq I^-$ we have $\bar{X}_{I'}\{\lambda\} \notin \bar{\mathcal{G}}$ iff $\emptyset \neq I' \subseteq I_j^-$ holds for $j = 1$ or $j = 2$,
- $I_{+-} = I_{+-}^1 \cap I_{+-}^2$, and $\bar{X}_{I'}\{\lambda\} \notin \bar{\mathcal{G}}$ holds for $I' \subseteq I_{+-} \cup I^-$ with $I' \cap I^- \neq \emptyset$ iff $I' = J_{+-} \cup J_-$ with $J_{+-} \subseteq I_{+-}^j$ and $J^- \subseteq I_j^-$ for $j = 1$ or $j = 2$.

Now we have to distinguish two more cases: $\bar{X}_{I_{+-}}\{\lambda\} \notin \bar{\mathcal{G}}$ or $\bar{X}_{I_{+-}}\{\lambda\} \in \bar{\mathcal{G}}$. In the first of these cases for $K \subseteq I_{+-}$ define

$$\begin{aligned}
\varrho_K &= \{(X_j : v_j) \mid j \in K\} \cup \{(X_j : v_j) \mid j \in I^-\}, \\
\varrho_K^1 &= \{(X_j : v_j) \mid j \in I_{+-} - K\} \cup \{(X_j : v_j) \mid j \in I_2^-\}, \text{ and} \\
\varrho_K^2 &= \{(X_j : v_j) \mid j \in I_{+-} - K\} \cup \{(X_j : v_j) \mid j \in I_1^-\}.
\end{aligned}$$

Take $J_1, \dots, J_\nu \subseteq I_{+-}$ maximal with $\bar{X}_{J_i}\{\lambda\} \in \bar{\mathcal{G}}$, let $\mathcal{I}$ be the generated ideal in $\mathfrak{P}(I_{+-})$, and let $\mathcal{J} = \mathfrak{P}(I_{+-}) - \mathcal{I}$ be the corresponding filter. Let $K_1, \dots, K_\mu$ be minimal in $\mathcal{J}$ and define the corresponding principal ideals $\mathcal{I}_i = K_i \downarrow$. Define $M_{1i} = \{\varrho_K \mid K \in \mathcal{I}_i \text{ with odd distance to } \emptyset\}$ and $M_{2i} = \{\varrho_K \mid K \in \mathcal{I}_i \text{ with even distance to } \emptyset\}$, and then define $M_1^-$ as the multiset union of all $M_{1i}$ for $i = 1, \dots, \mu$ and $\langle \varrho_{I_{+-} - I_{+-}^1}, \varrho_{I_{+-} - I_{+-}^2} \rangle$, and $M_2^-$ as the multiset union of all $M_{2i}$ for $i = 1, \dots, \mu$ and $\langle \varrho^1_{I^1_{+-}}, \varrho^2_{I^2_{+-}} \rangle$.

As in subcase 1 we obtain $M_1^+$ and $M_2^+$ inductively for $I_+ = \{i_1, \dots, i_k\}$, $\bar{X}^+ = \bar{X}(X_{i_1}\{X'_{i_1}\}, \dots \ \dots, X_{i_k}\{X'_{i_k}\})$ and $\bar{\mathcal{G}}^+ = \{Y \in \bar{\mathcal{G}} \mid \bar{X}^+ \geq Y\}$, so we define $M_j = \langle \tau^+ \cup \tau^- \mid \tau^+ \in M_j^+, \tau^- \in M_j^- \rangle$.

Now look at $Z = X\langle \bar{Z} \rangle \in \mathcal{S}(X)$. First assume $\bar{Z} = \bar{X}(Z_1, \dots, Z_n) \in \bar{\mathcal{G}}$, so we have $\pi_{\bar{Z}}^{\bar{X}}(\tau^+ \cup \tau^-) = \pi^{\bar{X}^+}_{\bar{X}(Z_{i_1}, \dots, Z_{i_k})}(\tau^+) \cup \pi^{\bar{X}_{+-}}_{\bar{X}(Z_{j_1}, \dots, Z_{j_\ell})}(\tau^-)$ using $I_{+-} = \{j_1, \dots, j_\ell\}$ and $\bar{X}_{+-} = \bar{X}(X_{j_1}\{X'_{j_1}\}, \dots \ \dots, X_{j_\ell}\{X'_{j_\ell}\})$.

Using the same arguments as in the case of degeneration depth 0 we obtain $\langle \pi^{\bar{X}_{+-}}_{\bar{X}(Z_{j_1}, \dots, Z_{j_\ell})}(\tau) \mid \tau \in M_1^- \rangle = \langle \pi^{\bar{X}_{+-}}_{\bar{X}(Z_{j_1}, \dots, Z_{j_\ell})}(\tau) \mid \tau \in M_2^- \rangle$ iff $\{j \in I_{+-} \mid J_j \neq \lambda\} \subseteq J_i$ holds, which follows from $\bar{Z} \in \bar{\mathcal{G}}$. For $\bar{Z} = \bar{X}(Z_1, \dots, Z_n) \notin \bar{\mathcal{G}}$ we must have $\{j \in I_{+-} \mid J_j \neq \lambda\} \not\subseteq J_i$ for all $i = 1, \dots, \nu$ or $Z_j \neq \lambda$ for some $j \in I^-$ or $\bar{X}(Z_{i_1}, \dots, Z_{i_k}) \notin \bar{\mathcal{G}}^+$. In all three cases the desired inequality is obvious.

Next look at $\bar{Z} = \bar{X}_I\{\lambda\}$. If $I \cap I_+ \neq \emptyset$ holds (hence $\bar{Z} \in \bar{\mathcal{G}}$), the desired equality follows from the construction of $M_1^+$ and $M_2^+$. So let us assume $I \subseteq I_{+-} \cup I^-$ – as in the previous subcase we may again ignore $I_-$. For $I \cap I^- \neq \emptyset$ we get $\bar{Z} \notin \bar{\mathcal{G}}$ iff $I \cap I_{+-} \subseteq I_{+-}^j$ and $I \cap I^- \subseteq I_j^-$ for $j = 1$ or $j = 2$. In this case only $\pi_{\bar{Z}}^{\bar{X}}(\varrho^j_{I^j_{+-}}) = \emptyset$ holds, which yields the desired inequality. Finally, for $I \subseteq I_{+-}$ we may concentrate on the multisets $M_{ji}$. For $K_i \subseteq I$ we obtain that $\langle \pi^{\bar{X}}_{\bar{Z}}(\tau) \mid \tau \in M_{1i} \rangle$ only contains non-empty sets, while $\langle \pi^{\bar{X}}_{\bar{Z}}(\tau) \mid \tau \in M_{2i} \rangle$ contains one empty set.

Now address the other case: $\bar{X}_{I_{+-}}\{\lambda\} \in \bar{\mathcal{G}}$. Here we use

$$\bar{X}^{+}_{+-} = \bar{X}(X_{i_1}\{X'_{i_1}\}, \dots, X_{i_k}\{X'_{i_k}\}, X_{j_1}\{X'_{j_1}\}, \dots, X_{j_\ell}\{X'_{j_\ell}\})$$

and $\bar{\mathcal{G}}^{+}_{+-} = \{Y \in \bar{\mathcal{G}} \mid \bar{X}^{+} \geq Y\}$ ignoring counter-attributes. So, inductively we obtain multisets $M_1^{+}$ and $M_2^{+}$ with $\langle \pi^{\bar{X}}_{\bar{Z}}(\tau) \mid \tau \in M_1^{+} \rangle = \langle \pi^{\bar{X}}_{\bar{Z}}(\tau) \mid \tau \in M_2^{+} \rangle$ iff $\bar{Z} \in \bar{\mathcal{G}}^{+}_{+-}$. Thus, $M_1^{+}$ and $M_2^{+}$ are sufficient for the desired equalities and inequalities with respect to all subattributes that can be written as record attributes. However, for $I \subseteq I_{+-}$ with $\bar{Z} = \bar{X}_I\{\lambda\} \in \bar{\mathcal{G}}$, but $\bar{X}(X_{j_1}\{X'_{j_1}\}, \dots, X_{j_\ell}\{X'_{j_\ell}\})$ for $I = \{j_1, \dots, j_\ell\}$ we still have $\langle \pi^{\bar{X}}_{\bar{Z}}(\tau) \mid \tau \in M_1^{+} \rangle \neq \langle \pi^{\bar{X}}_{\bar{Z}}(\tau) \mid \tau \in M_2^{+} \rangle$, but we need equality here. Note that the sets $I$ with this property form a filter $\mathcal{J}$ in $\mathfrak{P}(I_{+-})$ due to property 4(a) of Theorem 10. If $I$ is minimal in $\mathcal{J}$, then define $M_{I1} = \{\varrho_K \mid K \subseteq I$ with even distance to $I\}$ and $M_{I2} = \{\varrho_K \mid K \subseteq I$ with odd distance to $I\}$, where $\varrho_K = \{(X_j : \tau^{X'_j}_{X'_j}) \mid j \in I - K\}$, then add $M_{I1}$ and $M_{I2}$ to $M_1^{+}$ and $M_2^{+}$ (or the other way round), respectively, as many times as necessary to equalise the occurrences of $\emptyset$ in $\langle \pi^{\bar{X}}_{\bar{Z}}(\tau) \mid \tau \in M_j^{+} \rangle$ for $j = 1, 2$. This is possible, because $M_{I1}$ adds exactly one such occurrence. In doing so, none of the previously established equalities and inequalities will be destroyed, and we can continue the procedure with the filter $\mathcal{J} - \{I\}$. Finally, replace $\tau \in M_1^{+}$ (or $M_2^{+}$, respectively) by $\tau \cup \{(X_j : v_j) \mid j \in I^{-}\}$, and define $M_1 = M_1^{+} \uplus \langle \varrho_{I_{+-} - I^1_{+-}}, \varrho_{I_{+-} - I^2_{+-}} \rangle$ and $M_2 = M_2^{+} \uplus \langle \varrho^1_{I^1_{+-}}, \varrho^2_{I^2_{+-}} \rangle$ as in the previous subcase. This gives the necessary equalities and inequalities for $\bar{Z} = \bar{X}_I\{\lambda\}$ with $I \cap I^{-} \neq \emptyset$ thereby completing subcase 2.

Now let $\bar{X} = \bar{X}\langle X_1(X'_1) \oplus \cdots \oplus X_n(X'_n) \rangle \equiv \bar{X}(X_1\langle X'_1 \rangle, \dots, X_n\langle X'_n \rangle)$ with $dd(\bar{X}) = i + 1$. Without loss of generality we may assume again $X = X\langle \bar{X} \rangle$, so we take $\bar{\mathcal{G}} = \{Y \in \mathcal{S}(\bar{X}) \mid X\langle Y \rangle \in \mathcal{G}\}$. As in the base case with $dd(\bar{X}) = 0$ we consider the filter $\mathcal{H} = \mathcal{S}(\bar{X}) - \bar{\mathcal{G}}$. Basically, we apply the same construction as for the base case, i.e. for $j = 1, 2$ we construct $M_j$ as the multiset union of $M_{ji}$ for $i = 1, \dots, \nu$, where $M_{1i}$ and $M_{2i}$ are determined by a principal ideal $\mathcal{I}_i = Z_i \downarrow$ with $Z_i \in \mathcal{H}$; however, we adopt the following modifications:

1. For $Z_1, \dots, Z_\nu$ we take all elements in $\mathcal{H}$, not just the minimal ones. This does not do harm to the construction.

2. If $Z_i = \bar{X}(X_{j_1}\langle \lambda \rangle, \dots, X_{j_\mu}\langle \lambda \rangle)$ holds with $\bar{X}_{j_1,\dots,j_\mu}\langle \lambda \rangle \in \bar{\mathcal{G}}$, then we modify the construction such that $\langle \pi^{\bar{X}}_{\bar{X}_I\langle \lambda \rangle}(\tau) \mid \tau \in M_{1i} \rangle = \langle \pi^{\bar{X}}_{\bar{X}_I\langle \lambda \rangle}(\tau) \mid \tau \in M_{2i} \rangle$ holds for all $I$.

3. If $Z_i = \bar{X}(X_{j_1}\langle \lambda \rangle, \dots, X_{j_\mu}\langle \lambda \rangle)$ holds with $\bar{X}_{j_1,\dots,j_\mu}\langle \lambda \rangle \notin \bar{\mathcal{G}}$, then we modify the construction such that $\langle \pi^{\bar{X}}_{\bar{X}_I\langle \lambda \rangle}(\tau) \mid \tau \in M_{1i} \rangle = \langle \pi^{\bar{X}}_{\bar{X}_I\langle \lambda \rangle}(\tau) \mid \tau \in M_{2i} \rangle$ holds for all $I$ except $\{j_1, \dots, j_\mu\}$.

As the claimed properties 2 and 3 follow from Lemma 16 below, this completes the proof for this case.

Finally, let $\bar{X} = \bar{X}[X_1(X_1') \oplus \cdots \oplus X_n(X_n')]$ with $dd(\bar{X}) = i+1$, and without loss of generality assume $X = X\langle\bar{X}\rangle$. Then take $\bar{\mathcal{G}} = \{Y \in \mathcal{S}(\bar{X}) \mid X\langle Y\rangle \in \mathcal{G}\}$. First constructing multisets using the extended ideal $\bar{\mathcal{G}}_{\text{ext}} = \bar{\mathcal{G}} \cup \{\bar{X}(X_{j_1}[\lambda], \dots, X_{j_k}[\lambda])\}$ we may ignore all counter-attributes and obtain inductively multisets $M_1^{\sim}$ and $M_2^{\sim}$ with $\langle \pi_{\bar{Z}}^{\bar{X}}(\tau) \mid \tau \in M_1^{\sim}\rangle = \langle \pi_{\bar{Z}}^{\bar{X}}(\tau) \mid \tau \in M_2^{\sim}\rangle$ iff $\bar{Z} \in \bar{\mathcal{G}}_{\text{ext}}$ holds.

Secondly, we ignore the order and treat $\bar{X}$, as if it were a multiset attribute instead of a list attribute. Then the previous case gives us multisets $M_1^{\langle\rangle}$ and $M_2^{\langle\rangle}$ with the desired properties except for subattributes of the form $\bar{X}[X_{j_1}(Y_{j_1}) \oplus \cdots \oplus X_{j_k}(Y_{j_k})]$. We concatenate the elements in $M_j^{\langle\rangle}$ according to the order of the indices. Then $\langle \pi_{\bar{Z}}^{\bar{X}}(\tau) \mid \tau \in M_1^{\langle\rangle}\rangle = \langle \pi_{\bar{Z}}^{\bar{X}}(\tau) \mid \tau \in M_2^{\langle\rangle}\rangle$ holds iff either $\bar{Z} \in \bar{\mathcal{G}}$ and $\bar{Z} \neq \bar{X}[X_{j_1}(Y_{j_1}) \oplus \cdots \oplus X_{j_k}(Y_{j_k})]$ or $\bar{Z} = \bar{X}[X_{j_1}(Y_{j_1}) \oplus \cdots \oplus X_{j_k}(Y_{j_k})]$ and $\bar{X}(X_{j_1}[Y_{j_1}], \dots, X_{j_k}[Y_{j_k}]) \in \bar{\mathcal{G}}$. Hence, taking $M_j = M_j^{\langle\rangle} \uplus M_j^{\sim}$ $(j = 1, 2)$ completes the proof for this case and hence the theorem.

We still have to show the following lemma.

**Lemma 16.**

1. *For each $k \geq 2$ there exist finite multisets $M_1$, $M_2$ of $k$-tuples in $\mathbb{N}^k$ such that the following properties hold:*
   - $\langle \pi_I(\tau) \mid \tau \in M_1\rangle = \langle \pi_I(\tau) \mid \tau \in M_2\rangle$ *iff* $I \neq \{1, \dots, k\}$;
   - $\langle (1, \dots, 1) \bullet \tau \mid \tau \in M_1\rangle = \langle (1, \dots, 1) \bullet \tau \mid \tau \in M_2\rangle$.
2. *For each $k, \ell \in \mathbb{N}$ with $\ell < k$ there are finite multisets $M_1$, $M_2$ of $k$-tuples in $\mathbb{N}^k$ such that the following properties hold:*
   - $\langle \pi_I(\tau) \mid \tau \in M_1\rangle = \langle \pi_I(\tau) \mid \tau \in M_2\rangle$ *iff* $\{1, \dots, \ell\} \not\subseteq I$;
   - $\langle (1, \dots, 1) \bullet \pi_I(\tau) \mid \tau \in M_1\rangle = \langle (1, \dots, 1) \bullet \pi_I(\tau) \mid \tau \in M_2\rangle$ *iff* $I \neq \{1, \dots, \ell\}$.

Here $\bullet$ denotes the standard scalar product.

*Proof.* For the first statement we use the symmetric group $S_{k+1}$ and the alternating normal subgroup $A_{k+1}$ to define

$$M_1 = \langle (\sigma(1), \dots, \sigma(k)) \mid \sigma \in A_{k+1}\rangle \quad \text{and}$$
$$M_2 = \langle (\sigma(1), \dots, \sigma(k)) \mid \sigma \in S_{k+1} - A_{k+1}\rangle.$$

For $I \neq \{1, \dots, k\}$ there is some $j \notin I$. If $(\sigma(1), \dots, \sigma(k)) \in M_1$, then using $\tau = \sigma \circ (j, k+1) \notin A_{k+1}$ gives $\sigma(i) = \tau(i)$ for all $i \notin \{j, k+1\}$, hence the first property. The second property follows immediately from $(x_1, \dots, x_k) \in M_1$ iff $(x_2, x_1, x_3, \dots, x_k) \in M_2$.

For the second statement we may without loss of generality construct multisets of $k$-tuples in $\mathbb{Z}^k$ with the desired properties. Adding a sufficiently large positive constant $c$ to all values then gives the desired tuples in $\mathbb{N}^k$.

We use induction on $k - \ell$. So for the base case let $k = \ell + 1$. Let $A_\ell$ and $B_\ell$ be the sets of $0,1$-tuples of length $\ell$ with odd or even, respectively, occurrences of 1. We embed $A_\ell$ and $B_\ell$ into $k$-dimensional space yielding

$$\begin{aligned} A^1_{\ell+1} &= \{(x_1, \dots, x_\ell, 0) \mid (x_1, \dots, x_\ell) \in A_\ell\} \quad \text{and} \\ B^1_{\ell+1} &= \{(x_1, \dots, x_\ell, 0) \mid (x_1, \dots, x_\ell) \in B_\ell\}. \end{aligned}$$

Then define

$$A^2_{\ell+1} = B^1_{\ell+1} + (-1, \dots, -1, \ell) \text{ and } B^2_{\ell+1} = A^1_{\ell+1} + (-1, \dots, -1, \ell),$$

respectively, and $A_{\ell+1} = A^1_{\ell+1} \cup A^2_{\ell+1}$, $B_{\ell+1} = B^1_{\ell+1} \cup B^2_{\ell+1}$.

1. As $(-1, \dots, -1) \in \pi_{\{1,\dots,\ell\}}(A_{\ell+1}) - \pi_{\{1,\dots,\ell\}}(B_{\ell+1})$, we obviously have $\langle \pi_I(\tau) \mid \tau \in A_{\ell+1}\rangle \neq \langle \pi_I(\tau) \mid \tau \in B_{\ell+1}\rangle$ for $\{1, \dots, \ell\} \subseteq I$.

2. For $I$ with $\{1, \dots, \ell\} \nsubseteq I$ we may take $I = \{2, \dots, k\}$ without loss of generality. For $(0, x_2, \dots, x_\ell, 0) \in A^1_{\ell+1}$ we get $(1, x_2, \dots, x_\ell, 0) \in B^1_{\ell+1}$, and for $(1, x_2, \dots, x_\ell, 0) \in A^1_{\ell+1}$ we get $(0, x_2, \dots, x_\ell, 0) \in B^1_{\ell+1}$. Similarly, for $(0, x_2, \dots, x_\ell, \ell) \in A^2_{\ell+1}$ we must have $(1, x_2 + 1, \dots, x_\ell + 1, 0) \in B^1_{\ell+1}$, hence $(0, x_2 + 1, \dots, x_\ell + 1, 0) \in A^1_{\ell+1}$ and $(-1, x_2, \dots, x_\ell, \ell) \in B^2_{\ell+1}$. Analogously, $(-1, x_2, \dots, x_\ell, \ell) \in A^2_{\ell+1}$ implies $(0, x_2, \dots, x_\ell, \ell) \in B^2_{\ell+1}$ and vice versa. This shows $\langle \pi_I(\tau) \mid \tau \in A_{\ell+1}\rangle = \langle \pi_I(\tau) \mid \tau \in B_{\ell+1}\rangle$ for $I$ with $\{1, \dots, \ell\} \nsubseteq I$.

3. Due to 2 we already know $\langle (1, \dots, 1) \bullet \pi_I(\tau) \mid \tau \in A_{\ell+1}\rangle = \langle (1, \dots, 1) \bullet \pi_I(\tau) \mid \tau \in B_{\ell+1}\rangle$ for $I \neq \{1, \dots, \ell\}$ and $I \neq \{1, \dots, k\}$. Each $(1, \dots, 1) \bullet \tau$ with $\tau \in A^1_{\ell+1}$ gives an odd number with $i$ occurring $\binom{\ell}{i}$ times. The same holds for $B^2_{\ell+1}$, as $(1, \dots, 1) \bullet (-1, \dots, -1, \ell) = 0$. The analogue with even numbers holds for $A^2_{\ell+1}$ and $B^1_{\ell+1}$, so we obtain equality also for $I = \{1, \dots, k\}$. For the remaining $I = \{1, \dots, \ell\}$ we obtain $-\ell = (1, \dots, 1) \bullet \pi_{\{1,\dots,\ell\}}(\tau)$ for some $\tau \in A_{\ell+1}$, but not so for $B_{\ell+1}$.

Now assume the claimed properties hold for $\ell < k - 1$, so we can assume multisets $M'_1$, $M'_2$ with the desired properties. We first define

$$\begin{aligned} M^a_1 = \langle (x_1, \dots, x_{k-1}, 0) \mid (x_1, \dots, x_{k-1}) \in M'_1\rangle \uplus \\ \langle (x_1 - 1, \dots, x_{k-1} - 1, k - 1) \mid (x_1, \dots, x_{k-1}) \in M'_2\rangle \end{aligned}$$

and

$$M_2^a = \langle (x_1, \dots, x_{k-1}, 0) \mid (x_1, \dots, x_{k-1}) \in M_2' \rangle \uplus \\ \langle (x_1 - 1, \dots, x_{k-1} - 1, k-1) \mid (x_1, \dots, x_{k-1}) \in M_1' \rangle$$

similar to the base case.

1. The first property holds by induction for all $I$ with $k \notin I$. However, as $\langle \pi_I(\tau) \mid \tau \in M_1^a \rangle = \langle \pi_I(\tau) \mid \tau \in M_2^a \rangle$ iff $\langle \pi_{I-\{k\}}(\tau) \mid \tau \in M_1' \rangle = \langle \pi_{I-\{k\}}(\tau) \mid \tau \in M_2' \rangle$, which by induction is equivalent to $\{1, \dots, \ell\} \nsubseteq I - \{k\}$, it even holds for all $I$.

2. The second property holds by induction for all $I$ with $k \notin I$. According to the construction of $M_1^a$ and $M_2^a$ the equality immediately extends to all $I \cup \{k\}$, unless $I = \{1, \dots, \ell\}$ and $\ell < k - 1$.

3. For $I = \{1, \dots, \ell\}$ and $\ell < k - 1$ define

$$M_1^b = \langle (x_1, \dots, x_{k-1}, k-1-\ell) \mid (x_1, \dots, x_{k-1}) \in M_1' \rangle \uplus \\ \langle (x_1 - 1, \dots, x_{k-1} - 1, \ell) \mid (x_1, \dots, x_{k-1}) \in M_2' \rangle$$

and

$$M_2^b = \langle (x_1, \dots, x_{k-1}, k-1-\ell) \mid (x_1, \dots, x_{k-1}) \in M_2' \rangle \uplus \\ \langle (x_1 - 1, \dots, x_{k-1} - 1, \ell) \mid (x_1, \dots, x_{k-1}) \in M_1' \rangle,$$

for which we can apply the same arguments to show properties in 1 and 2. Consequently, $M_1 = M_1^a \uplus M_1^b$ and $M_2 = M_2^a \uplus M_2^b$ will satisfy the desired properties, if we can prove the second property for $I = \{1, \dots, \ell, k\}$. For this $I$ we obtain

$$\begin{aligned} &\langle (1, \dots, 1) \bullet \pi_I(\tau) \mid \tau \in M_1 \rangle = \\ &\langle (1, \dots, 1) \bullet \pi_{\{1,\dots,\ell\}}(\tau) \mid \tau \in M_1' \rangle \uplus \\ &\quad \langle (1, \dots, 1) \bullet \pi_{\{1,\dots,\ell\}}(\tau) + (k-1-\ell) \mid \tau \in M_2' \rangle \uplus \\ &\qquad \langle (1, \dots, 1) \bullet \pi_{\{1,\dots,\ell\}}(\tau) + (k-1-\ell) \mid \tau \in M_1' \rangle \uplus \\ &\qquad\quad \langle (1, \dots, 1) \bullet \pi_{\{1,\dots,\ell\}}(\tau) \mid \tau \in M_2' \rangle = \\ &\langle (1, \dots, 1) \bullet \pi_I(\tau) \mid \tau \in M_2 \rangle. \end{aligned}$$

This completes the proof of the lemma.

The result in Theorem 12 covers the case, for which structural induction in the proof of Theorem 17 breaks down. We can now show the main result of this section.

**Theorem 17 (Central Theorem).** *Let $\mathcal{F} \subseteq \mathcal{S}(X)$ be an ideal for the nested attribute $X \in \mathcal{N}$ such that the union constructor appears in $X$ only directly inside a set-, list or multiset-constructor, and the properties in Theorem 10 are satisfied. Then $\mathcal{F}$ is a coincidence ideal.*

*Proof.* We may assume $\mathcal{F} \neq \mathcal{S}(X)$ without loss of generality. Then use structural induction on $X$, the case $X = \lambda$ being trivial.

For a simple attribute $X = A$ we can only have $\mathcal{F} = \{\lambda\}$, so we take $t_1 = a$ and $t_2 = a'$ with arbitrary $a, a' \in dom(A)$ satisfying $a \neq a'$.

For record attributes let $X = X(X_1, \dots, X_n)$ with $X\{\lambda\}, X\langle\lambda\rangle, X[\lambda] \notin \mathcal{S}(X)$, i.e. not all $X_i$ are set, multiset or list attributes, in which case $X$ would be equivalent to (or a subattribute of) a set, multiset or list attribute with a union attribute as the component attribute – we deal with these cases separately. For $i = 1, \dots, n$ let $\mathcal{F}_i = \{Y_i \in \mathcal{S}(X_i) \mid X(\lambda, \dots, \lambda, Y_i, \lambda, \dots, \lambda) \in \mathcal{F}\}$, which by property 7(a) of Theorem 10 is a coincidence ideal on $\mathcal{S}(X_i)$. So by induction there exist complex values $t_{ij}$ $(i = 1, \dots, n,\ j = 1, 2)$ with $\pi^{X_i}_{Y_i}(t_{i1}) = \pi^{X_i}_{Y_i}(t_{i2})$ iff $Y_i \in \mathcal{F}_i$. Defining $t_j = (t_{1j}, \dots, t_{nj})$ for $j = 1, 2$ gives $\pi^X_{X(Y_1,\dots,Y_n)}(t_j) = (\pi^{X_1}_{Y_1}(t_{1j}), \dots, \pi^{X_n}_{Y_n}(t_{nj}))$, hence $\pi^X_{X(Y_1,\dots,Y_n)}(t_1) = \pi^X_{X(Y_1,\dots,Y_n)}(t_2)$ iff $Y_i \in \mathcal{F}_i$ for all $i = 1, \dots, n$ iff $X(Y_1, \dots, Y_n) \in \mathcal{F}$, as for $i \neq i'$ the subattributes $X(\lambda, \dots, \lambda, Y_i, \lambda, \dots, \lambda)$ and $X(\lambda, \dots, \lambda, Y_{i'}, \lambda, \dots, \lambda)$ are reconsilable.

For list attributes let $X = X[X']$ with $X'$ not being a union attribute – this case will be dealt with separately. For $\mathcal{F} = \{\lambda\}$ take $t_1 = [v]$ with any value $v \in dom(X')$ and $t_2 = []$. Then obviously $\pi^X_{X[\lambda]}(t_1) = [\top] \neq [] = \pi^X_{X[\lambda]}(t_2)$ holds. In case $\mathcal{F} \neq \{\lambda\}$ take the embedded coincidence ideal (by property 7(b) of Theorem 10) $\mathcal{G} = \{Y \in \mathcal{S}(X') \mid X[Y] \in \mathcal{F}\}$. By induction there are $t'_1, t'_2 \in dom(X')$ with $\pi^{X'}_Y(t'_1) = \pi^{X'}_Y(t'_2)$ iff $Y \in \mathcal{G}$. Now define $t_j = [t'_j]$ $(j = 1, 2)$, which gives $\pi^X_{X[Y]}(t_j) = [\pi^{X'}_Y(t'_j)]$ for all $Y \in \mathcal{S}(X')$. Hence we get $\pi^X_{X[Y]}(t_1) = \pi^X_{X[Y]}(t_2)$ iff $Y \in \mathcal{G}$ iff $X[Y] \in \mathcal{F}$.

Analogously, for a set attribute $X = X\{X'\}$ with $X'$ not being a union attribute we can choose $t_1 = \{v\}$ with any value $v \in dom(X')$ and $t_2 = \emptyset$ in case $\mathcal{F} = \{\lambda\}$. In case $\mathcal{F} \neq \{\lambda\}$ take the embedded defect coincidence ideal (by property 7(d) of Theorem 10) $\mathcal{G} = \{Y \in \mathcal{S}(X') \mid X\{Y\} \in \mathcal{F}\}$ on $\mathcal{S}(X')$. Using statement 1 of Theorem 12 there exist $t_1, t_2 \in dom(X)$ with $\{\pi^{X'}_Y(\tau) \mid \tau \in t_1\} = \{\pi^{X'}_Y(\tau) \mid \tau \in t_2\}$ iff $Y \in \mathcal{G}$, i.e. $\pi^X_{X\{Y\}}(t_1) = \pi^X_{X\{Y\}}(t_2)$ iff $X\{Y\} \in \mathcal{F}$.

Analogously, for a multiset attribute $X = X\langle X'\rangle$ with $X'$ not being a union attribute we can choose $t_1 = \langle v\rangle$ with any value $v \in dom(X')$ and $t_2 = \langle\rangle$ in case $\mathcal{F} = \{\lambda\}$. In case $\mathcal{F} \neq \{\lambda\}$ take the embedded defect coincidence ideal (by property 7(e) of Theorem 10) $\mathcal{G} = \{Y \in \mathcal{S}(X') \mid X\langle Y\rangle \in \mathcal{F}\}$ on $\mathcal{S}(X')$. Using statement 2 of Theorem 12 there exist $t_1, t_2 \in dom(X)$ with $\langle\pi^{X'}_Y(\tau) \mid \tau \in t_1\rangle = \langle\pi^{X'}_Y(\tau) \mid \tau \in t_2\rangle$ iff $Y \in \mathcal{G}$, i.e. $\pi^X_{X\langle Y\rangle}(t_1) = \pi^X_{X\langle Y\rangle}(t_2)$ iff $X\langle Y\rangle \in \mathcal{F}$.

Now let $X = X\{X_1(X'_1) \oplus \cdots \oplus X_n(X'_n)\} \equiv X(X_1\{X'_1\}, \dots, X_n\{X'_n\})$. For $\mathcal{F} = \{\lambda\}$ take $t_1 = \{(X_1 : v_1), \dots, (X_n : v_n)\}$ with arbitrary values $v_i \in dom(X'_i)$

for $i = 1, \ldots, n$ and $t_2 = \emptyset$. This gives $\pi^X_{X_I\{\lambda\}}(t_1) \neq \emptyset$ for $I \neq \emptyset$ and $\pi^X_{X_I\{\lambda\}}(t_2) = \emptyset$ for all $I$, from which the claim follows immediately. So it is sufficient to consider $\mathcal{F} \neq \{\lambda\}$, for which we define $I^+ = \{i \in \{1, \ldots, n\} \mid X_{\{i\}}\{\lambda\} \in \mathcal{F}\}$ and $I^- = \{i \in \{1, \ldots, n\} \mid X_{\{i\}}\{\lambda\} \notin \mathcal{F}\}$. Then we consider the following three cases:

1. $X_{\{1,\ldots,n\}}\{\lambda\} \in \mathcal{F}$ and $X_{I^-}\{\lambda\} \notin \mathcal{F}$;
2. $X_{\{1,\ldots,n\}}\{\lambda\} \in \mathcal{F}$ and $X_{I^-}\{\lambda\} \in \mathcal{F}$;
3. $X_{\{1,\ldots,n\}}\{\lambda\} \notin \mathcal{F}$.

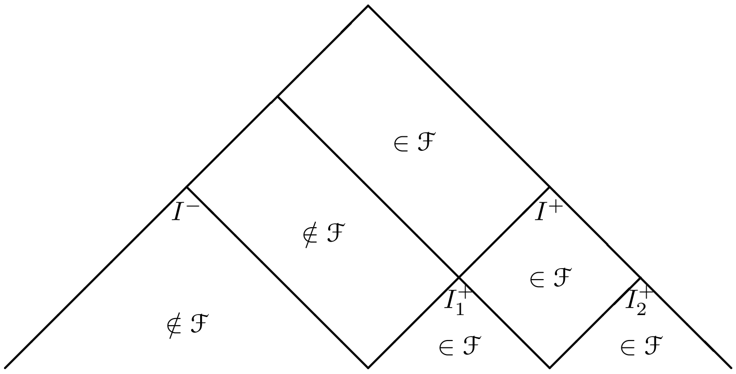


**Figure 7:** Counter Attributes for $X = X\{X_1(X_1') \oplus \cdots \oplus X_n(X_n')\}$ in Case 1

In case 1 we immediately obtain the following properties for counter attributes $X_I\{\lambda\}$:

– If $I \subseteq I^+$, then $X_I\{\lambda\} \in \mathcal{F}$. This follows immediately from property 5(a) in Theorem 10 and the definition of $I^+$.

– If $I \subseteq I^-$, then $X_I\{\lambda\} \notin \mathcal{F}$. Otherwise property 4(a) of Theorem 10 would imply $X(X_{i_1}\{X'_{i_1}\}, \ldots, X_{i_k}\{X'_{i_k}\}) \in \mathcal{F}$ for $I = \{i_1, \ldots, i_k\}$ and further $X(X_{i_j}\{\lambda\}) = X_{i_j}\{\lambda\} \in \mathcal{F}$ for $j = 1, \ldots, k$ by property 2 of coincidence ideals. However, this contradicts the definition of $I^-$.

Furthermore, we can partition $I^+$ into $I_1^+$ and $I_2^+$ defining $I_1^+ = \{i \in I^+ \mid \exists J_i \subseteq I^-.X_{J_i \cup \{i\}}\{\lambda\} \notin \mathcal{F}\}$ and $I_2^+ = I^+ - I_1^+$. Due to property 4(c) of Theorem 10 we have $I_2^+ \neq \emptyset$. This leads to the following two properties:

- For each non-empty $J \subseteq I^-$ and $I \subseteq I_1^+$ we must have $X_{I\cup J}\{\lambda\} \notin \mathcal{F}$. From property 4(d) of coincidence ideals in Theorem 10 it follows immediately that $X_{I\cup I^-}\{\lambda\} \notin \mathcal{F}$ holds. If we had $X_{I\cup J}\{\lambda\} \in \mathcal{F}$ for some $J \subseteq I^-$, then property 4(a) of Theorem 10 would lead again to the contradiction $X_{\{j\}}\{\lambda\} \in \mathcal{F}$ for $j \in J$.

- For each $J \subseteq I^-$ and each $I \subseteq I_1^+$ with $I \cap I_2^+ \neq \emptyset$ we obtain $X_{I\cup J}\{\lambda\} \in \mathcal{F}$. First, $X_{\{i\}\cup J}\{\lambda\} \in \mathcal{F}$ for $i \in I \cap I_2^+$ follows from the definition of $I_2^+$, then $X_{I-\{i\}}\{\lambda\} \in \mathcal{F}$ and property 5(a) imply the claimed property.

Figure 7 illustrates the various combinations of indices in $I^+$ and $I^-$ and the impact on $\mathcal{F}$.

For $I_2^+ = \{i_1, \dots, i_k\}$ define the subattribute $X^+$ by $X^+ = X(X_{i_1}\{X'_{i_1}\}, \dots$ $\dots, X_{i_k}\{X'_{i_k}\}) \equiv X\{X_{i_1}(X'_{i_1}) \oplus \dots \oplus X_{i_k}(X'_{i_k})\}$ and $\mathcal{F}^+ = \{Y \in \mathcal{F} \mid X^+ \geq Y\}$. Then $\mathcal{F}^+$ is a coincidence ideal on $\mathcal{S}(X^+)$. As for all $I \subseteq I_2^+$ we have $X_I\{\lambda\} \in \mathcal{F}$, properties 4, 5 and 6 of Theorem 10 follow immediately, while the other properties 1, 2, 3 and 7 follow from the corresponding properties of $\mathcal{F}$. As in the record case above define $\mathcal{F}_j = \{Y_j \in \mathcal{S}(X_{i_j}\{X'_{i_j}\}) \mid X(\lambda, \dots, \lambda, Y_j, \lambda, \dots, \lambda) \in \mathcal{F}^+\}$ for $j = 1, \dots, k$. Due to property 7(a) of coincidence ideals in Theorem 10 $\mathcal{F}_j$ is a coincidence ideal on $\mathcal{S}(X_{i_j}\{X'_{i_j}\})$. By induction there exist complex values $t_{i_j\ell} \in dom(X_{i_j}\{X'_{i_j}\})$ for $j = 1, \dots, k$ and $\ell = 1, 2$ such that $\pi_{Y_j}^{X_{i_j}\{X'_{i_j}\}}(t_{i_j1}) = \pi_{Y_j}^{X_{i_j}\{X'_{i_j}\}}(t_{i_j2})$ holds iff $Y_j \in \mathcal{F}_j$.

Define $t_\ell^+ = (t_{i_1\ell}, \dots, t_{i_k\ell}) \in dom(X^+)$ for $\ell = 1, 2$, which can be identified with $t_\ell^+ = \{(X_{i_j} : \tau_{i_j}) \mid \tau_{i_j} \in t_{i_j\ell}, j \in \{1, \dots, k\}\} \in dom(X)$. Then we obtain $\pi_Y^{X^+}(t_1^+) = \pi_Y^{X^+}(t_2^+)$ iff $Y \in \mathcal{F}^+$, because subattributes $X(\lambda, \dots, \lambda, Y_j, \lambda, \dots, \lambda)$ and $X(\lambda, \dots, \lambda, Y_{j'}, \lambda, \dots, \lambda)$ for $j \neq j'$ are reconsilable, $\mathcal{F}^+$ contains all counter attributes $X_I\{\lambda\}$ with $I \subseteq I_2^+$ and $X(X_{i_1}\{\lambda\}, \dots, X_{i_k}\{\lambda\}) \in \mathcal{F}$.

Finally, select arbitrary values $v_i \in dom(X'_i)$ for $i \in I^-$ and define $t_1^- = \{(X_i : v_i) \mid i \in I^-\}$, $t_1 = t_1^+ \cup t_1^-$ and $t_2 = t_2^+$. Then we obtain the following:

1. For $Y = X(Y_1, \dots, Y_n) \in \mathcal{F}$ we must have $Y_i = \lambda$ for all $i \in I^-$ and $Y^+ = X(Y_{i_1}, \dots, Y_{i_k}) \in \mathcal{F}^+$. This gives $\pi_Y^X(t_1) = \pi_{Y^+}^{X^+}(t_1^+) = \pi_{Y^+}^{X^+}(t_2^+) = \pi_Y^X(t_2)$ as desired.

2. For $Y = X(Y_1, \dots, Y_n) \notin \mathcal{F}$ we must have either $Y_i \neq \lambda$ for some $i \in I^-$ or $Y^+ = X(Y_{i_1}, \dots, Y_{i_k}) \notin \mathcal{F}^+$ in case this does not hold. In the first case we have $Y \geq X_{\{i\}}\{\lambda\}$ with $i \in I^-$ and from $\pi_{X_{\{i\}}\{\lambda\}}^X(t_1) \neq \emptyset = \pi_{X_{\{i\}}\{\lambda\}}^X(t_2)$ we obtain $\pi_Y^X(t_1) \neq \pi_Y^X(t_2)$ as desired.

3. For $Y = X_I\{\lambda\} \in \mathcal{F}$ we must have $I \subseteq I_1^+$ or $I \cap I_2^+ \neq \emptyset$. In the first case we have $\pi_Y^X(t_1) = \emptyset = \pi_Y^X(t_2)$, while in the second case due to the non-emptiness of $t_1^+$ and $t_2^+$ according to Theorem 12 we have $\pi_Y^X(t_1) = \{\top\} = \pi_Y^X(t_2)$.

4. For $Y = X_I\{\lambda\} \notin \mathcal{F}$ we must have $I \cap I_2^+ = \emptyset$ and $I \cap I^- \neq \emptyset$. Due to the second property we have $\pi_Y^X(t_1) \neq \emptyset$, while the first property gives $\pi_Y^X(t_2) = \emptyset$.

Taking these four cases together we have $\pi_Y^X(t_1) = \pi_Y^X(t_2)$ iff $Y \in \mathcal{F}$, which solves our claim in case 1.

In case 2 we immediately obtain the following properties for counter attributes $X_I\{\lambda\}$:

– If $I \subseteq I^+$, then $X_I\{\lambda\} \in \mathcal{F}$. This follows immediately from property 5(a) in Theorem 10 and the definition of $I^+$.

– There is a partition of $I^-$ into $I_1^-$ and $I_2^-$ such that $X_I\{\lambda\} \notin \mathcal{F}$ for all non-empty $I \subseteq I_1^-$ or $I \subseteq I_2^-$, whereas $X_I\{\lambda\} \in \mathcal{F}$ for $i \subseteq I^-$ with $I \cap I_1^- \neq \emptyset \neq I_2^-$. This follows directly from properties 4(b) and (a) in Theorem 10 together with the definition of $I^-$.

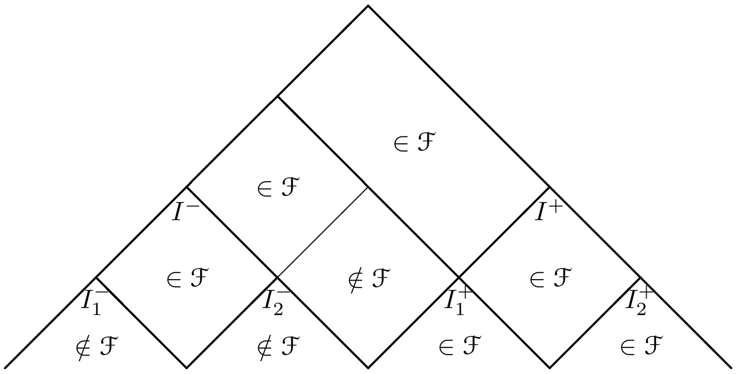


**Figure 8:** Counter Attributes for $X = X\{X_1(X_1') \oplus \cdots \oplus X_n(X_n')\}$ in Case 2

As in the previous case we partition $I^+$ into $I_1^+$ and $I_2^+$ defining $I_1^+ = \{i \in I^+ \mid \exists J_i \subseteq I^-.X_{J_i \cup \{i\}}\{\lambda\} \notin \mathcal{F}\}$ and $I_2^+ = I^+ - I_1^+$. This leads to the following properties:

– For each non-empty $J \subseteq I^-$ with $X_J\{\lambda\} \notin \mathcal{F}$ and $I \subseteq I_1^+$ we must have $X_{I \cup J}\{\lambda\} \notin \mathcal{F}$. This follows immediately from property 4(e) in Theorem 10.

– For each $J \subseteq I^-$ and each $I \subseteq I_1^+$ with $I \cap I_2^+ \neq \emptyset$ we obtain $X_{I \cup J}\{\lambda\} \in \mathcal{F}$. This follows from the definition of $I_2^+$ and property 5(a) in Theorem 10.

Figure 8 illustrates the various combinations of indices in $I^+$ and $I^-$ and the impact on $\mathcal{F}$.

As in the previous case define the subattribute $X^+$ by $X^+ = X(X_{i_1}\{X'_{i_1}\}, \dots, X_{i_k}\{X'_{i_k}\}) \equiv X\{X_{i_1}(X'_{i_1}) \oplus \dots \oplus X_{i_k}(X'_{i_k})\}$ and the coincidence ideal $\mathcal{F}^+ = \{Y \in \mathcal{F} \mid X^+ \geq Y\}$ on $\mathcal{S}(X^+)$, which allows us to use induction to obtain complex values $t_1^+, t_2^+$ with $\pi_Y^{X^+}(t_1^+) = \pi_Y^{X^+}(t_2^+)$ iff $Y \in \mathcal{F}^+$. Then select arbitrary values $v_i \in dom(X'_i)$ for $i \in I^-$ and define $t_1^- = \{(X_i : v_i) \mid i \in I_1^-\}$ and $t_2^- = \{(X_i : v_i) \mid i \in I_2^-\}$. Finally, take $t_j = t_j^+ \cup t_j^-$ for $j = 1, 2$. Then we obtain the following:

1. For $Y = X(Y_1, \dots, Y_n) \in \mathcal{F}$ we must have $Y_i = \lambda$ for all $i \in I^-$ and $Y^+ = X(Y_{i_1}, \dots, Y_{i_k}) \in \mathcal{F}^+$. This gives $\pi_Y^X(t_1) = \pi_{Y^+}^{X^+}(t_1^+) = \pi_{Y^+}^{X^+}(t_2^+) = \pi_Y^X(t_2)$ as desired.

2. For $Y = X(Y_1, \dots, Y_n) \notin \mathcal{F}$ we must have either $Y_i \neq \lambda$ for some $i \in I^-$ or $Y^+ = X(Y_{i_1}, \dots, Y_{i_k}) \notin \mathcal{F}^+$ – in case $I_2^+ = \emptyset$ the second case does not occur. In the first case we have $Y \geq X_{\{i\}}\{\lambda\}$ with $i \in I^-$ and from $\pi_{X_{\{i\}}\{\lambda\}}^X(t_1) \neq \emptyset = \pi_{X_{\{i\}}\{\lambda\}}^X(t_2)$ we obtain $\pi_Y^X(t_1) \neq \pi_Y^X(t_2)$ as desired.

3. For $Y = X_I\{\lambda\} \in \mathcal{F}$ we must have either $I \cap I_2^+ \neq \emptyset$ or $I \subseteq I_1^+$ or $I \cap I_1^- \neq \emptyset \neq I \cap I_2^-$ if none of these two hold. In the first case we have $\pi_Y^X(t_1) = \{\top\} = \pi_Y^X(t_2)$ due to the non-emptiness of $t_1^+$ and $t_2^+$ according to Theorem 12. In the second case we immediately get $\pi_Y^X(t_1) = \emptyset = \pi_Y^X(t_2)$ from the definition of $t_1$ and $t_2$. In the third case we have again $\pi_Y^X(t_1) = \{\top\} = \pi_Y^X(t_2)$ due to the definition of $t_1^-$ and $t_2^-$.

4. For $Y = X_I\{\lambda\} \notin \mathcal{F}$ we must have $I \cap I_2^+ = \emptyset$, $I \not\subseteq I_1^+$ and either $I \cap I^- \subseteq I_1^-$ or $I \cap I^- \subseteq I_2^-$. This implies $\pi_Y^X(t_j) = \emptyset$ for exactly one $j = 1$ or $j = 2$.

Taking these four cases together we have $\pi_Y^X(t_1) = \pi_Y^X(t_2)$ iff $Y \in \mathcal{F}$, which solves our claim in case 2.

In case 3 we must have $X_{I^-}\{\lambda\} \notin \mathcal{F}$, otherwise property 4(a) of Theorem 10 would lead to a contradiction. Furthermore, for $i \in I^+$ there must exist some $J_i \subseteq I^-$ with $X_{J_i \cup \{i\}}\{\lambda\} \notin \mathcal{F}$ for the same reason. Hence we obtain $X_I\{\lambda\} \in \mathcal{F}$ iff $I \subseteq I^+$, which is illustrated in Figure 9. Obviously, $I \subseteq I^+$ implies $X_I\{\lambda\} \in \mathcal{F}$ because of property 5(a), while $X_I\{\lambda\} \notin \mathcal{F}$ for $I \subseteq I^-$ follows from $X_{I^-}\{\lambda\} \notin \mathcal{F}$ and property 4(a). The remaining case $I = I' \cup J'$ with $\emptyset \neq I' \subseteq I^+$ and $\emptyset \neq J' \subseteq I^-$ leads to $X_I\{\lambda\} \notin \mathcal{F}$ due to property 4(d) of Theorem 10.

Now choose arbitrary values $v_i \in dom(X'_i)$ for $i \in I^-$ and define $t_1 = \{(X_i : v_i) \mid i \in I^-\}$ and $t_2 = \emptyset$. Then we get the following:

1. We have $Y = X(Y_1, \dots, Y_n) \in \mathcal{F}$ iff $Y_i = \lambda$ for all $i \in I^-$, which gives $\pi_Y^X(t_1) = \emptyset = \pi_Y^X(t_2)$ for $Y \in \mathcal{F}$, and $\pi_Y^X(t_1) \neq \emptyset = \pi_Y^X(t_2)$ for $Y \notin \mathcal{F}$.

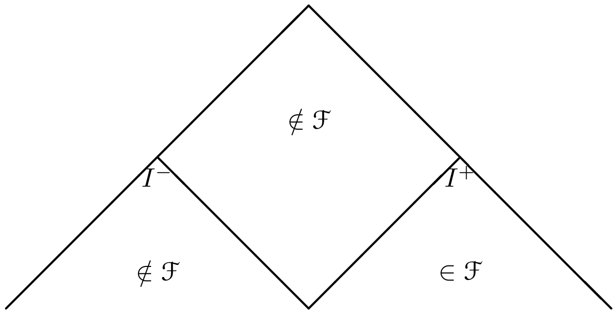


**Figure 9:** Counter Attributes for $X = X\{X_1(X_1') \oplus \cdots \oplus X_n(X_n')\}$ in Case 3

2. We have $X_I\{\lambda\} \in \mathcal{F}$ iff $I \subseteq I^+$, which gives $\pi_Y^X(t_1) = \emptyset = \pi_Y^X(t_2)$ for $Y \in \mathcal{F}$, and $\pi_Y^X(t_1) \neq \emptyset = \pi_Y^X(t_2)$ for $Y \notin \mathcal{F}$.

Both cases together give $\pi_Y^X(t_1) = \pi_Y^X(t_2)$ iff $Y \in \mathcal{F}$, which solves our claim in case 3.

Now let $X = X\langle X_1(X_1') \oplus \cdots \oplus X_n(X_n')\rangle \equiv X(X_1\langle X_1'\rangle, \ldots, X_n\langle X_n'\rangle)$. For $\mathcal{F} = \{\lambda\}$ take $t_1 = \langle (X_1 : v_1), \ldots, (X_n : v_n)\rangle$ with arbitrary values $v_i \in dom(X_i')$ for $i = 1, \ldots, n$ and $t_2 = \langle\rangle$. This gives $\pi^X_{X_I\langle\lambda\rangle}(t_1) \neq \langle\rangle = \pi^X_{X_I\langle\lambda\rangle}(t_2)$ for all $I \neq \emptyset$ and hence $\pi_Y^X(t_1) = \pi_Y^X(t_2)$ iff $Y = \lambda$. So it is sufficient to assume $\mathcal{F} \neq \{\lambda\}$. For this define $I^+ = \{i \in \{1, \ldots, n\} \mid X_{\{i\}}\langle\lambda\rangle \in \mathcal{F}\}$ and $I^- = \{i \in \{1, \ldots, n\} \mid X_{\{i\}}\langle\lambda\rangle \notin \mathcal{F}\}$. Due to property 5(c) of Theorem 10 we have $X_I\langle\lambda\rangle \in \mathcal{F}$ for all $I \subseteq I^+$. Furthermore, for $I \subseteq I^+$ and $J \subseteq I^-$ we have $X_{I\cup J}\langle\lambda\rangle \in \mathcal{F}$ iff $X_J\langle\lambda\rangle \in \mathcal{F}$ due to properties 5(c) and (e) of Theorem 10.

For $I^+ = \{i_1, \ldots, i_k\}$ define the subattribute $X^+$ as $X^+ = X(X_{i_1}\langle X_{i_1}'\rangle, \ldots, X_{i_k}\langle X_{i_k}'\rangle) \equiv X\langle X_{i_1}(X_{i_1}') \oplus \cdots \oplus X_{i_k}(X_{i_k}')\rangle$ and $\mathcal{F}^+ = \{Y \in \mathcal{F} \mid X^+ \geq Y\}$. As $\mathcal{F}^+$ contains all counter attributes $X_I\langle\lambda\rangle$ with $I \subseteq I^+$, it must be a coincidence ideal on $\mathcal{S}(X^+)$. In particular, due to property 7(a) of Theorem 10 all $\mathcal{F}_j = \{Y_j \in \mathcal{S}(X_{i_j}\langle X_{i_j}'\rangle) \mid X(\lambda, \ldots, \lambda, Y_j, \lambda, \ldots, \lambda) \in \mathcal{F}^+\}$ for $j = 1, \ldots, k$ are coincidence ideals on $\mathcal{S}(X_{i_j}\langle X_{i_j}'\rangle)$, respectively.

By induction there exist complex values $t_{i_j\ell} \in dom(X_{i_j}\langle X_{i_j}'\rangle)$ for $j = 1, \ldots, k$ and $\ell = 1, 2$ such that $\pi_{Y_j}^{X_{i_j}\langle X_{i_j}'\rangle}(t_{i_j1}) = \pi_{Y_j}^{X_{i_j}\langle X_{i_j}'\rangle}(t_{i_j2})$ iff $Y_j \in \mathcal{F}_j$. Define $t_\ell^+ = (t_{i_1\ell}, \ldots, t_{i_k\ell})$ for $\ell = 1, 2$, which can be identified with $\langle (X_{i_j} : \tau_{i_j}) \mid \tau_{i_j} \in t_{i_j\ell}, j = 1, \ldots, k\rangle \in dom(X)$. For these values we obtain $\pi_Y^{X^+}(t_1^+) = \pi_Y^{X^+}(t_2^+)$ iff $Y \in \mathcal{F}^+$ analogously to the set case above.

Now let $I^- = \{j_1, \ldots, j_\ell\}$ and construct positive integers $x_p, y_p$ $(p = 1, \ldots, \ell)$

such that for $J = \{j_{m_1}, \dots, j_{m_{|J|}}\} \subseteq I^-$ the equation

$$\sum_{p=1}^{|J|} x_{m_p} = \sum_{p=1}^{|J|} y_{m_p}$$

holds iff $X_J\langle\lambda\rangle \in \mathfrak{F}$ holds. For the selection of these $x_p$, $y_p$ we can take the following procedure:

```
for p = 1,...,ℓ :
    choose x_p, y_p such that all equations and inequations containing
        only x_i, y_i with 1 ≤ i ≤ p are satisfied;
    replace x_p, y_p in the remaining equations and inequations by the
        chosen values
endfor
```

Properties 5(c) and (g) of Theorem 10 ensure that this procedure always produces a solution for the given equations and inequations. Then define

$$t_1^- = \langle \underbrace{(X_{j_1} : v_{j_1})}_{x_{j_1}\text{-times}}, \dots, \underbrace{(X_{j_\ell} : v_{j_\ell})}_{x_{j_\ell}\text{-times}} \rangle \qquad \text{and} \qquad t_2^- = \langle \underbrace{(X_{j_1} : v_{j_1})}_{y_{j_1}\text{-times}}, \dots, \underbrace{(X_{j_\ell} : v_{j_\ell})}_{y_{j_\ell}\text{-times}} \rangle$$

with arbitrary values $v_i \in dom(X_i')$ for $i \in I^-$.

Finally, define $t_1 = t_1^+ \uplus t_1^-$ and $t_2 = t_2^+ \uplus t_2^-$ using multiset union $\uplus$ that adds multiplicities. For these values the following holds:

1. For $Y \leq X^+$ we have $\pi_Y^X(t_\ell) = \pi_Y^{X^+}(t_\ell^+)$ for $\ell = 1, 2$, which implies $\pi_Y^X(t_1) = \pi_Y^X(t_2)$ iff $Y \in \mathfrak{F}^+$.

2. For $Y \not\leq X^+$ we either have $Y \geq X_{\{j\}}\langle\lambda\rangle$ for some $j \in I^-$ or $Y = X_I\langle\lambda\rangle$ with $I \not\subseteq I^+$. In the first case we have $Y \notin \mathfrak{F}$ and $\pi^X_{X_{\{j\}}\langle\lambda\rangle}(t_1) = \pi^X_{X_{\{j\}}\langle\lambda\rangle}(t_1^-) \neq \pi^X_{X_{\{j\}}\langle\lambda\rangle}(t_2^-) = \pi^X_{X_{\{j\}}\langle\lambda\rangle}(t_2)$, hence also $\pi_Y^X(t_1) \neq \pi_Y^X(t_2)$.

   We have $\pi_Y^X(t_1) = \pi_Y^X(t_2)$ iff $\pi^X_{X_{I\cap I^-}\langle\lambda\rangle}(t_1^-) = \pi^X_{X_{I\cap I^-}\langle\lambda\rangle}(t_2^-)$ in the second case, as $\pi^X_{X_{I\cap I^+}\langle\lambda\rangle}(t_1) = \pi^X_{X_{I\cap I^+}\langle\lambda\rangle}(t_1^+) = \pi^X_{X_{I\cap I^+}\langle\lambda\rangle}(t_2^+) = \pi^X_{X_{I\cap I^+}\langle\lambda\rangle}(t_2)$ holds due to the construction of $t_1^+$ and $t_2^+$. Due to property 5(e) of Theorem 10 we have $Y \in \mathfrak{F}$ iff $X_{I\cap I^-}\langle\lambda\rangle \in \mathfrak{F}$. Then due to the construction of $t_1^-$ and $t_2^-$ we have $X_{I\cap I^-}\langle\lambda\rangle \in \mathfrak{F}$ iff $\pi^X_{X_{I\cap I^-}\langle\lambda\rangle}(t_1^-) = \pi^X_{X_{I\cap I^-}\langle\lambda\rangle}(t_2^-)$.

Both cases together imply $\pi_Y^X(t_1) = \pi_Y^X(t_2)$ iff $Y \in \mathfrak{F}$, which completes the case of a multiset attribute with a component union attribute.

Now let $X = X[X_1(X_1') \oplus \cdots \oplus X_n(X_n')] \geq X(X_1[X_1'], \dots, X_n[X_n'])$. For $\mathfrak{F} = \{\lambda\}$ take $t_1 = [(X_1 : v_1), \dots, (X_n : v_n)]$ with arbitrary values $v_i \in dom(X_i')$ for $i = 1, \dots, n$ and $t_2 = []$. This gives $\pi^X_{X_I[\lambda]}(t_1) \neq [] = \pi^X_{X_I[\lambda]}(t_2)$ for all $I \neq \emptyset$

and hence $\pi_Y^X(t_1) = \pi_Y^X(t_2)$ iff $Y = \lambda$. So it is sufficient to assume $\mathcal{F} \neq \{\lambda\}$. For this define $I^+ = \{i \in \{1,\dots,n\} \mid X_{\{i\}}[\lambda] \in \mathcal{F}\}$ and $I^- = \{i \in \{1,\dots,n\} \mid X_{\{i\}}[\lambda] \notin \mathcal{F}\}$. Due to property 5(b) of Theorem 10 we have $X_I[\lambda] \in \mathcal{F}$ for all $I \subseteq I^+$. Furthermore, for $I \subseteq I^+$ and $J \subseteq I^-$ we have $X_{I\cup J}[\lambda] \in \mathcal{F}$ iff $X_J[\lambda] \in \mathcal{F}$ due to properties 5(b) and (d) of Theorem 10.

For $I^+ = \{i_1,\dots,i_k\}$ define the subattributes $X^+ = X[X_{i_1}(X'_{i_1}) \oplus \cdots \oplus X_{i_k}(X'_{i_k})]$ and $\tilde{X} = X(X_{i_1}[X'_{i_1}],\dots,X_{i_k}[X'_{i_k}])$ with $X^+ \geq \tilde{X}$ and the coincidence ideals $\mathcal{F}^+ = \{Y \in \mathcal{F} \mid X^+ \geq Y\}$ and $\tilde{\mathcal{F}} = \{Y \in \mathcal{F} \mid \tilde{X} \geq Y\}$ on $\mathcal{S}(X^+)$ and $\mathcal{S}(\tilde{X})$, respectively. Due to property 7(a) of Theorem 10 all $\mathcal{F}_j = \{Y_j \in \mathcal{S}(X_{i_j}[X'_{i_j}]) \mid X(\lambda,\dots,\lambda,Y_j,\lambda,\dots,\lambda) \in \tilde{\mathcal{F}}\}$ for $j = 1,\dots,k$ are coincidence ideals on $\mathcal{S}(X_{i_j}[X'_{i_j}])$, respectively.

By induction there exist complex values $t_{i_j\ell} \in dom(X_{i_j}[X'_{i_j}])$ for $j = 1,\dots,k$ and $\ell = 1,2$ such that $\pi_{Y_j}^{X_{i_j}[X'_{i_j}]}(t_{i_j1}) = \pi_{Y_j}^{X_{i_j}[X'_{i_j}]}(t_{i_j2})$ iff $Y_j \in \mathcal{F}_j$. Now concatenate these lists $t_{i_j\ell}$ in the order of the indices $i_j$ to define lists $t_1^+, t_2^+ \in dom(X^+)$, respectively. For these values we obtain $\pi_Y^{\tilde{X}}(t_1^+) = \pi_Y^{\tilde{X}}(t_2^+)$ iff $Y \in \tilde{\mathcal{F}}$ analogously to the set case above.

For $k \leq 1$ we have $X^+ \equiv \tilde{X}$ and $\mathcal{F}^+ = \tilde{\mathcal{F}}$, so we get $\pi_Y^{X^+}(t_1^+) = \pi_Y^{X^+}(t_2^+)$ iff $Y \in \mathcal{F}^+$. For $k \geq 2$ we will modify $t_1^+$ and $t_2^+$ to achieve this equivalence. We exploit that the property just shown for $t_1^+$ and $t_2^+$ does not change, if for any $j$ we replace $t_{i_j1}$ and $t_{i_j2}$ by the concatenated lists $t_{i_j1}{}^\frown t_{i_j1}$ and $t_{i_j2}{}^\frown t_{i_j1}$, respectively. Now let $K = \{k_1,\dots,k_m\} \subseteq I^+$ be maximal such that $X[X_{k_1}(X'_{k_1}) \oplus \cdots \oplus X_{k_m}(X'_{k_m})] \in \mathcal{F}$. Then for $k \in I^+ - K$ we must have $X(X_k[X'_k]) \notin \mathcal{F}$, otherwise also $X[X_{k_1}(X'_{k_1}) \oplus \cdots \oplus X_{k_m}(X'_{k_m}) \oplus X_k(X'_k)] \in \mathcal{F}$ due to property 3 of Theorem 10 and the fact that the two subattributes are reconsilable. Therefore, $K$ is uniquely determined.

Now, if $X(X_{i_1}[Y'_{i_1}],\dots,X_{i_\mu}[Y'_{i_\mu}]) \in \mathcal{F}$, but $X[X_{i_1}(Y'_{i_1}) \oplus \cdots \oplus X_{i_\mu}(Y'_{i_\mu})] \notin \mathcal{F}$, then the uniqueness of $K$ implies $X(X_{i_1}[X'_{i_1}],\dots,X_{i_\mu}[X'_{i_\mu}]) \notin \mathcal{F}$. Hence there must be some $\iota \in \{i_1,\dots,i_\mu\}$ with $t_{\iota1} \neq t_{\iota2}$. We therefore replace $t_{\iota1}$ and $t_{\iota2}$ by the concatenated lists $t_{\iota1}{}^\frown t_{\iota1}$ and $t_{\iota2}{}^\frown t_{\iota1}$, respectively, changing $t_1^+$ and $t_2^+$ accordingly. This gives $\pi^{X^+}_{X[X_{i_1}(Y'_{i_1})\oplus\cdots\oplus X_{i_\mu}(Y'_{i_\mu})]}(t_1^+) \neq \pi^{X^+}_{X[X_{i_1}(Y'_{i_1})\oplus\cdots\oplus X_{i_\mu}(Y'_{i_\mu})]}(t_2^+)$ without destroying previously established equalities and inequalities. This implies $\pi_Y^{X^+}(t_1^+) = \pi_Y^{X^+}(t_2^+)$ iff $Y \in \mathcal{F}^+$ as claimed.

Now let $I^- = \{j_1,\dots,j_\ell\}$ and construct positive integers $x_p$, $y_p$ $(p = 1,\dots,\ell)$ such that for $J = \{j_{m_1},\dots,j_{m_{|J|}}\} \subseteq I^-$ the equation

$$\sum_{p=1}^{|J|} x_{m_p} = \sum_{p=1}^{|J|} y_{m_p}$$

holds iff $X_J\langle\lambda\rangle \in \mathcal{F}$ holds. For the selection of these $x_p$, $y_p$ we can take the same procedure as in the multiset case above. Properties 5(b) and (f) of Theorem 10 ensure that this procedure always produces a solution for the given equations and inequations. Then define

$$t_1^- = [\underbrace{(X_{j_1} : v_{j_1})}_{x_{j_1}\text{-times}}, \dots, \underbrace{(X_{j_\ell} : v_{j_\ell})}_{x_{j_\ell}\text{-times}}] \qquad \text{and} \qquad t_2^- = [\underbrace{(X_{j_1} : v_{j_1})}_{y_{j_1}\text{-times}}, \dots, \underbrace{(X_{j_\ell} : v_{j_\ell})}_{y_{j_\ell}\text{-times}}]$$

with arbitrary values $v_i \in dom(X'_i)$ for $i \in I^-$. In both lists let the elements appear in the order given by the indices.

Finally, define $t_1 = t_1^+ {}^\frown t_1^-$ and $t_2 = t_2^+ {}^\frown t_2^-$ using list concatenation $^\frown$. For these list values the following holds:

1. For $Y \leq X^+$ we have $\pi_Y^X(t_\ell) = \pi_Y^{X^+}(t_\ell^+)$ for $\ell = 1, 2$, which implies $\pi_Y^X(t_1) = \pi_Y^X(t_2)$ iff $Y \in \mathcal{F}^+$.

2. For $Y \not\leq X^+$ we either have $Y \geq X_{\{j\}}[\lambda]$ for some $j \in I^-$ or $Y = X_I[\lambda]$ with $I \not\subseteq I^+$. In the first case we have $Y \notin \mathcal{F}$ and $\pi_{X_{\{j\}}[\lambda]}^X(t_1) = \pi_{X_{\{j\}}[\lambda]}^X(t_1^-) \neq \pi_{X_{\{j\}}[\lambda]}^X(t_2^-) = \pi_{X_{\{j\}}[\lambda]}^X(t_2)$, hence also $\pi_Y^X(t_1) \neq \pi_Y^X(t_2)$.

   In the second case we have $\pi_Y^X(t_1) = \pi_Y^X(t_2)$ iff $\pi_{X_{I \cap I^-}[\lambda]}^X(t_1^-) = \pi_{X_{I \cap I^-}[\lambda]}^X(t_2^-)$, because $\pi_{X_{I \cap I^+}[\lambda]}^X(t_1) = \pi_{X_{I \cap I^+}[\lambda]}^X(t_1^+) = \pi_{X_{I \cap I^+}[\lambda]}^X(t_2^+) = \pi_{X_{I \cap I^+}[\lambda]}^X(t_2)$ due to the construction of $t_1^+$ and $t_2^+$. Due to property 5(d) of Theorem 10 we have $Y \in \mathcal{F}$ iff $X_{I \cap I^-}[\lambda] \in \mathcal{F}$. Then due to the construction of $t_1^-$ and $t_2^-$ we have $X_{I \cap I^-}[\lambda] \in \mathcal{F}$ iff $\pi_{X_{I \cap I^-}[\lambda]}^X(t_1^-) = \pi_{X_{I \cap I^-}[\lambda]}^X(t_2^-)$.

Both cases together imply $\pi_Y^X(t_1) = \pi_Y^X(t_2)$ iff $Y \in \mathcal{F}$, which completes this final case of a list attribute with a component union attribute.

## 4 Conclusions

In this article we laid the foundations to complete our work on the axiomatisation of functional dependencies and weak functional dependencies on trees with restructuring. These trees arise from constructors for complex values comprising arbitrarily nesting of finite sets, multisets, lists, disjoint unions and records and a "null" attribute. Restructuring, i.e. non-trivial equivalence between these attributes are mainly due to the presence of the union constructor.

Our previous work in [Sali and Schewe, 2006] captured the case, where so called counter-attributes were excluded. The generalisation in [Sali and Schewe, 2008] requires a very deep and very technical investigation of certain ideals in the algebra of subattributes, which is what we presented in this article. We proved the central theorem on coincidence ideals, which gives an exact characterisation of sets of subattributes, on which two complex values coincide. This result is essential for the completeness proof in [Sali and Schewe, 2008].

Thus, in a sense the work presented in this article is mainly a stepping stone for continuing the work on dependency theory, but it may have other application, e.g. for research on the existence of Armstrong instances (see e.g. [Sali and Schewe, 2006]).

## References

[Abiteboul et al., 2000] Abiteboul, S., Buneman, P., and Suciu, D. (2000). *Data on the Web: From Relations to Semistructured Data and XML*. Morgan Kaufmann Publishers.

[Abiteboul and Hull, 1988] Abiteboul, S. and Hull, R. (1988). Restructuring hierarchical database objects. *Theoretical Computer Science*, 62(1-2):3–38.

[Batini et al., 1992] Batini, C., Ceri, S., and Navathe, S. B. (1992). *Conceptual Database Design: An Entity-Relationship Approach*. Benjamin Cummings.

[Chen, 1976] Chen, P. P. (1976). The Entity-Relationship model: Towards a unified view of data. *ACM Transactions Database Systems*, 1:9–36.

[Chen, 1983] Chen, P. P. (1983). English sentence structure and Entity-Relationship diagrams. *Information Science*, 29:127–149.

[Hartmann, 2001] Hartmann, S. (2001). Decomposing relationship types by pivoting and schema equivalence. *Data & Knowledge Engineering*, 39:75–99.

[Hartmann et al., 2004] Hartmann, S., Link, S., and Schewe, K.-D. (2004). Weak functional dependencies in higher-order datamodels. In Seipel, D. and Turull Torres, J. M., editors, *Foundations of Information and Knowledge Systems*, volume 2942 of *LNCS*, pages 116–133. Springer Verlag.

[Hartmann et al., 2005] Hartmann, S., Link, S., and Schewe, K.-D. (2005). Functional dependencies over XML documents with DTDs. *Acta Cybernetica*, 17(1):153–171.

[Hartmann et al., 2006] Hartmann, S., Link, S., and Schewe, K.-D. (2006). Axiomatisation of functional dependencies in the presence of records, lists, sets and multisets. *Theoretical Computer Science*, 355:167–196.

[Hull and King, 1987] Hull, R. and King, R. (1987). Semantic database modeling: Survey, applications and research issues. *ACM Computing Surveys*, 19(3).

[Mok et al., 1996] Mok, W. Y., Ng, Y. K., and Embley, D. W. (1996). A normal form for precisely characterizing redundancy in nested relations. *ACM Transactions on Database Systems*, 21:77–106.

[Özsoyoglu and Yuan, 1987] Özsoyoglu, Z. M. and Yuan, L. Y. (1987). A new normal form for nested relations. *ACM Transactions on Database Systems*, 12:111–136.

[Paredaens et al., 1989] Paredaens, J., De Bra, P., Gyssens, M., and Van Gucht, D. (1989). *The Structure of the Relational Database Model*. Springer-Verlag.

[Sali, 2004] Sali, A. (2004). Minimal keys in higher-order datamodels. In Seipel, D. and Turull Torres, J. M., editors, *Foundations of Information and Knowledge Systems*, volume 2942 of *LNCS*, pages 242–251. Springer Verlag.

[Sali and Schewe, 2006] Sali, A. and Schewe, K.-D. (2006). Counter-free keys and functional dependencies in higher-order datamodels. *Fundamenta Informaticae*, 70(3):277–301.

[Sali and Schewe, 2008] Sali, A. and Schewe, K.-D. (2008). Weak functional dependencies on trees with restructuring. submitted for publication.

[Schewe and Thalheim, 1993] Schewe, K.-D. and Thalheim, B. (1993). Fundamental concepts of object oriented databases. *Acta Cybernetica*, 11(4):49–85.

[Tari et al., 1997] Tari, Z., Stokes, J., and Spaccapietra, S. (1997). Object normal forms and dependency constraints for object-oriented schemata. *ACM Transactions on Database Systems*, 22:513–569.

[Thalheim, 1992] Thalheim, B. (1992). Foundations of entity-relationship modeling. *Annals of Mathematics and Artificial Intelligence*, 6:197–256.

[Thalheim, 2000] Thalheim, B. (2000). *Entity-Relationship Modeling: Foundations of Database Technology*. Springer-Verlag.

[Tjoa and Berger, 1993] Tjoa, A. M. and Berger, L. (1993). Transformation of requirement specifications expressed in natural language into an EER model. In *Entity-Relationship Approach*, volume 823 of *LNCS*. Springer-Verlag.